CHAPTER 10
HARMONIC UNIVALENT FUNCTIONS WITH FIXED POINT AND NEGATIVE COEFFICIENTS

10.1 Let $U$ denote the open unit disc $U = \{ z : |z| < 1 \}$ and $S_H$ denote the class of all complex-valued, harmonic, sense-preserving univalent function $f$ in $U$ normalized by $f(0) = 0$, $f'(0) = 1$. Each $f \in S_H$ can be expressed as

$$f = h + g$$

where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n.$$  \hspace{1cm} (10.1.1)

We call $h$ the analytic part and $g$ the co-analytic part of $f$. We observe that $S_H$ reduces to $S$, the class of normalized univalent analytic functions, if the co-analytic part of $f$ is zero.

Let $S_H^{*,0}$ and $K_H^0$ be the subclasses of $S_H$ consisting of functions $f$ that map $U$ onto starlike and convex domains, respectively. We further denote by $T_H^{*,0}$ and $TK_H^0$ the subclasses of $S_H^{*,0}$ and $K_H^0$, respectively, whose coefficients $f = h + g$ take the forms

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0; \quad g(z) = -\sum_{n=2}^{\infty} b_n z^n, b_n \geq 0.$$  \hspace{1cm} (10.1.2)

Silverman [101] proved that the coefficient conditions

$$\sum_{n=2}^{\infty} n \left( |a_n| + |b_n| \right) \leq 1 \quad \text{and} \quad \sum_{n=2}^{\infty} n^2 \left( |a_n| + |b_n| \right) \leq 1,$$  \hspace{1cm} (10.1.3)
are necessary and sufficient conditions for functions \( f = h + g \) to be harmonic starlike with negative coefficients and harmonic convex with negative coefficients, respectively.

We shall need above mentioned results (10.1.3) throughout this chapter. Several authors, such as ([18], [37], [57], [60], [98]) studied the subclasses of analytic univalent functions with fixed points. In this chapter, an attempt has been made to study the subclasses of harmonic univalent functions with fixed point in the following way

A function \( f = h + g \), where

\[
h(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = -\sum_{n=2}^{\infty} b_n z^n,\]

\( a_1 > 0, \quad a_n \geq 0 \) and \( b_n \geq 0 \),

is said to be in the family \( T_h^{*,0}(z_0) \) and \( TK_h^0(z_0) \) if there exist functions that map \( U \) onto starlike and convex domains respectively and

\[
f(z_0) = z_0, -1 < z_0 < 1, z_0 \neq 0.\]

In this chapter, we investigate sense-preserving harmonic univalent function of the form (10.1.4) with a fixed point (10.1.5) and determine the coefficients bounds, distortion and extreme points for these types of functions.

### 10.2 Main Results

**Theorem 10.2.1.** For \( f \) of the form (10.1.4), \( f \in T_h^{*,0}(z_0) \) if and only if

\[
\sum_{n=2}^{\infty} n(a_n + b_n) \leq a_1,\]

(10.2.1)

where \( a_1 = 1 + \sum_{n=2}^{\infty} (a_n z_0^{n-1} + b_n z_0^{n+1}) \).

**Proof.** Note first that
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\[ |h'(z)| \geq a_1 - \sum_{n=2}^{\infty} na_n r^{n-1} \]

\[ > a_1 - \sum_{n=2}^{\infty} na_n \]

\[ \geq \sum_{n=2}^{\infty} nb_n \]

\[ > \sum_{n=2}^{\infty} nb_n r^{n-1} \]

\[ \geq |g'(z)|. \]

so that \( f \) is locally univalent and sense-preserving. It suffices to show that

\[ \frac{\partial}{\partial \theta} \left( \arg f(re^{i\theta}) \right) > 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 < r < 1. \]

