Chapter - 4

GENERATING AND BILATERAL GENERATING FUNCTIONS INVOLVING GE GENBOUR POLYNOMIALS WITH PRODUCT OF SEQUENCE OF INTEGRALS
CHAPTER-4

GENERATING AND BILATERAL
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PRODUCT OF SEQUENCE OF INTEGRALS

In this chapter, we define some sequence of integrals and then
investigate some of sequence of functions involving Gegenbour
polynomials with product of sequence of integrals and thus on
applying them, we derive their generating and bilateral generating
functions. Also, we make their applications to obtain generating
relations.

4.1 Introduction

In our work, we introduce following sequence of integrals.

\[
I_r(x,T) = \int_0^{\pi T} \left[ \sum_{\mu=0}^{\infty} \left( x - iT \right)^{\mu} \right] \frac{\Gamma \left( \frac{\alpha + 1}{2}, 2 \right)}{\Gamma \left( \frac{\alpha}{2}, -4\pi T \right) (x - iT)^\alpha} \frac{4\pi T}{(x - iT)^\alpha} \left[ \frac{1}{x - iT} \right] \frac{1}{x - iT} \left[ \frac{1}{1 - iT (x - iT)} \right] \frac{1}{x - iT} \left[ \frac{1}{1 - iT (x - iT)} \right] \frac{dx}{x - iT},
\]

r=0,1,2...

...... (4.1.1)

In this chapter, the paper entitled "ON SOME GENERATING AND BILATERAL GENERATING FUNCTIONS THROUGH SOME SEQUENCE OF FUNCTIONS INVOLVING GEGENBOUR POLYNOMIALS WITH PRODUCT OF SEQUENCE OF INTEGRALS" is communicated with Jour., Pure Math. Calcutta, India

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provided that \( x, y, \lambda, \mu > 0; \rho, \sigma, \gamma, a, b, c \) are real or complex 
\( a, b \neq 0, -1, \ldots; T \neq 0, 0 \leq t < \infty; |\arg(\gamma x^n)| < \pi. \text{Re}(\sigma) > \text{Re}(\alpha), \nu > \gamma, \text{Re}(\rho) > 0. \)

\[
J_r(x,T) = \int_0^{\infty} e^{-\alpha \tau} (1 - \alpha \tau)^{-1} I_{\frac{\gamma + 1}{2}} \left[ \frac{4\sqrt{\alpha \tau}}{x \gamma + 1} \right] \times _2 F_1 \left[ \frac{\gamma + 1}{2}, \frac{\gamma + 1}{2}; \frac{\gamma + 1}{\gamma + 1}; \frac{4\sqrt{\alpha \tau}}{x \gamma + 1} \right] \times H_{\rho q}^{m n} \left[ \frac{y}{1 - \alpha x \tau + \tau^2} \right] (x - \alpha \tau)^{\alpha} \left( \frac{a_j A_j}{b_j B_j} \right) \, d\tau,
\]

\( r = 0, 1, 2, \ldots \)

..... (4.1.2)

where all conditions of (4.1.1) are included and \( H_{\rho q}^{m n}[z] \) is the Fox's [1] H-function (see, Srivastava and Manocha [9]) defined by

\[
H_{\rho q}^{m n} \left[ \left( \frac{a_j A_j}{b_j B_j} \right) \right] = \frac{1}{(2\pi i)} \int_{-\infty}^{\infty} \Gamma(1 - a_j + A_j, s) \prod_{j=1}^{m} \Gamma(b_j - B_j, s) \prod_{j=1}^{n} \Gamma(1 - b_j + B_j, s) \prod_{j=m+1}^{n} \Gamma(a_j - A_j, s) \, ds.
\]

..... (4.1.3)

where an empty product is interpreted as unity, the integers \( m, n, p, q \) satisfy the inequalities \( 0 \leq m \leq q \) and \( 0 \leq n \leq p \), the coefficients \( A_1, \ldots, A_p \) and \( B_1, \ldots, B_q \) are positive real numbers such that

\[
\Omega = \sum_{j=1}^{p} A_j - \sum_{j=m+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{n} B_j > 0, \quad |\arg(z)| < \frac{1}{2} \Omega \pi,
\]

the complex parameters \( a_1, \ldots, a_p \) and \( b_1, \ldots, b_q \) are such that

\( A_j (b_j + \nu) \neq B_j (a_j - \eta - 1), (\nu, \eta = 0, 1, 2, \ldots; h = 1, \ldots, m; i = 1, \ldots, l). \) The point \( z = 0 \) being tacitly excluded.
where \( z \neq 0, \xi \geq 0 \) and the conditions given in (4.1.1) and (4.1.2) are satisfied.

