CHAPTER 8
AN APPLICATION OF FRACTIONAL CALCULUS TO HARMONIC UNIVALENT FUNCTIONS

8.1 A continuous complex-valued function \( f = u + iv \) defined in a simply connected domain \( D \) is said to be harmonic in \( D \) if both \( u \) and \( v \) are real harmonic in \( D \). In any simply connected domain we can write \( f = h + \bar{g} \), where \( h \) and \( g \) are analytic in \( D \). We call \( h \) the analytic part and \( g \) the co-analytic part of \( f \). A necessary and sufficient condition for \( f \) to be locally univalent and sense-preserving in \( D \) is that 

\[ |h'(z)| > |g'(z)|, \quad z \in D. \]

See Clunie and Sheil-Small [16].

Denote by \( S_h \) the class of functions \( f = h + \bar{g} \) that are harmonic univalent and sense-preserving in the open unit disk \( U = \{ z : |z| < 1 \} \) for which \( f(0) = f'(0) - 1 = 0 \). Then for \( f = h + \bar{g} \in S_h \), we may express the analytic functions \( h \) and \( g \) as

\[

g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_k| < 1. \tag{8.1.1}
\]

Recently, Jahangiri [46] defined the class \( S_h^*(\alpha) \) consisting of functions \( f \) of the form (8.1.1) satisfying the condition

\[
\Re \left( \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right) \geq \alpha, \quad (0 \leq \alpha < 1). \tag{8.1.2}
\]

The case when \( \alpha = 0 \) is given in [104] and for \( \alpha = b_1 = 0 \), see [101].

The class \( S_h \) reduces to class \( S \) of normalized analytic univalent functions if co-analytic part of \( f \) i.e. \( g = 0 \), for this class \( f(z) \) may be expressed as

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{8.1.3}
\]

Several authors, such as ([15], [20], [67], [68], [70], [75], [106], [107], [110]) studied the subclasses of analytic univalent functions by using fractional calculus operator. In this
Chapter 8

Chapter 8 has been made to study the subclass of harmonic univalent functions by using fractional calculus.

### 8.2 Fractional Operator

The following definitions of fractional derivatives and fractional integrals are due to Owa [70] and Srivastava and Owa [110].

**Definition 8.2.1.** The fractional integral of order $\lambda$ is defined for a function $f(z)$ by

$$D_{z}^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta,$$

(8.2.1)

where $\lambda > 0$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

**Definition 8.2.2.** The fractional derivative of order $\lambda$ is defined for a function $f(z)$ by

$$D_{z}^{\lambda}f(z) = \frac{\Gamma(n+\lambda)}{\Gamma(n)z^{n+\lambda}} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{1+\lambda}} d\zeta,$$

(8.2.2)

where $0 < \lambda < 1$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 8.2.1 above.

**Definition 8.2.3.** Under the hypothesis of Definition 8.2.2 the fractional derivative of order $n + \lambda$ is defined for a function $f(z)$ by

$$D_{z}^{n+\lambda}f(z) = \frac{d^{n}}{dz^{n}} D_{z}^{\lambda}f(z),$$

(8.2.3)

where $0 < \lambda < 1$ and $n \in N_{0} = \{0,1,2,\ldots\}$.

Using the Definition 8.2.1 and its known extension involving fractional derivatives, Owa and Srivastava [75] introduced the operator $\Omega^{\lambda} : A \rightarrow A$ defined by

$$\Omega^{\lambda}f(z) = \Gamma(2-\lambda)z^{-1} D_{z}^{\lambda}f(z), \quad (\lambda \neq 2,3,4,\ldots).$$

(8.2.4)

where $A$ is the class of functions of the form (8.1.3) which are analytic in $U$.

Let $S_{H,\lambda}(\alpha)$ denote the subclass of $S_{H}$ consisting of functions $f$ of the form (8.1.1) satisfying the following condition
where $0 \leq \alpha < 1, 0 \leq \lambda < 1$.

Further, let the subclass $T S_{s, \lambda}^* (\alpha)$ consist of harmonic functions $f = h + g$ in $S_{s, \lambda} (\alpha)$, so that $h$ and $g$ are of the form

$$h(z) = z - \sum_{k=1}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k.$$  \hfill (8.2.6)

We note that for $\lambda = 0$ the class $S_{s, \lambda}^* (\alpha)$ reduces to the class $S_{s}^* (\alpha)$ studied by Jahangiri [46].