We have

\[ f(re^{i\theta}) = a_re^{i\theta} - \sum_{n=2}^{\infty} \left( a_n e^{i(n-1)\theta} + b_n e^{-i(n+1)\theta} \right) r^{n-1}, \]

and

\[ \frac{\partial}{\partial \theta} \left( \arg f(re^{i\theta}) \right) = \text{Im} \left( \frac{\partial}{\partial \theta} \log f(re^{i\theta}) \right) \]

\[ = \frac{\partial}{\partial \theta} \left( \text{Im} \left( \frac{\partial}{\partial \theta} f(re^{i\theta}) \right) \right) \]

\[ = \text{Re} \left\{ \frac{a_1 - \sum_{n=2}^{\infty} \left( a_n e^{i(n-1)\theta} - b_n e^{-i(n+1)\theta} \right) r^{n-1}}{a_1 - \sum_{n=2}^{\infty} \left( a_n e^{i(n-1)\theta} + b_n e^{-i(n+1)\theta} \right) r^{n-1}} \right\} \]

\[ = \text{Re} \left\{ \frac{a_1 + A(z)}{a_1 + B(z)} \right\}. \]

Setting

\[ \frac{a_1 + A(z)}{a_1 + B(z)} = \frac{1 + \omega(z)}{1 - \omega(z)^{-1}}, \]
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so that

\[ \omega(z) = \frac{A(z) - B(z)}{2a_1 + A(z) + B(z)}. \]

We will have \( \frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) > 0, \) if \( |\omega(z)| \leq r. \)

But

\[ \omega(z) = \frac{A(z) - B(z)}{2a_1 + A(z) + B(z)} \]

\[ = \frac{\sum_{n=2}^{\infty} \left[ -(n-1)a_n e^{i(n-1)\theta} + (n+1)b_n e^{-i(n+1)\theta} \right] r^{n-1}}{2a_1 - \sum_{n=2}^{\infty} \left[ (n+1)a_n e^{i(n+1)\theta} - (n-1)b_n e^{-i(n-1)\theta} \right] r^{n-1}}. \]

So that

\[ |\omega(z)| \leq \frac{\sum_{n=2}^{\infty} [(n-1)a_n + (n+1)b_n]r}{2a_1 - \sum_{n=2}^{\infty} [(n+1)a_n + (n-1)b_n]} \]

This last expression is bounded above by \( r, \) if and only if

\[ \sum_{n=2}^{\infty} n(a_n + b_n) \leq a_1. \]  \hspace{1cm} (10.2.2)

Conversely, we need only show that \( f \not\in T_{n0}^{\infty}(z_0) \) if the coefficient condition does not hold, for this case, we will show that \( f \) is not even univalent. Setting \( z = r > 0, \) we have

\[ f(r) = a_1 r - \sum_{n=2}^{\infty} (a_n + b_n) r^n \quad \text{and} \quad f'(r) = a_1 - \sum_{n=2}^{\infty} n(a_n + b_n) r^{n-1}. \]
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Since \( f'(0) = a_1 > 0 \) and \( f'(1) = a_1 - \sum_{n=2}^{\infty} n (a_n + b_n) < 0 \), there must exist an \( r_0, r_0 < 1 \), for which \( f'(r_0) = 0 \). Hence, \( r_0 \) is a local max for \( f(r) \) and \( f(r) \) is not one-one on the real interval \((0,1)\). Therefore \( f \in T_{n}^{*0}(z_0) \).

Now, using the condition \( f(z_0) = z_0 \), we have

\[
a_1 z_0 - \sum_{n=2}^{\infty} (a_n z_0^n + b_n z_0^n) = z_0,
\]

or

\[
a_1 = 1 + \sum_{n=2}^{\infty} a_n z_0^{n-1} + b_n z_0^{n-1}.
\]

**Theorem 10.2.2.** If \( f \in T_{n}^{*0}(z_0) \), then

\[
a_1 \left( r - r^3/2 \right) \leq |f(z)| \leq a_1 \left( r + r^3/2 \right), \quad (|z| = r).
\]

The result is sharp, with equality for \( a_i \left( z - z^2/2 \right) \) and \( a_i \left( z + z^2/2 \right) \),

where \( a_1 = 1 + \sum_{n=2}^{\infty} a_n z_0^{n-1} + b_n z_0^{n-1} \).