The general class of polynomials is given by (see, Srivastava [7])

\[
S_N^M[z] = \sum_{k=0}^{[N/M]} (-N)_{\lambda k} \Phi_{N,k} \frac{z^k}{k!}
\]  

provided that \( M \) is an arbitrary positive integer and the coefficients \( \Phi_{N,k} (N, k \geq 0) \) are arbitrary constants, real or complex.

We use these integrals ((4.1.1)-(4.1.3)) to investigate some of sequence of functions and thus by them, we evaluate various analytic functions for generating and bilateral generating function formulae.

The generating function of Gegenbaur polynomial (see, Rainville [5]) is defined as

\[
\sum_{r=0}^{\infty} C_r(x) t^r = (1 - 2xt + t^2)^{-\nu}
\]  

 Particularly, for \( \nu = 1/2 \), (4.1.6) gives
\[ \sum_{n=0}^{\infty} P_n(x) t^n = (1 - 2xt + t^2)^{-1/2} \]  \hspace{1cm} \text{......... (4.1.7)}

Again, for \( \nu = 1 \), (4.1.6) becomes

\[ \sum_{n=0}^{\infty} U_n(x) t^n = (1 - 2xt + t^2)^{-1} \]  \hspace{1cm} \text{......... (4.1.8)}

In the present work, we investigate some of sequence of functions involving Gegenbouer polynomials with product of sequence of integrals defined (4.1.1)-(4.1.3) and apply them to obtain various results on analytic continuation of the generating and bilateral generating functions. Finally, we make their applications for obtaining various generating relations and other interesting relations. This work is also the generalization of the work already done due to Kumar and Srivastava [3].

4.2 Analytic Formulae for Generating and Bilateral Generating Functions

**Theorem 1**: For \( x, y, \lambda, \mu > 0; \rho, \sigma, \gamma, a, b, c \) be real or complex \( a, b \neq 0, -1, -2, \ldots; T \neq 0, 0 \leq t < \infty; \) take \( I_r(x, T) = I_{r,\nu}(x, T), r = 0, 1, \ldots \) defined in (4.1.1), exist for \( |\arg(yx^\nu)| < \pi, C_r^\nu(.) \) is the Gegenbouer polynomial given by (4.1.6) and if the sequence of functions are given by

\( \text{(a)} \quad A_r(x, T) = \sum_{k=0}^{\infty} C_{r-k}^{\nu} \left( \frac{x}{2} \right) C_k^\nu \left( \frac{x}{2} \right) I_{k, \rho-r+4}(x, T), \quad r = 0, 1, 2, \ldots \)  \hspace{1cm} \text{......... (4.2.1)}
Then, the generating function

\[
\sum_{r=0}^{\infty} A_r(x,T) T^r = \frac{\Gamma(c)}{\Gamma(d) \Gamma(b)} x^{\mu-1} T^r \left[ \frac{1}{(1-\rho - \alpha, \mu, 0, 0)} \right]_{\text{E}_{\text{2121u}}^{2121u}} \left( \frac{\mu \nu}{x} \right) \left( \frac{-(\sigma - \alpha, \mu, 0, 0)}{x} \right) \left( \frac{(1-\sigma - \alpha, \mu, 0, 0)}{x} \right) \left( \frac{(1-\rho - \alpha, \mu, 0, 0)}{x} \right)
\]

provided that \( \arg(yx^\nu) < \pi \) and \( \arg(-x^2) < \pi \). \( \text{H}[x,y] \) is two variable
H-function defined by Mittal and Gupta [4] (see also Saxena and
Nishimoto [6] and Srivastava, Gupta and Goyal [8]).

