Chapter - 2

SEQUENCE OF INTEGRALS
AND THEIR APPLICATIONS
SEQUENCE OF INTEGRALS AND THEIR APPLICATIONS

In this chapter, we introduce some sequence of integrals and apply them make to evaluate an infinite product representation of Legendre function when its non-zero zeros are given. Also, we derive some analytic formula for obtaining bilateral generating functions which may be used to compute various mathematical, physical and engineering problems.

2.1 Introduction:

The generating function of Legendre polynomials $P_n(x)$ is defined by

$$
\sum_{n=0}^{\infty} P_n(x) t^n = \left(1 - 2xt + t^2\right)^{-1/2} \quad \ldots \quad (2.1.1)
$$

we define the sequence of integrals, given by

$$
I_{\nu} = \frac{1}{2\pi i} \int_{c-} \frac{F(t)}{(t-z)^{\nu+1}} dt, \quad \omega = \sqrt{-1}, \quad \nu = 1, 2, \ldots
$$

and suppose that $G(z) = z^{-\nu} P_n(z)$ so that

$$
F(t) = \frac{t G'(t)}{G(t)} = \frac{P_{n-1}}{P_n(t)} \quad \ldots \quad (2.1.2)
$$

In this chapter, the paper entitled "ON SOME SEQUENCES OF INTEGRALS AND THEIR APPLICATIONS" is published in the journal Bull. Cal. Math. Soc. 100, (5) (2008), 563-572.
The recurrence relation
\[ P_{n-1}'(x) = xP_{n}'(x) - nP_n(x), \quad \left( P_n'(x) = \frac{d}{dx} P_n(x) \right), \quad \ldots \quad (2.1.3) \]
implies
\[ G'(z) = z^{-n} P_{n-1}'(z). \quad \ldots \quad (2.1.4) \]

Let \( z_k \neq 0 \), for all \( k=1,2,...,n \), are zeros of \( P_n(z) \) which are simple.

Then from (2.1.3), we get
\[ P_{n-1}'(z_k) = z_k P_n'(z_k), \quad k = 1, 2, ..., n \]
and \( P_{n-1}'(-z_k) = -z_k P_n'(-z_k), \quad k = 1, 2, ..., n \)

Motivated by this work, we evaluate the infinite product representation of Legendre function when its non-zero zeros are known.

\( H \)-function of two variables is defined by Mittal and Gupta (1972) (see also Saxena and Nishimoto (1994)) and Srivastava, Gupta and Goyal (1982) as
\[
H \left[ \begin{array}{c} x \\ y \\ \end{array} \right] = H^{b_1, m_1; b_2, m_2; n_1, n_2}_{p_1, q_1; p_2, q_2; n_1} \left[ \begin{array}{c} (a_j; \alpha_j, A_j)_{l_1, p_1} : (c_j; \gamma_j)_{l_2, p_2} : (e_j; E_j)_{l_3, p_3} \\ (b_j; \beta_j, B_j)_{l_4, q_1} : (d_j; \delta_j)_{l_5, q_2} : (f_j; F_j)_{l_6, q_3} \\ \end{array} \right]
\]
\[
= \frac{1}{(2\pi \omega)^{\frac{n}{2}}} \int_{u} \int_{v} \psi(u,v)\psi(u,v) x^y y^\nu du dv, \quad \omega = \sqrt{-1}, \quad \ldots \quad (2.1.6)
\]
where
\[
\psi(u,v) = \frac{\prod_{j=1}^{n} \Gamma(1-a_j + \alpha_j u + A_j v)}{\prod_{j=1}^{n} \Gamma(a_j - \alpha_j u - A_j v) \prod_{j=1}^{n} \Gamma(1-b_j + \beta_j u + B_j v)}
\]
\[
\psi_1(u) = \frac{\prod_{j=1}^{m_1} \Gamma(d_j - \delta_j u) \prod_{j=1}^{n_1} \Gamma(1 - c_j + \gamma_j u)}{\prod_{j=m_1+1}^{q_1} \Gamma(1 - d_j + \delta_j u) \prod_{j=n_1+1}^{p_1} \Gamma(c_j - \gamma_j u)}
\]

and

\[
\psi_2(v) = \frac{\prod_{j=1}^{m_2} \Gamma(f_j - F_j v) \prod_{j=1}^{n_2} \Gamma(1 - e_j + E_j v)}{\prod_{j=m_2+1}^{q_2} \Gamma(1 - f_j + F_j v) \prod_{j=n_2+1}^{p_2} \Gamma(e_j - E_j v)}
\]  