### 8.3 Main Results

We begin with a sufficient coefficient condition for functions in $S_{s, \lambda}^* (\alpha)$.

**Theorem 8.3.1.** Let $f = h + g$ be such that $h$ and $g$ are given by (8.1.1). Furthermore, let

$$\sum_{k=1}^{\infty} \left( \frac{k - \alpha}{1 - \alpha} |a_k| + \frac{k + \alpha}{1 - \alpha} |b_k| \right) \varphi(k, \lambda) \leq 2,$$  \hfill (8.3.1)

where $a_i = 1, 0 \leq \alpha < 1, 0 \leq \lambda < 1$ and $\varphi(k, \lambda) = \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)}$. Then $f$ is sense-preserving, harmonic univalent in $U$ and $f \in S_{s, \lambda}^* (\alpha)$.

**Proof.** First we note that $f$ is locally univalent and sense-preserving in $U$. This is because

$$|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k |a_k| z^{k-1}$$

$$> 1 - \sum_{k=2}^{\infty} k |a_k|$$

$$\geq 1 - \sum_{k=2}^{\infty} \frac{k - \alpha}{1 - \alpha} \varphi(k, \lambda) |a_k|$$

$$\geq \sum_{k=1}^{\infty} \frac{k + \alpha}{1 - \alpha} \varphi(k, \lambda) |b_k|$$

$$\geq \sum_{k=1}^{\infty} k |b_k|$$
\[
> \sum_{k=1}^{\infty} k |b_k| r^{k-1} \\
\geq |g'(z)|.
\]

To show that \( f \) is univalent in \( U \), suppose \( z_1, z_2 \in U \) such that \( z_1 \neq z_2 \), then
\[
\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{|z_1 - z_2 + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)|}
\]
\[
= 1 - \frac{\sum_{k=1}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|}
\]
\[
\geq 1 - \frac{\sum_{k=1}^{\infty} k + \alpha \varphi(k, \lambda) |b_k|}{1 - \sum_{k=2}^{\infty} k - \alpha \varphi(k, \lambda) |a_k|}
\]
\[
\geq 0.
\]

Now, we show that \( f \in S^*_{H, \alpha}(\alpha) \). Using the fact that \( \Re \omega \geq \alpha \), if and only if
\[
|1 - \alpha + \omega| \geq |1 + \alpha - \omega|,
\]
it suffices to show that
\[
|A(z) + (1 - \alpha) B(z)| - |A(z) - (1 + \alpha) B(z)| \geq 0,
\]  
(8.3.2)
where \( A(z) = z \left( \Omega^4 h(z) \right)' - z \left( \Omega^4 g(z) \right)' \) and \( B(z) = \Omega^4 h(z) + \Omega^4 g(z) \).

Substituting for \( A(z) \) and \( B(z) \) in L.H.S. of (8.3.2) and making use of (8.3.1), we obtain
\[
\left| \frac{z (\Omega^4 h(z))' - z (\Omega^4 g(z))'}{(\Omega^4 h(z) + \Omega^4 g(z))} \right| + (1 - \alpha) (\Omega^4 h(z) + \Omega^4 g(z)) \]
Chapter 8

\[ -\left[ z(\Omega^h(z))' - z(\Omega^g(z))' \right] - (1+\alpha)(\Omega^h(z) + \Omega^g(z)) \]

\[ = (2-\alpha)z + \sum_{k=2}^{\infty} (k+1-\alpha) \varphi(k,\lambda) a_k z^k - \sum_{k=1}^{\infty} (k-1+\alpha) \varphi(k,\lambda) b_k z^k \]

\[ \geq (2-\alpha)|z| - \sum_{k=2}^{\infty} (k+1-\alpha) \varphi(k,\lambda) |a_k| |z|^k - \sum_{k=1}^{\infty} (k+1+\alpha) \varphi(k,\lambda) |b_k| |z|^k \]

\[ = 2(1-\alpha)|z| \left\{ 1 - \sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} \varphi(k,\lambda) |a_k| |z|^{k-1} - \sum_{k=1}^{\infty} \frac{k+\alpha}{1-\alpha} \varphi(k,\lambda) |b_k| |z|^{k-1} \right\} \]

\[ = 2(1-\alpha)|z| \left\{ 1 - \left( \sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} \varphi(k,\lambda) |a_k| + \sum_{k=1}^{\infty} \frac{k+\alpha}{1-\alpha} \varphi(k,\lambda) |b_k| \right) \right\} \]

\[ \geq 0. \quad \text{(Using (8.3.1))} \]