Again, if the sequence of functions are

\[
(b) E_r(x,T) = \sum_{i=0}^{\infty} C_{i+r} \left( \frac{x}{2} \right) I_i(x,T) T^{i+r}, \quad r=0,1,2,...
\]

Then, the bilateral generating function \( \sum_{r=0}^{\infty} E_r(x,T) C_r \left( \frac{x}{2} \right) T^{-r} \) exists and
there holds the formula

\[
\sum_{r=0}^{\infty} E_r(x,T) C_r \left( \frac{x}{2} \right) T^{-r} = \frac{\Gamma(c)}{\Gamma(d) \Gamma(b)} x^{\mu-1} T^r \left[ \frac{1}{(1-\rho - \alpha, \mu, 0, 0)} \right]_{\text{E}_{\text{2121u}}^{2121u}} \left( \frac{\mu \nu}{x} \right) \left( \frac{-(\sigma - \alpha, \mu, 0, 0)}{x} \right) \left( \frac{(1-\sigma - \alpha, \mu, 0, 0)}{x} \right) \left( \frac{(1-\rho - \alpha, \mu, 0, 0)}{x} \right)
\]

provided that \( \arg(yx^\nu) < \pi \) and \( \arg(-x^2) < \pi \).
Proof of (4.2.2)

Replace \( \rho \) by \( \rho + k \) in (4.1.1) and then, both sides to it multiply by \( C_r^{(\frac{x}{2})} \) and again, sum \( r \) from 0 to \( \infty \), we find a bilateral generating function formula such that

\[
\sum_{r=0}^{\infty} C_r^{(\frac{x}{2})} I_{r,\rho+k}(x, T) T^r = \frac{1}{(2\pi i)} \int \int_{-\infty}^{\infty} \frac{\Gamma(c) e^{-xT}}{\Gamma(a+s)\Gamma(b+s)\Gamma(c+s)} (-s)^{r} T^{r+k-1} \times (x-tT)^{a+s-1} \left(1 - \frac{tT}{x}\right)^{a} \{1 - xtT + t^2T^2\}^{-\beta-\lambda+\gamma} dt \, ds
\]

...... (4.2.5)

Now, multiply \( C_r^{(\frac{x}{2})} T^k \) in both sides of the equ. (4.2.5) and in that of sum \( k \) from 0 to \( \infty \) and then use the formula (4.1.6) in the right hand side of it, we get

\[
\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} C_r^{(\frac{x}{2})} C_r^{(\frac{x}{2})} I_{r,\rho+k}(x, T) T^{r+k} = \frac{1}{(2\pi i)} \int \int_{-\infty}^{\infty} \frac{\Gamma(c) e^{-xT}}{\Gamma(a+s)\Gamma(b+s)\Gamma(c+s)} (-s)^{r} T^{r+k} \times T^{r-1} (x-tT)^{a+s-1} \left(1 - \frac{tT}{x}\right)^{a} \{1 - xtT + t^2T^2\}^{-\beta-\lambda+\gamma} dt \, ds
\]

...... (4.2.6)

Then, interchange \( k \) and \( r \) in the equ.(4.2.6) and replace \( r \) by \( r-k \) in it and use the formula (4.2.1) in left hand side of it. Again, make the solutions and thus define \( H \)-function of two variables in right hand side of it, we obtain required result (4.2.2).
Proof of (4.2.4)

Make an appeal to (4.1.1) in (4.2.3) and then, multiply $C_r^0 \left( \frac{x}{2} \right) T^{-r}$ in both sides of it and then sum $r$ from 0 to $\infty$, we find

$$
\sum_{r=0}^{\infty} C_r^0 \left( \frac{x}{2} \right) E_r(x,T) T^{-r} = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} r^{k+1} T^k C_r^0 \left( \frac{x}{2} \right) (x-iT)^{-r}
$$

$$
= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \left[ \frac{\alpha + 1}{2} + 2 \frac{\gamma}{2} - \frac{4\alpha T (x-iT)}{1-iT (x-iT)^2} \right] T^k \left[ \frac{\alpha + 1}{2} + 2 \frac{\gamma}{2} - \frac{4\alpha T (x-iT)^2}{(x-iT)^2} \right]
$$

$$
\times \text{B} \left[ \frac{\alpha}{2}, \frac{\beta}{2} ; -y \left( 1-\alpha T + T^2 \right)^{-2}(x-iT)^{n} \right] dt
$$

...... (4.2.7)

Now, replace $k$ by $k+r$ in the equ.(4.2.7) and use the formula (4.1.6) in the right hand side of it and thus on solving it define $H$-function of two variables we arrive the required result of (4.2.4).