..... (2.1.7)

an empty product is interpreted as unity, the integers \(p_1, p_2, p_3, q_1, q_2, q_3, n_1, n_2, n_3, m_2, m_3\) are non-negative integers such that

\[0 \leq n_1 \leq p_1; 0 \leq n_2 \leq p_2; 0 \leq n_3 \leq p_3; q_1 \geq 0, 0 \leq m_2 \leq q_2; 0 \leq m_3 \leq q_3\]

and the coefficients \(\alpha's, \beta's, \gamma's, \delta's\), as also \(A's, B's, E's, F's\) are all positive. The contour \(L_1\) lies in the complex \(u\)-plane and runs from \(-\omega \infty\) to \(+\omega \infty\) with loops, if necessary, to ensure that the poles of \(\Gamma(d_j - \delta_j u), (j = 1, 2, ..., m_2)\) lie to the right, and the poles of 
\(\Gamma(1 - c_j + \gamma_j u), (j = 1, 2, ..., n_1)\) and \(\Gamma(1 - a_j + \alpha_j u + A_j v), (j = 1, 2, ..., n_1)\) lie to the left of \(L_1\).

Whereas, the contour \(L_2\) lies in the complex \(v\)-plane running from \(-\omega \infty\) to \(\omega \infty\) with loops, if necessary, to ensure that the poles of 
\(\Gamma(f_j - F_j v), (j = 1, 2, ..., m_3), \) lie to the right and the poles of 
\(\Gamma(1 - e_j + E_j v), (j = 1, 2, ..., n_3), \Gamma(1 - a_j + \alpha_j u + A_j v), (j = 1, 2, ..., n_3), \) lie to the left of \(L_2\). All poles are simple poles. The function \(H \left[ \begin{array}{c} x \\ y \end{array} \right] \) is an analytic function of \(x\) and \(y\),
If

\[ \rho_1 = \sum_{j=1}^{n_i} \alpha_j + \sum_{j=1}^{n_i} \gamma_j - \sum_{j=1}^{n_i} \beta_j - \sum_{j=1}^{n_i} \delta_j \leq 0 \]

and

\[ \rho_2 = \sum_{j=1}^{n_i} A_j + \sum_{j=1}^{n_i} E_j - \sum_{j=1}^{n_i} B_j - \sum_{j=1}^{n_i} F_j \leq 0 \] ...... (2.1.8)

The double integral in (2.1.6) converges absolutely under the conditions

\[ \Delta_1 = \sum_{j=1}^{n_i} \alpha_j - \sum_{j=n_i+1}^{n} \alpha_j - \sum_{j=1}^{n_i} \beta_j + \sum_{j=n_i+1}^{n} \beta_j - \sum_{j=n_i+1}^{n} \delta_j - \sum_{j=1}^{n_i} \delta_j + \sum_{j=n_i+1}^{n} \gamma_j - \sum_{j=1}^{n_i} \gamma_j > 0 \]

and

\[ \Delta_2 = \sum_{j=1}^{n_i} A_j - \sum_{j=n_i+1}^{n} A_j - \sum_{j=1}^{n_i} B_j + \sum_{j=n_i+1}^{n} B_j + \sum_{j=1}^{n_i} F_j - \sum_{j=n_i+1}^{n} F_j + \sum_{j=1}^{n_i} E_j - \sum_{j=n_i+1}^{n} E_j > 0 \]

such that

\[ |\arg(x)| < \frac{1}{2} \Delta_1 \pi \quad \text{and} \quad |\arg(y)| < \frac{1}{2} \Delta_2 \pi. \] ...... (2.1.9)

We use following formulae for our investigations.