The Coefficient bound (8.3.1) is sharp for the function

\[ f(z) = z + \sum_{k=2}^{\infty} \frac{1-\alpha}{(k-\alpha) \varphi(k,\lambda)} x_k z^k + \sum_{k=1}^{\infty} \frac{1-\alpha}{(k+\alpha) \varphi(k,\lambda)} y_k z^k. \]  

(8.3.3)

where \( \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1. \)

This completes the proof of the theorem.

If we put \( \lambda = 0 \) in above theorem, we obtain the following result given by Jahangiri [46].

Corollary 8.3.1. Let the function \( f = h + \overline{g} \) be so that \( h \) and \( g \) are given by (8.1.1). Furthermore, let

\[ \sum_{k=1}^{\infty} \left( \frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k| \right) \leq 2, \]

(8.3.4)

where \( a_k = 1, \ 0 \leq \alpha < 1. \) Then \( f \) is sense-preserving, harmonic univalent in \( U \) and \( f \in S^*_u(\alpha) \).
Chapter 8

In the following theorem, it is proved that the condition (8.3.1) is also necessary for functions \( f = h + \tilde{g} \), where \( h \) and \( g \) are of the form (8.2.6).

**Theorem 8.3.2.** Let the functions \( f = h + \tilde{g} \) be so that \( h \) and \( g \) are given by (8.2.6). Then \( f \in TS_{H,A}^*(\alpha) \), if and only if

\[
\sum_{k=1}^{\infty} \left( (k-\alpha) \varphi(k,\lambda) |a_k| + (k+\alpha) \varphi(k,\lambda) |b_k| \right) \leq 2(1-\alpha),
\]

(8.3.5)

where \( a_i = 1, 0 \leq \alpha < 1, 0 \leq \lambda < 1 \) and \( \varphi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \).

**Proof.** Since \( TS_{H,A}^*(\alpha) \subset S_{H,A}^*(\alpha) \), we only need to prove the "only if" part of the theorem. To this end, for functions \( f \) of the form (8.1.2) we notice that condition

\[
\text{Re} \left\{ \frac{z \left( \Omega^2 h(z) \right)' - z \left( \Omega^2 g(z) \right)'}{\Omega^2 h(z) + \Omega^2 g(z)} \right\} \geq \alpha
\]

is equivalent to

\[
\text{Re} \left\{ \frac{(1-\alpha) z - \sum_{k=2}^{\infty} (k-\alpha) \varphi(k,\lambda) |a_k| z^k - \sum_{k=1}^{\infty} (k+\alpha) \varphi(k,\lambda) |b_k| z^k}{z - \sum_{k=2}^{\infty} |a_k| \varphi(k,\lambda) z^k + \sum_{k=1}^{\infty} |b_k| \varphi(k,\lambda) z^k} \right\} \geq 0.
\]

The above condition must hold for all values of \( z, |z| = r < 1 \). Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq z = r < 1 \), we must have

\[
(1-\alpha) - \sum_{k=2}^{\infty} (k-\alpha) \varphi(k,\lambda) |a_k| r^{k-1} - \sum_{k=1}^{\infty} (k+\alpha) \varphi(k,\lambda) |b_k| r^{k-1}
\]

\[
1 - \sum_{k=2}^{\infty} |a_k| \varphi(k,\lambda) r^{k-1} + \sum_{k=1}^{\infty} |b_k| \varphi(k,\lambda) r^{k-1}
\]

\[
\geq 0.
\]

If the condition (8.3.5) does not hold then the numerator in (8.3.6) is negative for \( r \) sufficiently close to 1. Thus there exist a \( z_0 = r_0 \) in \((0,1)\) for which the quotient in (8.3.6) is negative. This contradicts the required condition for \( f \in S_{H,A}^*(\alpha) \) and so the proof is complete. □
Next, we determine the extreme points of closed convex hulls of $TS_{H,x}'(\alpha)$ denoted by $\text{clco } TS_{H,x}'(\alpha)$.