Theorem 2: For $x, y, \lambda, \mu > 0, \rho, \sigma, \gamma$ be real or complex; $T \neq 0, 0 \leq t < \infty$; the coefficients $A_1, ..., A_p$ and $B_1, ..., B_q$ are positive real numbers such that

$$
\Omega = \sum_{j=1}^{n} A_j - \sum_{j=m+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0;
$$

the complex numbers $a_1, ..., a_p$ and $b_1, ..., b_q$ are s.t. $A_i(b_n + \nu) = B_h(a_i - \eta - 1), (\nu, \eta = 0, 1, 2, ..., h = 1, ..., m; i = 1, ..., n)$; on taking $J_r(x, T) = J_{r, \rho}(x, T), r = 0, 1, 2, ...$, defined in (4.1.2), exist for $\left| \text{arg}(\rho x^\nu) \right| < \frac{1}{2} \pi C_r^\nu(x)$ is the Gegenbour polynomial and if the sequence of functions are given by
(a) \[ B_r(x, T) = \sum_{k=0}^{\infty} C_{r-k}^\nu \left( \frac{x}{2} \right) C_t^\beta \left( \frac{x}{2} \right) J_{k,r+r-1}(x, T), \quad r = 0, 1, 2, \ldots \]

...... (4.2.8)

Then, the generating function \[ \sum_{r=0}^{\infty} B_r(x, T)T^r \] exists and there holds the formula

\[
\sum_{r=0}^{\infty} B_r(x, T)T^r = \frac{x^{\alpha - 1}}{T^\rho} \times \left[ \frac{(1 - \nu - \beta + \gamma, \lambda)(1 - \sigma - \alpha, \mu, 1)}{x^\mu - x^2} \right] \times \left[ (1 - \rho - \sigma - \alpha, \mu, 2) : (b, B) \right] (1 - \nu - \beta + \gamma, \lambda): (0, 1) \]

...... (4.2.9)

provided that \[ \arg(yx^\nu) \mid < \frac{1}{2} \Omega \pi \] \[ \arg(-x^2) \mid < \pi \] and \[ \sum_{j=1}^{\kappa} B_j - \sum_{j=1}^{\kappa} A_j \geq 0. \]

Again, if the sequence of functions are

(b) \[ F_r(x, T) = \sum_{k=r}^{\infty} C_k^\nu \left( \frac{x}{2} \right) J_k(x, T)T^{k+r}, \quad r = 0, 1, 2, \ldots \]

...... (4.2.10)

Then, the bilateral generating function \[ \sum_{r=0}^{\infty} F_r(x, T)C_t^\beta \left( \frac{x}{2} \right) T^{-r} \] exists and there holds the formula
Proof of (4.2.9)

In the similar manner as analysed to prove (4.2.2), from (4.1.2) we find the bilateral generating function formula

\[
\sum_{k=0}^{\infty} F_k(xT) C^\beta \left( \frac{x}{2} \right) T^k = \frac{x^{\alpha-1}}{\Gamma^0}
\]

\[
\times \frac{(1-\nu-\beta+\gamma, \lambda, \lambda)}{(1-\sigma-\alpha, \mu_2)}
\]

\[
\times H_{\alpha,\beta,\gamma,\lambda}^{\mu,\nu,\sigma,\tau}
\]

provided that \( \arg(x^\mu) < \frac{1}{2} \Omega \pi, \arg(-x^\nu) < \pi \) and \( \sum_{j=1}^{\infty} B_j - \sum_{j=1}^{\infty} A_j \geq 0. \)

**Proof of (4.2.9)**

In the similar manner as analysed to prove (4.2.2), from (4.1.2) we find the bilateral generating function formula

\[
\sum_{r=0}^{\infty} C_r^u \left( \frac{x}{2} \right) J_{r,\sigma} \left( x, T \right) T^r = \int_{0}^{\infty} \left( 1 - \frac{t}{x} \right)^{\sigma-1} \left( 1 - t\sigma - T^2 \right)^{\tau-\beta} dt
\]

\[
\times \frac{y(x-tT)^\mu}{\left( 1 - xtT + t^2T^2 \right)^{\frac{1}{2}}} \left( a_j, A_j \right)_{\lambda,\rho} \left( b_j, B_j \right)_{\mu,\nu}
\]

\[
\text{..... (4.2.11)}
\]