Fox's H-functions (1982) (Also, see Srivastava and Manocha (1984)).
\[ H_{m,n}^{p,q} \left[ \frac{(a_j, A_j)_{(p)}}{(b_j, B_j)_{(q)}} \right] = \frac{1}{2\pi i} \left[ \prod_{j=1}^{m} \Gamma(b_j - B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s) \right] ds, \]

where \( m, n, p, q \) satisfy the inequalities

\[ 0 \leq m \leq q \quad \text{and} \quad 0 \leq n \leq p, \]

the coefficients \( A_1, \ldots, A_p \) and \( B_1, \ldots, B_q \) are positive real numbers and the complex parameters \( a_1, \ldots, a_p \) and \( b_1, \ldots, b_q \) are so constrained that no poles of the integrand coincide.

Further, if

\[ \Omega = \sum_{j=1}^{p} A_j - \sum_{j=m+1}^{n} B_j > 0 \]

then, the integral in (2.1.10) is absolutely convergent and analytic in the sector \(|\arg(z)| < \frac{1}{2} \Omega \pi\), the point \( z = 0 \) being tacitly excluded.

The formula for Gaussian hypergeometric function,

\[ _2 F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \]

in Mellin-Barnes integral form (see, Hochstadt 1986)
\[ _2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-\infty}^{\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)(-z)^s}{\Gamma(c+s)} ds, \]

...... (2.1.12)

\[ a, b \neq 0, -1, -2, \ldots \text{ and } |\text{arg}(z)| < \pi. \]

The path in (2.1.12) is so chosen, that all poles of \( \Gamma(a+s)\Gamma(b+s) \) lie on the left and those of \( \Gamma(-s) \) lie on the right. For \( \text{Re}(c-a-b) \), we can close the contour on the right and recapture the power series (2.1.11) by means of contour integration.

The formula

\[ (1 + y)^{-\delta} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(\delta + i t)\Gamma(-i t) y^i}{\Gamma(\delta)} dt, \]

...... (2.1.13)

provided that \( \delta > 0 \) and \( |\text{arg}(y)| < \pi. \)

Here, further in our work, we also define following sequence of integrals formulae

\[ I^r_{x,T} = \int_0^{x/T} t^{a+r-1}(x-t T)^{\rho-1} \left( \frac{y}{2} + \frac{1}{2}, \frac{\gamma}{2}; y + 1; \frac{-4xtT}{(x-t T)^2} \right) \times _2F_1(a,b;c;-y(1-xtT + t^2 T^2))^{x-t T} (x-t T)^{\alpha} dt, \text{ } r = 0, 1, 2, \ldots \]

...... (2.1.14)

provided that \( y > 0, \lambda > 0, \mu > 0; a, b, c, \rho, \alpha, \gamma \) are real or complex, \( x > 0 \) and \( T \neq 0 \), for \( 0 \leq t < \infty. \) It may be noted that conditions stated in (2.1.12) are also valid for (2.1.14).
\[ J'_{x/T} = \sum_{r=0}^{\infty} t^{r+1}(x-tT)^{q-1} \binom{\gamma}{r+1} F_1 \left( \frac{\gamma}{2} + \frac{1}{2}, \frac{\gamma}{2}, \gamma + 1; \frac{-4xtT}{(x-tT)^2} \right) \]
\[
\times H^{m}_{p,q'} \left[ \gamma(1-xtT+t^2T^2)^{1/2} (x-tT)^{r} \right] \left[ (a_j, A_j)_{r,p} (b_j, B_j)_{r,q} \right] dt,
\]

\[
J'_{x/T} = \sum_{r=0}^{\infty} t^{r+1}(x-tT)^{q-1} \binom{\gamma}{r+1} F_1 \left( \frac{\gamma}{2} + \frac{1}{2}, \frac{\gamma}{2}, \gamma + 1; \frac{-4xtT}{(x-tT)^2} \right) \]
\[
\times H^{m}_{p,q'} \left[ \gamma(1-xtT+t^2T^2)^{1/2} (x-tT)^{r} \right] \left[ (a_j, A_j)_{r,p} (b_j, B_j)_{r,q} \right] S_{N}^{\mu} [zt^\nu] dt,
\]