**Proof.** From Theorem 10.2.1, we have

\[
2 \sum_{n=2}^{\infty} (a_n + b_n) \leq \sum_{n=2}^{\infty} n (a_n + b_n) \leq a_i.
\]

Thus

\[
\sum_{n=2}^{\infty} (a_n + b_n) \leq \frac{a_i}{2}.
\]

Now
\[ |f(z)| \leq a_r + \sum_{n=2}^{\infty} (a_n + b_n) r^n \]
\[
\leq a_r + r^2 \sum_{n=2}^{\infty} (a_n + b_n) \\
\leq a_r + r^2 \frac{a_1}{2}
\]
and
\[
|f(z)| \geq a_r - \sum_{n=2}^{\infty} (a_n + b_n) r^n \\
\geq a_r - r^2 \sum_{n=2}^{\infty} (a_n + b_n) \\
\geq a_r - r^2 \frac{a_1}{2} \\
= a_1 \left( r - \frac{r^2}{2} \right). \]

**Theorem 10.2.3.** For \( f \) of the form (10.1.4), \( f \in TK^0_H(z_0) \), if and only if
\[
\sum_{n=2}^{\infty} n^2 (a_n + b_n) \leq a_1, \quad (10.2.3)
\]
where \( a_1 = 1 + \sum_{n=2}^{\infty} (a_n z_0^{n-1} + b_n z_0^{n-1}) \).

**Proof.** For \( f \in S^0_H \) it is given in [101] that \( \int_0^z \frac{f(\zeta)}{\zeta} d\zeta \in K^0_H \), where integration is along the line segment from 0 to \( z \), applying the same reasoning we have \( f \in T^0_H(z_0) \) then \( \int_0^z \frac{f(\zeta)}{\zeta} d\zeta \in TK^0_H(z_0) \). Now the “if part” follows from Theorem 10.2.1.
For "only if" part we need only show that \( f \notin TK^0_H(z_o) \), if the coefficient inequality does not hold. A necessary and sufficient condition for \( f \) to map \( |z| = r \) onto the convex domain is that

\[
\frac{\partial}{\partial \theta} \left( \arg \left( \frac{\partial}{\partial \theta} f(re^{i\theta}) \right) \right) = \text{Re} \left\{ \frac{a_i - \sum_{n=2}^{\infty} n^2 \left( a_n e^{i(n-1)\theta} + b_n e^{-i(n+1)\theta} \right) r^{n-1}}{a_i - \sum_{n=2}^{\infty} n \left( a_n e^{i(n-1)\theta} - b_n e^{-i(n+1)\theta} \right) r^{n-1}} \right\} > 0.
\]

If we set \( \theta = 0 \), the last expression is negative for \( \sum_{n=2}^{\infty} n^2 (a_n + b_n) > a_i \) and \( r \) sufficiently close to 1. Hence \( f \in TK^0_H(z_o) \) and the proof is complete.

\[ \square \]

**Theorem 10.2.4.** If \( f \in TK^0_H(z_o) \), then

\[
a_i \left( r - \frac{r^2}{4} \right) \leq |f(z)| \leq a_i \left( r + \frac{r^2}{4} \right), \quad (|z| = r).
\]

The result is sharp with equality for \( a_i \left( z - \frac{z^2}{4} \right) \) and \( a_i \left( z - \frac{z^2}{4} \right) \).

**Proof.** From Theorem 10.2.3, we note that

\[
4 \sum_{n=2}^{\infty} (a_n + b_n) \leq \sum_{n=2}^{\infty} n^2 (a_n + b_n) \leq a_i
\]

\[
\Rightarrow \sum_{n=2}^{\infty} (a_n + b_n) \leq \frac{a_i}{4}.
\]

Now

\[
|f(z)| \leq a_i r + \sum_{n=2}^{\infty} (a_n + b_n) r^n
\]

\[
\leq a_i r + r^2 \sum_{n=2}^{\infty} (a_n + b_n)
\]
\[ a_i r + r^2 \leq \frac{a_i}{4} \]

\[ = a_i \left( r + \frac{r^2}{4} \right) \]

and

\[ |f(z)| \geq a_i r - \sum_{n=2}^{\infty} (a_n + b_n) r^n \]

\[ \geq a_i r - r^2 \sum_{n=2}^{\infty} (a_n + b_n) \]

\[ \geq a_i r - \frac{a_i}{4} r^2 \]

\[ = a_i \left( r - \frac{r^2}{4} \right). \]

Next, we determine the extreme points for the classes \( T \) and \( \mathcal{T} \).