Now, multiply \( C_k^u \left( \frac{x}{2} \right) T^k \) in both sides of the equ.(4.2.12) and in that of sum \( k \) from 0 to \( \infty \) and then, apply the formula (4.1.6) in right hand side of it, we get

\[
\sum_{k=0}^{\infty} \sum_{r=0}^{\infty} C_k^u \left( \frac{x}{2} \right) C_r^\beta \left( \frac{x}{2} \right) J_{r,\sigma} \left( x, T \right) T^{r+k} = \int_{0}^{\infty} \left( 1 - \frac{t}{x} \right)^{\sigma-1} \left( 1 - xtT + t^2T^2 \right)^{\tau-\beta} dt
\]

\[
\times \frac{y(x-tT)^\mu}{\left( 1 - xtT + t^2T^2 \right)^{\frac{1}{2}}} \left( a_j, A_j \right)_{\lambda,\rho} \left( b_j, B_j \right)_{\mu,\nu}
\]

\[
\text{..... (4.2.13)}
\]
Now, follow the similar techniques in left hand side of (4.2.13) as applied in that of the equ. (4.2.6) and again, there use the equ. (4.2.8). Further, in right hand side of it use the formula (4.1.3) and then after on solving it define H-function of two variables [cf.(3)], we obtain the required result (4.2.9).

**Proof of (4.2.11)**

Make an appeal to (4.1.2) in (4.2.10) and multiply \( C_\nu^\beta (\xi T^{-r}) \) in both side of it and then, there sum r from 0 to \( \infty \), to get

\[
\sum_{r=0}^{\infty} F_r(x,\nu) C_\nu^\beta \left( \frac{x}{2} \right) T^{-r} = \sum_{r=0}^{\infty} \int_0^{\pi T} \rho^{\nu-k+1} T^k C_{\nu-r} \left( \frac{x}{2} \right) C_\nu^\beta \left( \frac{x}{2} \right) (x-tT)^{\sigma-1} \left( 1 - \frac{tT}{x} \right) dt \times \left\{ 1 - xtT + t^2 T^2 \right\}^{\gamma-1} H_{p,q}^{m,n} \left[ \frac{y(x-tT)^{\nu}}{(1-xtT+t^2 T^2)^{\nu}} \right] \left[ \frac{(a_j, A_j)_{p,q}}{(b_j, B_j)_{p,q}} \right] \]

...... (5.2.14)

Now, replace k by k+r in the equ. (4.2.14) and in the right hand side use the equ. (4.1.6) and after use the formula (4.1.3) and then on solving it define H-function of two variables we derive the required result of (4.2.11).

**Theorem 3**: For \( x, y, \lambda, \mu > 0; p, \sigma, \gamma \) be real or complex \( z \neq 0, T \neq 0, 0 \leq t < \infty \); the coefficients \( A_1, \ldots, A_p \) and \( B_1, \ldots, B_q \) are positive real numbers such that \( \Omega = \sum_{j=1}^{p} A_j - \sum_{j=1}^{m} B_j - \sum_{j=1}^{q} B_j > 0 \) and the complex numbers \( a_1, \ldots, a_p \) and \( b_1, \ldots, b_q \) are such that
\[ A_h(a, \eta) \neq B_h(a, -\eta - 1), (\nu, \eta = 0, 1, 2, \ldots; h = 1, 2, \ldots m; i = 1, 2, \ldots n); \]

take

\[ K_r(x, T) = K_{r,0}(x, T), r = 0, 1, 2, \ldots \]

defined in (4.1.3), exist for

\[ \arg(x^\nu) < \frac{1}{2} \Omega \pi, C^\omega_r(.) \] be Gegenbour polynomial and if the sequence of functions are defined by

\[ (a) \quad y_r(x, T) = \sum_{k=0}^{r} C^\omega_{r-k} \left( \frac{x}{2} \right) C^\beta_k \left( \frac{x}{2} \right) K_{k+r,r}(x, T), r = 0, 1, 2, \ldots \]

\[ \quad \ldots \quad (4.2.15) \]

Then, the generating function \( \sum_{r=0}^{\infty} y_r(x, T) T^r \) exists and there holds the formula

\[ \sum_{r=0}^{\infty} y_r(x, T) T^r = \frac{x^{\nu+1-|\lambda|}}{T^r} \sum_{r=0}^{\infty} (-\lambda)^r \Phi_{\lambda, r} \left( \frac{x}{T} \right)^r \left( \frac{x}{2} \right) \]

\[ \times \mathcal{H}_{2,1}^{\beta, \gamma, \mu, \lambda} \left[ \begin{array}{c}
(1-\beta-\nu+\gamma, \lambda) \\
(1-\sigma-\alpha, \mu)
\end{array} \right] (a_j, A_j)_{\lambda, \mu} : (1-\rho-\xi_1, 1) \]

\[ \times \mathcal{H}_{2,1}^{\beta, \gamma, \mu, \lambda} \left[ \begin{array}{c}
(1-\beta-\nu+\gamma, \lambda) \\
(1-\sigma-\alpha, \mu)
\end{array} \right] (b_j, B_j)_{\lambda, \mu} : (1-\rho-\xi_2, 1) \]

\[ \quad \ldots \quad (4.2.16) \]

provided that \( \arg(x^\nu) < \frac{1}{2} \Omega \pi, \arg(-x^2) < \pi \) and \( \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \)