\[
K'_{x/T} = \sum_{r=0}^{\infty} t^{r+1}(x-tT)^{q-1} \binom{\gamma}{r+1} F_1 \left( \frac{\gamma}{2} + \frac{1}{2}, \frac{\gamma}{2}, \gamma + 1; \frac{-4xtT}{(x-tT)^2} \right) \]
\[
\times H^{m}_{p,q'} \left[ \gamma(1-xtT+t^2T^2)^{1/2} (x-tT)^{r} \right] \left[ (a_j, A_j)_{r,p} (b_j, B_j)_{r,q} \right] S_{N}^{\mu} [zt^\nu] dt,
\]

\[
r=0,1,2,..., \quad \text{provided that all conditions of (2.1.10) and (2.1.14) are satisfied.}
\]

\[
S_{N}^{M} [z] = \sum_{k=0}^{[N/M]} (-1)^{M-k} \frac{z^{k}}{K!}
\]

\[
\text{where } M \text{ is an arbitrary positive integer and the coefficients } \Phi_{N,k} (N,K > 0) \text{ are arbitrary constants, real or complex.}
\]

In this chapter, making an appeal to above integral formulae (2.1.14)-(2.1.16), we evaluate analytic formulae for bilateral generating functions,
which may be applied to various problems in physical sciences.

2.2. An Infinite Product Representation For Legendre Functions

We have

\[ F(z) = z \left( \frac{G'(z)}{G(z)} \right) = \frac{P_{n+1}(z)}{P_n(z)} \]  

...... (2.2.1)

(By means of (2.1.2) and (2.1.4)).

If we consider a sequence of circles \( \{C_u\} \) about the points such that \( C_u \) includes all points \( \pm z_1, \pm z_2, \ldots, \pm z_n \) and the point \( z \) twice and not passing through any zeros, then \( F \) is uniformly bounded on these, so that \( |F(t)| < M, M > 0 \).

The sequence of integrals

\[ I_u = \frac{1}{2\pi i} \int_{C_u} \frac{F(t)}{(t-z)^2} \, dt \]  

...... (2.2.2)

must converge to zero in view of the fact that for fixed \( z \neq z_n \) and large \( t \),

\[ \left| \frac{F(t)}{(t-z)^2} \right| \leq \frac{M}{|t|^2} \]  

...... (2.2.3)

It follows that if \( R_u \) denotes the radius of \( C_u \), then
\[ |I_u| \leq \frac{M}{R_o} \] ...... (2.2.4)

so that

\[ \lim_{u \to \infty} I_u = 0. \] ...... (2.2.5)

Further, by virtue of calculus of residues, (2.2.2) implies

\[ I_u = \lim_{t \to \infty} \frac{d}{dt} F(t) + \sum_{k=1}^{\infty} \left[ \lim_{t \to z_k} (t - z_k) \frac{F(t)}{(t - z_k)^2} + \lim_{t \to -z_k} (t + z_k) \frac{F(t)}{(t - z_k)^2} \right]. \] ...... (2.2.6)

Employing (2.2.1) into (2.2.6), and solving it by using (2.1.5), and taking \( u \to \infty \), (2.2.6) yields the following, upon integration

\[ \frac{G'(z)}{G(z)} = \sum_{i=1}^{\infty} \left( \frac{2z}{z^2 - z_i^2} \right) + \frac{c-2}{z}. \] ...... (2.2.7)

From the formula

\[ P_n(z) = \frac{1}{n!} \left[ z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \ldots \right], \]

It, indeed, gives

\[ \lim_{z \to \infty} \frac{zP'_n(z)}{P_n(z)} = n \]

and so that with the aid of (2.1.3),(2.2.1) and (2.2.7), we find that

\[ c = 0 \text{ as } z \to \infty. \] ...... (2.2.8)
Hence, from (2.2.7) and (2.2.8) we get

\[
\frac{G'(z)}{G(z)} = \sum_{k=1}^{\infty} \left( \frac{2z}{z^2 - z_k^2} \right) - \frac{2}{z}. \quad \text{...... (2.2.9)}
\]