**Theorem 10.2.5.** Set \( h_i(z) = z \), \( h_n(z) = z - \frac{z^n}{n} \), \( n = 2, 3, 4, \ldots \) and \( g_n(z) = z - \frac{z^n}{n} \), \( n = 2, 3, \ldots \). Then \( f \in T \) if and only if it can be expressed in the form

\[ f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n), \]

where \( \lambda_n \geq 0 \), \( \gamma_n \geq 0 \), \( \lambda_1 = a_i - \sum_{n=2}^{\infty} (\lambda_n + \gamma_n) \geq 0 \) and \( \gamma_1 = 0 \).

In particular, the extreme points of \( T \) are \( \{h_n\} \) and \( \{g_n\} \).

**Proof.** Suppose that

\[ f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) \]

\[ = \lambda_1 h_1 + \gamma_1 g_1 + \sum_{n=2}^{\infty} \lambda_n h_n + \gamma_n g_n \]
\[ = \left( a_1 - \sum_{n=2}^{\infty} \left( \lambda_n + \gamma_n \right) \right) z + \sum_{n=2}^{\infty} \lambda_n \left( z - \frac{z^n}{n} \right) + \gamma_n \left( z - \frac{z^n}{n} \right) \]

\[ = a_1 z - \sum_{n=2}^{\infty} \left( \frac{\lambda_n}{n} z^n + \frac{\gamma_n}{n} z^n \right) . \]

Then

\[ \sum_{n=2}^{\infty} \left( \frac{\lambda_n}{n} + \frac{\gamma_n}{n} \right) = \sum_{n=2}^{\infty} (\lambda_n + \gamma_n) \]

\[ = a_1 - \lambda_1 \]

\[ \leq a_1 , \]

and \( f \in T^*_{H}(z_0) \).

Conversely, if \( f \in T^*_{H}(z_0) \), then \( a_n \leq \frac{a_1}{n} \) and \( b_n \leq \frac{a_1}{n} \).

Set \( \lambda_n = na_n \), \( \gamma_n = nb_n \), \( (n = 2, 3, 4, \ldots) \).

\[ \lambda_1 = a_1 - \sum_{n=2}^{\infty} (\lambda_n + \gamma_n) \]

and

\[ f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n + b_n \overline{z}^n \]

\[ = \left( \lambda_1 + \sum_{n=2}^{\infty} (\lambda_n + \gamma_n) \right) z - \sum_{n=2}^{\infty} \left( \frac{\lambda_n}{n} z^n + \frac{\gamma_n}{n} \overline{z}^n \right) \]

\[ = \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n \left( z - \frac{z^n}{n} \right) + \gamma_n \left( z - \frac{\overline{z}^n}{n} \right) \]
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\[ \lambda_i h_i + \gamma g_i + \sum_{n=2}^{\infty} \lambda_n h_n + \gamma_n g_n \]

\[ = \sum_{n=1}^{\infty} \lambda_n h_n(z) + \gamma_n g_n(z). \]

**Theorem 10.2.6.** Set \( h_i(z) = z \), \( h_n(z) = z - \frac{z^n}{n^2} \) \((n = 2, 3, 4, \ldots)\) and \( g_n(z) = z - \frac{z^n}{n^2} \) \((n = 2, 3, \ldots)\). Then \( f \in TK_H^0(z_0) \), if and only if it can be expressed in the form \( f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) \), where \( \lambda_n \geq 0, \gamma_n \geq 0, \lambda_1 = a_1 - \sum_{n=2}^{\infty} (\lambda_n + \gamma_n) \), and \( \gamma_1 = 0 \). In particular, the extreme points of \( TK_H^0(z_0) \) are \( \{h_n\} \) and \( \{g_n\} \).