Again if the sequence of functions are such that

\[ (b) \quad N_r(x, T) = \sum_{k=r}^{\infty} C^\omega_{k-r} \left( \frac{x}{2} \right) N_{k+r}(x, T) T^{k+r}, r = 0, 1, 2, \ldots \]

\[ \quad \ldots \quad (4.2.17) \]

then, the bilateral generating function \( \sum_{k=r}^{\infty} N_r(x, T) C^\beta_k \left( \frac{x}{2} \right) T^{-r} \) exists and there holds the formula

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\[
\sum_{r,s} N_r(x,T) e^{\nu r} \left( \frac{x}{2} \right) T^r = \frac{x^{m-1} N_M}{T^p} \sum_{s=0}^{N_M} (-N) \Phi \Phi_{x,s} \left( \frac{y(x/T)^{\nu}}{s!} \right) \times H_{\nu+\mu,1}^{0,2m,n+1} \left[ (1-\beta-\nu+\gamma,1) \mid (1-\sigma-\alpha,1) ; \right.
\]
\[
\left( a_j, A_j \right)_{h_p} ; \quad (1-\rho-\xi,1) \left(1-\xi s-\rho-\sigma-\alpha,1) ; \right.
\]
\[
(\xi_1, B_j)_{i_q} (1-\beta-\nu+\gamma,1) ; \quad (0,1) \right] \]

provided that all conditions of (4.2.16) are satisfied.

**Proof of (4.2.16)**

From (4.1.4), we obtain the bilateral generating function in the form

\[
\sum_{r,s} C_{r,s} \left( \frac{x}{2} \right) K_{r,s}^\nu (x,T) T^r = \sum_{s=0}^{N_M} (-N) \Phi \Phi_{x,s} \left( \frac{x}{T} \right)^{\nu+\mu} \int_0^1 \left( \frac{y(x-tT)^{\nu}}{s!} \right) \left( \frac{y(x-tT)^{\nu}}{s!} \right) dT \]

Then, multiply \( C_{\nu} \left( \frac{x}{2} \right) \) in both sides of the equ.(4.2.19) and there sum

\[
k \text{ from } 0 \text{ to } \infty \text{ and use formula (4.1.6) in right hand side of it, we get}
\]

\[
\sum_{r,s} C_{r,s} \left( \frac{x}{2} \right) K_{r,s}^\nu (x,T) T^r \int_0^1 \left( \frac{y(x-tT)^{\nu}}{s!} \right) \left( \frac{y(x-tT)^{\nu}}{s!} \right) dT
\]

Again, follow the techniques in left hand side of it as employed in

that of the equ.(4.2.6) and then recall the function (4.2.15) and

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further, in right hand side of it make use of the formula (4.1.3) and then on solving it define the H-function of two variables [cf.(3)] we evaluate the required result (4.2.16).

**Proof of (4.2.18)**

Make an application of the function (4.1.3) in (4.2.17) and multiply $C_r(x, T) T^{-r}$ in both sides of it and then, sum $r$ from $0$ to $\infty$, to get

$$
\sum_{r=0}^{\infty} N_r(x, T) C_r(x, T) T^{-r} = \sum_{r=0}^{\infty} \sum_{k=0}^{nT} \int_{0}^{x} \tau^{\alpha+1-k-T} C_{r-k} \left( \frac{x}{2} \right) C_r \left( \frac{x}{2} \right)
$$

$$
\times (x-tT)^{\alpha-1} \left( 1 - \frac{tT}{x} \right) \left\{ 1 - tT(x-tT) \right\}^y
$$

$$
\times H^{m,n}_{p,q} \left[ (a_j, A_j)_{q,p} \left( \frac{y(x-tT)^a}{1-tT(x-tT)} \right)^{\lambda} \right] S^{M}[zt]\ dt
$$

...... (4.2.21)

Now, employ the techniques in right hand side of the equ. (4.2.21) as followed in that of the equ. (4.2.14) and then make an appeal to the formula (4.1.3) and use the definition of H-function of two variables [cf.(3)], we get the required result (4.2.18).