Now, on integrating (2.2.9), we get

\[
G(z) = c_1 z^{-3} \prod_{k=1}^{\infty} (z^2 - z_k^2). \quad \text{...... (2.2.10)}
\]

Again, on putting \( G(z) = z^{-n} P_n(z) \) by (2.1.2) in (2.2.10), we evaluate

\[
P_n(z) = c_1 z^{-n-2} \prod_{k=1}^{\infty} (z^2 - z_k^2). \quad \text{...... (2.2.11)}
\]

Then, on setting \( z=1 \), in (2.2.11) and using \( P_n(1) = 1 \) in it, we get

\[
c_1 = \frac{1}{\prod_{k=1}^{\infty} (1 - z_k^2)} \quad \text{...... (2.2.12)}
\]

Finally, from (2.2.11) and (2.2.12), we obtain the infinite product representation for Legendre function as

\[
P_n(z) = z^{n-2} \prod_{k=1}^{\infty} \left( \frac{1 - \frac{1}{z^2}}{1 - \frac{1}{z_k^2}} \right), \quad \text{...... (2.2.13)}
\]

where \( z_k \neq 0 \) and \( \pm 1 \), for all \( k=1,2,... \) and \( 1 \leq z < \infty \).

2.3. Formulae For Bilateral Generating Functions-

In this section, we evaluate following analytic formulae for bilateral generating functions,
\[ \sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) I'_{x/r} T' = \frac{x^{r+\alpha-1}}{T^{\rho} \Gamma(a) \Gamma(b)} H^{0.23,23,1}_{2,12,23,1} \left[ \begin{array}{c} \lambda x, \mu, 1 \\ \mu x, \mu, 1 \\ 1/2 : \lambda, 1, 1/2 : \lambda, 1 \\ 1/2 : \lambda, 1, 1/2 : \lambda, 1 \\ \lambda x, \mu, 1 \\ \mu x, \mu, 1 \end{array} \right] \]

Provided that \(|\arg(yx^\nu)| < \pi\) and \(|\arg(-x^2)| < \pi\) and all conditions mentioned in (2.1.14) are satisfied.

**Proof:**

Multiply in \( P_r \left( \frac{x}{2} \right) T' \) both sides of (2.1.14), take its sum from \( r=0 \) to \( \infty \) and then in the right-hand side of it imply the change of order of summation and integration, we obtain

\[ \sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) I'_{x/r} T' = \int_0^{\infty} t^{\rho-1} (x-tT)^{\alpha-1} \left( 1 - \left( \frac{tT}{x} \right)^\gamma \right)^{1/2} \left( 1 - xtT + t^2T^2 \right)^{-1/2} x^2 F_1 \left( a, b; c; -y(1-xtT+t^2T^2)^{-\nu} (x-tT)^\nu \right) dt. \]

Using (2.1.12), we get

\[ \sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) I'_{x/r} T' = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)2\pi\omega} \int_{-\infty}^{\infty} \Gamma(a+u)\Gamma(b+u)\Gamma(-u) y^u \Gamma(c+u) \left( 1 - tT(x-tT) \right)^{1/2} \left( 1 - \frac{tT}{x} \right)^\gamma \left( 1 - xtT + t^2T^2 \right)^{-\nu} (x-tT)^\nu dt du. \]
Further, on applying (2.1.13) in the right-hand side, we get

\[
\sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) I_{\frac{\pi}{2}, T'} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)(2\pi\omega)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(a+u)\Gamma(b+u) \\
\times \frac{\Gamma(-u)\Gamma(1/2 + \lambda u + \nu)\Gamma(-v)(-T)^{\nu} y^{\nu}}{\Gamma(1/2 + \lambda u)} \\
\times \int_{0}^{xT} t^r u^{r+1} (x-tT)^{\alpha+\beta u+r-1} \left(1 - \frac{tT}{x}\right) \, dt \, du \, dv
\]
\[
...... (2.3.4)
\]

Finally, in the right-hand side of (2.3.4) we get \( t = \frac{XS}{T} \), use Beta function and invoke (2.1.6)-(2.1.9), we obtain (2.3.1).