**Theorem 8.3.3.** If $f \in \text{clco } TS_{H,x}'(\alpha)$, if and only if

\[
f(z) = \sum_{k=1}^{\infty} \left( x_k h_k(z) + y_k g_k(z) \right),
\]

where $h_k(z) = z$, $h_k(z) = z - \frac{1-\alpha}{(k-\alpha) \varphi(k, \lambda)} z^k$, $(k = 2, 3, 4, \ldots)$,

\[
g_k(z) = z + \frac{1-\alpha}{(k+\alpha) \varphi(k, \lambda)} z^k, \quad (k = 1, 2, 3, \ldots), \quad x_k \geq 0, \quad y_k \geq 0, \quad \sum_{k=1}^{\infty} (x_k + y_k) = 1.
\]

In particular, the extreme points of $TS_{H,x}'(\alpha)$ are $\{h_k\}$ and $\{g_k\}$.

**Proof.** For functions $f$ of the form (8.3.7), we have

\[
f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z))
\]

\[
= \sum_{k=1}^{\infty} (x_k + y_k) z - \sum_{k=2}^{\infty} \frac{1-\alpha}{(k-\alpha) \varphi(k, \lambda)} x_k z^k + \sum_{k=1}^{\infty} \frac{1-\alpha}{(k+\alpha) \varphi(k, \lambda)} y_k z^k.
\]

Then

\[
\sum_{k=2}^{\infty} \frac{k \alpha}{1-\alpha} \varphi(k, \lambda) \left( - \frac{1-\alpha}{(k-\alpha) \varphi(k, \lambda)} x_k \right) + \sum_{k=1}^{\infty} \frac{k \alpha}{1-\alpha} \varphi(k, \lambda) \left( \frac{1-\alpha}{(k+\alpha) \varphi(k, \lambda)} y_k \right)
\]

\[
= \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1 \leq 1,
\]

and so $f \in TS_{H,x}'(\alpha)$.

Conversely, suppose that $f \in \text{clco } TS_{H,x}'(\alpha)$ then set $x_k = \frac{k \alpha}{1-\alpha} \varphi(k, \lambda) a_k$, $(k = 2, 3, 4, \ldots)$ and $y_k = \frac{k \alpha}{1-\alpha} \varphi(k, \lambda) b_k$, $(k = 1, 2, 3, \ldots)$. Note that by Theorem 8.3.2, $0 \leq x_k \leq 1$, $(k = 2, 3, 4, \ldots)$ and $0 \leq y_k \leq 1$, $(k = 1, 2, 3, \ldots)$. We define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$ and note that by Theorem 8.3.2, $x_i \geq 0$. Consequently, we obtain $f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z))$ as required. $\square$
Chapter 8

The following theorem gives the bounds for functions in $TS_{H, \lambda}^*(\alpha)$, which yields a covering result for this class.

**Theorem 8.3.4.** Let $f \in TS_{H, \lambda}^*(\alpha)$, then

$$|f(z)| \leq (1 + |b_1|)r + \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right) \frac{2-\lambda}{2} r^2, \quad |z| = r < 1$$

and

$$|f(z)| \geq (1 - |b_1|)r - \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right) \frac{2-\lambda}{2} r^2, \quad |z| = r < 1.$$

**Proof.** We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in TS_{H, \lambda}^*(\alpha)$. Taking the absolute value of $f$, we obtain

$$|f(z)| \leq (1 + |b_1|)r + \sum_{k=2}^\infty (|a_k| + |b_k|)r^k$$

$$\leq (1 + |b_1|)r + \sum_{k=2}^\infty (|a_k| + |b_k|)r^2$$

$$= (1 + |b_1|)r + \frac{1-\alpha}{2-\alpha}\varphi(2, \lambda) \sum_{k=2}^\infty \frac{2-\alpha}{1-\alpha} \frac{2+\alpha}{1-\alpha} |a_k| + |b_k|)r^2$$

$$\leq (1 + |b_1|)r + \frac{(1-\alpha)}{2(2-\alpha)} \sum_{k=2}^\infty \frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k|) \varphi(k, \lambda)r^2$$

$$\leq (1 + |b_1|)r + \frac{(1-\alpha)}{2(2-\alpha)} \left(1 - \frac{1+\alpha}{1-\alpha} |b_1|\right) r^2$$

$$= (1 + |b_1|)r + \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right) \frac{2-\lambda}{2} r^2. \quad \square$$

The following covering result follows from the left hand inequality in Theorem 8.3.4.