**Proof.** The proof of above theorem is much akin to that of Theorem 10.2.5. Therefore, we omit details involved.

The following theorem brings out a closure property of the classes \( T_H^*(z_0) \) and \( TK_H^0(z_0) \).

**Theorem 10.2.7.** The class \( T_H^*(z_0) \) is closed under convex linear combinations.

**Proof.** For \( i = 1, 2, 3, \ldots \), let \( f_i(z) \in T_H^*(z_0) \), where \( f_i(z) \) is given by

\[ f_i(z) = a_i z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=2}^{\infty} b_n z^n. \quad (a_i > 0, a_n \geq 0, b_n \geq 0) \]

Then by Theorem 10.2.1, we have

\[ \sum_{n=2}^{\infty} n(a_n + b_n) \leq a_i. \] (10.2.4)
For \( \sum_{i=1}^{n} t_i = 1, 0 \leq t_i \leq 1 \), the convex combination of \( f_i \) may be written as
\[
\sum_{i=1}^{n} t_i f_i(z) = a_i z - \sum_{n=2}^{\infty} \left( \sum_{i=1}^{n} t_i a_{n_i} \right) z^n + \sum_{n=2}^{\infty} \left( \sum_{i=1}^{n} t_i b_{n_i} \right) z^n.
\]

Then by (10.2.4), we have
\[
\sum_{n=2}^{\infty} \left[ \sum_{i=1}^{n} t_i a_{n_i} + \sum_{i=1}^{n} t_i b_{n_i} \right] = \sum_{i=1}^{n} \left( \sum_{n=2}^{\infty} \left( a_{n_i} + b_{n_i} \right) \right) \leq a_i \sum_{i=1}^{n} t_i = a_i.
\]

This is the condition required by Theorem 10.2.1 and so \( \sum_{i=1}^{n} t_i f_i(z) \in T^{*0}_{h}(z_0) \). \( \square \)

**Theorem 10.2.8.** The class \( T^0_{h}(z_0) \) is closed under convex linear combinations.

**Proof.** The proof of this theorem is similar to that of Theorem 10.2.7, therefore we omit details involved. \( \square \)

In the next Theorem, we discuss a class preserving integral operator for these classes studied in this chapter.

Let \( f(z) = h(z) + \overline{g(z)} \) be defined by (10.1.1), then \( F(z) \) defined by the relation
\[
F(z) = \frac{c+1}{z^c} \int_0^{c-1} h(t) dt + \frac{c+1}{z^c} \int_0^{c-1} g(t) dt, \quad (c > -1).
\] (10.2.5)

**Theorem 10.2.9.** Let \( f(z) = h(z) + \overline{g(z)} \) be given by (10.1.4) and \( f \in T^{*0}_{h}(z_0) \) then \( F(z) \) be defined by (10.2.5) also belong to \( T^{*0}_{h}(z_0) \).
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Proof. Let \( f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=2}^{\infty} b_n z^n \) be in \( T_{H}^{*,0}(z_0) \) then by Theorem 10.2.1, we have

\[
\sum_{n=2}^{\infty} n(a_n + b_n) \leq a_1.
\]

By definition of \( F(z) \), we have

\[
F(z) = a_1 z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n - \sum_{n=1}^{\infty} \frac{c+1}{c+n} b_n z^n.
\]

Now

\[
\sum_{n=2}^{\infty} \left( \frac{c+1}{c+n} a_n + \frac{c+1}{c+n} b_n \right)
\]

\[
\leq \sum_{n=2}^{\infty} n(a_n + b_n)
\]

\[
\leq a_1.
\]

Thus \( F(z) \in T_{H}^{*,0}(z_0) \).

\[\square\]

Theorem 10.2.10. Let \( f(z) = h(z) + g(z) \) be given by (10.1.4) and \( f \in TK_{H}^{0}(z_0) \) then \( F(z) \) be defined by (10.2.5) also belong to \( TK_{H}^{0}(z_0) \).

Proof. The proof of above theorem is similar to that of Theorem 10.2.8, so we omit details involved. \[\square\]