The second bilateral generating function formula is

\[
\sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) I_{\frac{\pi}{2}, T'} = \frac{x^{\alpha+\beta-1}}{T^\rho} H_{\alpha, \beta, \rho+1, 1}^{0, 2, m, n+1, 1} \\
\left[ \begin{array}{c}
\begin{array}{c}
x x^\rho \\
-x^2
\end{array}
\end{array} \right] \\
\left[ \begin{array}{c}
\begin{array}{c}
(1/2 : \lambda, 1), (1 - \alpha - \gamma : \mu, 1); \\
(1 - \alpha - \rho : \mu, 2);
\end{array}
\end{array} \right]
\left[ \begin{array}{c}
\begin{array}{c}
(a_j : A_j), (b_j : B_j); \\
(0, 1)
\end{array}
\end{array} \right]
\]
\[
...... (2.3.5)
\]

provided that

\[
|\text{arg}(yx^\rho)| < \Delta, \frac{\pi}{2}
\]

and

\[
|\text{arg}(-x^2)| < \pi, \sum_{j=m}^{q} B_j - \sum_{j=n}^{l} A_j \geq 0,
\]

\[
\Delta = \sum_{j=1}^{m} A_j - \sum_{j=m+1}^{n} A_j + \sum_{j=1}^{n} B_j - \sum_{j=n+1}^{m} B_j > 0.
\]
Proof:

On multiplying by \( P_r \left( \frac{x}{2} \right) T^r \), both sides of (2.1.15) and then, consider the sum \( r \) from 0 to \( \infty \), we get

\[
\sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) J_{x/r} T^r = \int_0^{xT} t^{\rho-1} (x-tT)^{\alpha-1} \left( \frac{1-tT}{x} \right)^\gamma \left( 1-xtT+t^2T^2 \right)^{-1/2} \times H_{p,q}^{m,n} \left[ \left( \frac{z(x)}{T} \right)^\mu \right] dt.
\]

Now, using (2.1.10) and following the analysis those used for (2.3.3) and (2.3.4), we obtain (2.3.5).

Third bilateral generating function formula is

\[
\sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) K_{x/r} T^r = \frac{x^{\rho-1} \left( \frac{W/M}{\lambda} \right)}{T^\rho} \sum_{k=0}^{\infty} (-N)_{\lambda,k} \Phi_{\lambda,k} \left( \frac{z(x)}{T} \right)^k \times H_{2,m,\mu}^{2,p+1,1} \left[ \begin{array}{c}
(1/2, 1), (1-\alpha-\gamma : \mu, 1), (a_j : A_j)_{\lambda,p}, (1-\rho-\mu : 1), (1-\alpha-\gamma-\rho-\mu : 2), (1/2, \mu, 1), (b_j, B_j)_{\lambda,q} : (0 : 1) : \mu, \xi^2 \end{array} \right],
\]

provided that \( |\arg(yx^\mu)| < \Delta, \frac{\pi}{2} \) and \( |\arg(-x^2)| < \pi \), where

\[
\Delta = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{m} A_j + \sum_{j=1}^{n} B_j - \sum_{j=m+1}^{l} B_j > 0, \quad \sum_{j=1}^{l} B_j - \sum_{j=1}^{n} A_j \geq 0,
\]

including the conditions stated for (2.1.14).
Proof:

On multiplying by $P_r \left( \frac{x}{2} \right) T^r$, both sides of (2.1.16) and sum it for $r$ from 0 to $\infty$, we get

$$\sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) K_{\nu}^T T^r = \int_0^{xT} t^{a-1} (x-tT)^{a-1} \left( \frac{1-tT}{x} \right)^r (1-xtT+t^2T^2)^{-1/2}$$

$$\times H_{p,q}^{m,n} \left[ y(1-xtT+t^2T^2)^{-\lambda} (x-tT)^{\mu} \left( a_j A_j \right)_{i,p} \left( b_j B_j \right)_{i,q} \right] \int S_{x}^{M} [zt^n] dt.$$

...... (2.3.8)