**4.3. Generating Relations**

In this section, we employ our results obtained in previous sections and then, derive some generating relations.

Making an application of the equs. (4.2.1), (4.2.2), (4.2.3) and (4.2.4), we derive the generating relation
provided that all conditions of the theorem (1) are satisfied.

Employing the equs. (4.2.8), (4.2.9), (4.2.10) and (4.2.11), we obtain the generating relation

\[
\sum_{r=0}^{\infty} C_{r+1}^\alpha \left( \frac{x}{2} \right) \sum_{k=0}^{\infty} C_{k+1}^\beta \left( \frac{x}{2} \right) J_r(x, T) T^r = \sum_{r=0}^{\infty} T^r J_r(x, T) \sum_{k=0}^{\infty} C_{k+1}^\beta \left( \frac{x}{2} \right) C_{r+k}^\alpha \left( \frac{x}{2} \right)
\]

\[
= x^{2\alpha+1} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{x}{2} \right)^{r+k} \left( \frac{x}{2} \right)^k C_{r+k}^\alpha \left( \frac{x}{2} \right) C_{r+k}^\beta \left( \frac{x}{2} \right)
\]

\[
\times F_{2,1,2,3}^{2,1,2,3} \left[ \begin{array}{c}
\begin{array}{c}
(1 - u - \beta + \gamma, l)(1 - \sigma - \alpha, \mu, l) :
(1 - a, l)(1 - b, l),
(1 - \rho - r, l) \\
(1 - \rho - r - \sigma - \alpha, \mu, 2):
(0, l)(1 - u - \beta + \gamma, l)(1 - c, l),
(0, l)
\end{array}
\end{array} \right]
\]

... (4.3.1)

provided that all conditions of the theorem (2) are satisfied.

Making use of the equs. (4.2.15), (4.2.16), (4.2.17) and (4.2.18), we evaluate the generating relation

\[
\sum_{r=0}^{\infty} C_{r+1}^\alpha \left( \frac{x}{2} \right) \sum_{k=0}^{\infty} C_{k+1}^\beta \left( \frac{x}{2} \right) K_r(x, T) T^r = \sum_{r=0}^{\infty} T^r K_r(x, T) \sum_{k=0}^{\infty} C_{k+1}^\beta \left( \frac{x}{2} \right) C_{r+k}^\alpha \left( \frac{x}{2} \right)
\]

\[
= x^{2\alpha+1} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{x}{2} \right)^{r+k} \left( \frac{x}{2} \right)^k C_{r+k}^\alpha \left( \frac{x}{2} \right) C_{r+k}^\beta \left( \frac{x}{2} \right)
\]

\[
\times F_{2,1,2,3}^{2,1,2,3} \left[ \begin{array}{c}
\begin{array}{c}
(1 - u - \beta + \gamma, l)(1 - \sigma - \alpha, \mu, l) :
(a_j, A), (1 - \rho - r, l) \\
(1 - \rho - r - \sigma - \alpha, \mu, 2):
(b_j, B), (l - \nu - \beta + \gamma, l)
\end{array}
\end{array} \right]
\]

... (4.3.2)

provided that all conditions of the theorem (2) are satisfied.
provided that all conditions of the theorem (3) are satisfied.

4.4 Applications

In the equ. (4.2.5) replace \( k \) by \( k_1 + \cdots + k_n \) and then multiply \( \prod_{i=1}^{n} C_{k_i}^{\nu}(x)T^{k_i} \) in both sides of it and again sum \( k_1, \ldots, k_n \) from 0 to \( \infty \) and then apply the formula (4.1.6) in right hand side of it, we get

\[
\sum_{k_0=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \left( \prod_{i=1}^{n} C_{k_i}^{\nu}(x) \right) C_{r}^{\beta}(x) I_{r,\beta+k_1+\cdots+k_n}(x,T) T^{r+k_1+\cdots+k_n}
\]

\[
= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)(2\pi\omega)} \int_{0}^{\infty} \int_{0}^{x} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} t^{s-1}(x-tT)^{\sigma+\mu-1} y^s
\]

\[
\times \left( 1 - \frac{tT}{x} \right) \left( 1 - xtT + t^2T^2 \right)^{-\beta-\lambda-s-\mu} dt \, ds
\]