**Corollary 8.3.2.** Let $f$ of the form (8.2.6) be so that $f \in TS_{H, \lambda}^*(\alpha)$.

Then

$$\left\{ \omega \mid \omega < \frac{1}{4 - 2\alpha} \left(2 + \lambda - \alpha \lambda - |b_1| (2 - 4\alpha + \lambda + \alpha \lambda) \right) \right\} \subset f(U).$$

**Remark 8.3.1.** If we put $\lambda = 0$ in above corollary, we obtain the covering result of Jahangiri [46].
**Theorem 8.3.5.** If \( f \in TS^*_{H,\lambda}(\alpha) \) then \( f \) is convex in the disc

\[
|z| \leq \min_k \left\{ \frac{(1-\alpha)(1-|b_1|)}{k \left[ (1-\alpha)-(1+\alpha)|b_1| \right]} \right\}^{\frac{1}{k-1}}, \quad (k = 2,3,4,\ldots).
\]

**Proof.** Let \( f \in TS^*_{H,\lambda}(\alpha) \), and let \( r(0 < r < 1) \) be fixed. Then \( r^{-1}f(rz) \in TS^*_{H,\lambda}(\alpha) \) and we have

\[
\sum_{k=2}^{\infty} k^2 (|a_k|+|b_k|) r^{k-1} = \sum_{k=2}^{\infty} k (|a_k|+|b_k|) (kr^{k-1}) \\
\leq \sum_{k=2}^{\infty} \frac{(k-\alpha)}{1-\alpha} \varphi(k,\lambda) |a_k| + \frac{k+\alpha}{1-\alpha} \varphi(k,\lambda) |b_k| kr^{k-1} \\
\leq 1-b_1,
\]

provided

\[
kr^{k-1} \leq \frac{1-|b_1|}{1-\frac{1+\alpha}{1-\alpha} |b_1|}
\]

which is true if

\[
r \leq \min_k \left\{ \frac{(1-\alpha)(1-|b_1|)}{k \left[ (1-\alpha)-(1+\alpha)|b_1| \right]} \right\}^{\frac{1}{k-1}}, \quad (k = 2,3,4,\ldots).
\]

\[\square\]

**8.4 Convolution and Convex Combinations**

In this section, we show that the class \( TS^*_{H,\lambda}(\alpha) \) is closed under convolution and convex combinations. We need the following definition of convolution of two harmonic functions.

Let the functions \( f(z) \) be defined by

\[
f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| z^k
\]

and

\[
F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| z^k,
\]

we define the convolution of two harmonic functions \( f \) and \( F \) as
\((f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty} |a_k A_k|^z^k + \sum_{k=1}^{\infty} |b_k B_k|\bar{z}^k. \) (8.4.1)

Using this definition, we show that the class \(TS_{H,\lambda}^*(\alpha)\) is closed under convolution.

**Theorem 8.4.1.** For \(0 \leq \beta \leq \alpha < 1\), let \(f \in TS_{H,\lambda}^*(\alpha)\) and \(F \in TS_{H,\lambda}^*(\beta)\).

Then \(f * F \in TS_{H,\lambda}^*(\alpha) \subset TS_{H,\lambda}^*(\beta)\).

**Proof.** Let \(f(z) = z - \sum_{k=2}^{\infty} |a_k A_k|^z^k + \sum_{k=1}^{\infty} |b_k B_k|^z^k\) be in \(TS_{H,\lambda}^*(\alpha)\)

and

\(F(z) = z - \sum_{k=2}^{\infty} |A_k|^z^k + \sum_{k=1}^{\infty} |B_k|^z^k\), be in \(TS_{H,\lambda}^*(\beta)\).

Then the convolution \(f * F\) is given by (8.4.1). We wish to show that the coefficients of \(f * F\) satisfy the required condition given in Theorem 8.3.2. For \(F \in TS_{H,\lambda}^*(\beta)\), we note that \(|A_k| \leq 1\) and \(|B_k| \leq 1\). Now, for the convolution function \(f * F\), we obtain

\[
\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} \varphi(k, \lambda)|a_k A_k| + \sum_{k=1}^{\infty} \frac{k+\alpha}{1-\alpha} \varphi(k, \lambda)|b_k B_k|
\]

\[\leq \sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} \varphi(k, \lambda)|a_k| + \sum_{k=1}^{\infty} \frac{k+\alpha}{1-\alpha} \varphi(k, \lambda)|b_k|\]

\[\leq 1. \quad \text{(Since } f \in TS_{H,\lambda}^*(\alpha).\text{)}
\]

Therefore \(f * F \in TS_{H,\lambda}^*(\alpha) \subset TS_{H,\lambda}^*(\beta)\). \(\square\)

Next, we show that \(TS_{H,\lambda}^*(\alpha)\) is closed under convex combinations of its members.

**Theorem 8.4.2.** The class \(TS_{H,\lambda}^*(\alpha)\) is closed under convex combination.

**Proof.** For \(i = 1, 2, 3, \ldots\), let \(f_i \in TS_{H,\lambda}^*(\alpha)\), where \(f_i(z)\) is given by

\[f_i(z) = z - \sum_{k=2}^{\infty} |a_k|^z^k + \sum_{k=1}^{\infty} |b_k|^z^k.\]

Then by (8.3.5), we have

\[
\sum_{k=1}^{\infty} \left( \frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k| \right) \varphi(k, \lambda) \leq 2. \quad \text{(8.4.2)}
\]
For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of $f_i$ may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_k| \right) z^k + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_k| \right) z^k.$$  \hspace{1cm} (8.4.3)

Then by (8.4.2), we have

$$\sum_{k=1}^{\infty} \left( \frac{k-\alpha}{1-\alpha} \sum_{i=1}^{\infty} t_i |a_k| \right) + \frac{k+\alpha}{1-\alpha} \left( \sum_{i=1}^{\infty} t_i |b_k| \right) \phi(k, \lambda)$$

$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=1}^{\infty} \left( \frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k| \right) \phi(k, \lambda) \right\}$$

$$\leq 2 \sum_{i=1}^{\infty} t_i = 2.$$  

This is the condition required by (8.3.5) and so $\sum_{i=1}^{\infty} t_i f_i \in TS_{*,\lambda}^*(\alpha)$. $\Box$

### 8.5 A Family of Class Preserving Integral Operator

Let $f(z) = h(z) + g(z)$ be defined by (8.1.1), then $F(z)$ defined by the relation

$$F(z) = c + z^{-c-1} \int_{c}^{1} h(t) dt + z^{-c-1} \int_{c}^{1} g(t) dt, \quad (c > -1).$$  \hspace{1cm} (8.5.1)

**Theorem 8.5.1.** Let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by (8.2.6) and $f \in TS_{*,\lambda}^*(\alpha)$ then $F(z)$ defined by (8.5.1) is also in the class $TS_{*,\lambda}^*(\alpha)$.

**Proof.** From the representation (8.5.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} \frac{c+1}{c+k} |a_k| z^k + \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k| z^k.$$  

Since $f \in TS_{*,\lambda}^*(\alpha)$, we have

$$\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} \phi(k, \lambda) |a_k| + \sum_{k=1}^{\infty} \frac{k+\alpha}{1-\alpha} \phi(k, \lambda) |b_k| \leq 1.$$  \hspace{1cm} (8.5.2)

Now

$$\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} \phi(k, \lambda) \left( \frac{c+1}{c+k} \right) |a_k| + \sum_{k=1}^{\infty} \frac{k+\alpha}{1-\alpha} \phi(k, \lambda) \left( \frac{c+1}{c+k} \right) |b_k|$$
Thus \( F(z) \in TS_{H,\lambda}^*(\alpha) \).