Employing (2.1.17), we get

$$\sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) K_{\nu}^T T^r = \sum_{k=0}^{\infty} (-N)^{N,k} \frac{z^k}{k!} \int_0^{xT} t^{(a+k-1)} (x-tT)^{a-1} \left( \frac{1-tT}{x} \right)^r$$

$$\times \left\{ 1-tT(x-tT) \right\}^{-1/2} H_{p,q}^{m,n} \left[ y \left\{ 1-tT \right\}^{\lambda} (x-tT)^{\mu} \left( a_j A_j \right)_{i,p} \left( b_j B_j \right)_{i,q} \right] dt.$$

...... (2.3.9)

Proof is completed upon following that of employed for (2.3.6).

Further, setting $\frac{x}{T} = \theta$ in (2.3.1),(2.3.5) and (2.3.7), we get the bilateral generating functions in the forms,
\[
\sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) I_r^\prime T' = T^{a-1} \Theta^{\alpha+\alpha-1} \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} H_{2,1,2,3,1,1}^0 \left[ \begin{array}{c}
\left( \frac{1}{2} : \lambda, 1 \right), (1 - \alpha - \gamma : \mu, 1) ; (1 - a : 1), (1 - b : 1) ; (1 - \rho : 1) \\
y X^\mu \\
-x \\
(1 - \alpha - \rho - \gamma : \mu, 2) ; (0 : 1), (1 - c : 1), \left( \frac{1}{2} : \lambda \right) ; (0, 1) 
\end{array} \right],
\]

(2.3.10)

provided that \( |\arg(yx^\mu)| < \pi \) and \( |\arg(-x^2)| < \pi \) and all conditions of (2.1.14) are satisfied.

\[
\sum_{r=0}^{\infty} P_r \left( \frac{x}{2} \right) J_r^\prime T' = T^{a-1} \Theta^{\alpha+\alpha-1} H_{2,1,2,3,1,1}^0 \left[ \begin{array}{c}
\left( 1/2 : \lambda, 1 \right), (1 - \alpha - \gamma : \mu, 1) ; (a_j : A_j)_{l,p} ; (1 - \rho : 1) \\
y X^\mu \\
-x \\
(1 - \alpha - \rho - \gamma : \mu, 2) ; (1/2 : \lambda), (b_j : B_j)_{l,q} ; (0, 1) 
\end{array} \right],
\]

(2.3.11)

provided that \( |\arg(yx^\mu)| < \Delta, \frac{\pi}{2} \) and \( |\arg(-x^2)| < \pi \), where
\[ \Delta_i = \sum_{j=1}^{n} A_j - \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \sum_{j=1}^{m} B_j > 0, \sum_{j=1}^{m} B_j - \sum_{j=1}^{m} A_j \geq 0. \]

and

\[ \sum_{r=0}^{P} \left( \frac{x^r}{2} \right) K_{q}^{r} = T^{n+1} \theta^{n+1} \sum_{k=0}^{[N/k]} (-N)_{N,k} \Phi_{N,k} \left( \frac{z(\theta)^x}{k!} \right) \]

\[ \times H_{2.2.2.2}^{0.0.0.0} \left[ \begin{array}{c} (1/2: \lambda, l) \left( 1 - \alpha - \nu : \mu, l \right) : (a_j : A_j)_{l_0} : (1 - \rho - u k : l) : J x^\mu, -x^\nu \left( 1 - \alpha \gamma - \rho u k : \mu, 2 \right) : (1/2: \lambda) (b_j)_{l_0} : (0, l) : \end{array} \right] \]

provided that \[ |\arg(\arg(x))| < \Delta_i \frac{\pi}{2} \] and \[ |\arg(-x^2)| < \pi, \] where \( \Delta_i \) is defined above and \( \sum_{j=1}^{m} B_j - \sum_{j=1}^{m} A_j \geq 0. \)

Due to the most general character of the general class of polynomials \( S^M_N \), appearing in (2.1.16), (2.3.7) and (2.3.12), special cases involving orthogonal and nonorthogonal polynomials, scattered in literature can be derived, as special cases.
References