...... (4.4.1)

Now, interchange \( k_1 \) and \( r \) in the left hand side of (4.4.1) and then replace \( r \) by \( r-k_1 \) and next, there, interchange \( k_2 \) and \( r \) and again replace \( r \) by \( r-k_2 \) and thus do this process up to \( k_n \). Also in right hand side of it follow the techniques, applied in that of (4.2.6), we yield the function

\[
\sum_{r=0}^{\infty} \cdots \sum_{r-k_1=0}^{\infty} \cdots \sum_{r-k_n=0}^{\infty} C_{r-k_1}^{\nu}(x) \left( \prod_{i=1}^{n} C_{k_i}^{\nu}(x) \right) C_{r}^{\beta}(x) I_{r,\beta+k_1+\cdots+k_n}(x,T) T^{r-k_1+\cdots+k_n}
\]

\[
= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{0}^{\infty} \int_{0}^{x} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} t^{s-1}(x-tT)^{\sigma+\mu-1} y^s
\]

\[
\times \left( 1 - \frac{tT}{x} \right) \left( 1 - xtT + t^2T^2 \right)^{-\beta-\lambda-s-\mu} dt \, ds
\]

...... (4.4.2)
for all, \( n \geq 2 \), provided that all conditions of the theorem (1) are satisfied and for \( n = 1 \), it is equivalent to the formula (4.2.2) along with (4.2.1).

In the same manner, from (4.2.12) and (4.2.19) we derive other functions such that

\[
\sum_{r=0}^{\infty} \sum_{k=0}^{k_r} \sum_{k_{t, r=0}^{t=1}} C_{r, k} \left( \frac{x}{2} \right) \prod_{t=1}^{m} K_{k_{t, r=0}^{t=1}} \left( \frac{x}{2} \right) C_{p, r} \left( \frac{x}{2} \right) J_{k_{t, r=0}^{t=1}}(x,T) T^r
\]

\[
= \frac{x^{n-1}}{T^n} \times H_{2^n+1 \times 1}^x \times H_{2^n+1 \times 1}^y
\]

\[
\begin{bmatrix}
(1-nv-\beta+y, \lambda)(1-\sigma-\alpha, \mu) & (a, A)_{k_p} & (1-\rho, 1)
\end{bmatrix}
\]

\[
(1-\rho-\sigma-\alpha, \mu) & (b, B)_{k_q} & (1-nv-\beta+y, \lambda)
\]

\[
(0, 1)
\]

...... (4.4.3)

for all \( n \geq 2 \), provided that all conditions of the theorem (2) are satisfied; for \( n = 1 \), it is equivalent to the formula (4.2.9) along with (4.2.8);

and

\[
\sum_{r=0}^{\infty} \sum_{k=0}^{k_r} \sum_{k_{t, r=0}^{t=1}} C_{r, k} \left( \frac{x}{2} \right) \prod_{t=1}^{m} K_{k_{t, r=0}^{t=1}} \left( \frac{x}{2} \right) C_{p, r} \left( \frac{x}{2} \right) K_{k_{t, r=0}^{t=1}}(x,T) T^r
\]

\[
= \frac{x^{n-1}}{T^n} \times \sum_{s=0}^{N} (-N)_{k_s} \Phi_{k_s}(\frac{z(x/T)}{s})
\]

\[
\begin{bmatrix}
(1-nv-\beta+y, \lambda)(1-\sigma-\alpha, \mu) & (a, A)_{k_p} & (1-\rho-\xi, 1)
\end{bmatrix}
\]

\[
(1-\rho-\sigma-\alpha-\xi; \mu) & (b, B)_{k_q} & (1-nv-\beta+y, \lambda)
\]

\[
(0, 1)
\]

...... (4.4.4)
for all $n \geq 2$, provided that all conditions of the theorem (3) are satisfied; for all $n=1$, it is equivalent to the formula (4.2.16) along with (4.2.15).

4.5. Special Cases

1. Setting $\beta = \frac{1}{2}, \nu = 1$ and $\gamma = 0$ in (4.2.1)-(4.4.4), we get all the results due to Kumar and Srivastava [2] derived in section 2-4.

2. Setting $\beta = \frac{1}{2}, \nu = 0$ and $\gamma = 0$ we get various results obtained by Kumar and Srivastava [3].

References:


