CHAPTER 6
APPLICATIONS OF CONVOLUTION OPERATORS INVOLVING HYPERGEOMETRIC FUNCTIONS

Section 1

6.1 Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (6.1.1)

which are analytic in the open unit disc $U = \{ z : |z| < 1 \}$ and $S$ denote the subclass of $A$ which are univalent in $U$.

In 1999, Kanas and Wisniowska [52] (See also Kanas and Srivastava [54]) studied the class of $k$-Uniformly convex functions denoted by $k-UCV$, $0 \leq k < \infty$ so that

$$f \in k-UCV, \text{ if and only if }$$

$$\Re\{ 1 + (z - \zeta) \frac{f''(z)}{f'(z)} \geq 0, |\zeta| \leq k, z \in U \}.$$  \hspace{1cm} (6.1.2)

For real $\phi$ we may let $\zeta = -kze^{i\phi}$. Then condition (6.1.2) can be rewritten as

$$\Re\left\{ 1 + (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)} \right\} \geq 0,$$  \hspace{1cm} (6.1.3)

and $k-UCV(\alpha)$ denote the subclass of $S$, if and only if

$$\Re\left\{ 1 + (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)} \right\} \geq \alpha, \hspace{1cm} (0 \leq \alpha < 1).$$  \hspace{1cm} (6.1.4)

Further, the class $k-S_{\alpha}(\alpha)$ denotes the subclass of $S$, if and only if
For $1 < \beta \leq \frac{4+k}{3}$ and $z \in U$, let

$$k - UCV^*(\beta) = \left\{ f \in S : \text{Re}\left\{1 + \left(1 + ke^{i\theta}\right)\frac{zf''(z)}{f'(z)} - ke^{i\theta}\right\} < \beta \right\},$$

and

$$k - S_p^+(\beta) = \left\{ f \in S : \text{Re}\left\{1 + \left(1 + ke^{i\theta}\right)\frac{zf''(z)}{f'(z)} - ke^{i\theta}\right\} < \beta \right\}.$$

Further, let $V$ be the subclass of $S$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} |a_n|z^n.$$  \hfill (6.1.8)

Let $k - PUCV^*(\beta) = k - UCV^*(\beta) \cap V$, $k - PS_p^+(\beta) = k - S_p^+(\beta) \cap V$.

In particular, when $k = 0$, we obtain $0 - UCV^*(\beta) = L(\beta)$, $0 - S_p^+(\beta) = M(\beta)$, $0 - PUCV^*(\beta) = U(\beta)$ and $0 - PS_p^+(\beta) = V(\beta)$. These classes $L(\beta)$, $M(\beta)$, $U(\beta)$ and $V(\beta)$ have been extensively study by Uralegaddi et. al. [115].

Several authors such as ([52], [53], [54]) studied the classes of $k$-Uniformly convex and $k$-Uniformly starlike functions with negative coefficients only. In the present chapter, analogues to these results an attempt has been made to study above mentioned classes of functions with positive coefficients. Furthermore, connections between various subclasses of analytic univalent functions have been established by applying certain convolution operators involving hypergeometric functions.
6.2 Coefficient Inequalities

In this section, coefficient inequalities are determined for the classes $k - PUCV^*(\beta)$ and $k - PS^*_p(\beta)$. This leads to extreme points, distortion bounds and covering theorems.

First, we give a sufficient coefficient condition for functions of form (6.1.1) belonging to the class $k - UCV^*(\beta)$.

**Theorem 6.2.1.** Let $f(z) = z + \sum_{n=2}^\infty a_n z^n$ be in $S$. If $\sum_{n=2}^\infty n(n + nk - k - \beta)|a_n| \leq \beta - 1$ then $f \in k - UCV^*(\beta)$.

**Proof.** Let $\sum_{n=2}^\infty n(n + nk - k - \beta)|a_n| \leq \beta - 1$. It suffices to show that

$$\left| \frac{1 + (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)} - 1}{1 + (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)} - (2\beta - 1)} \right| < 1, \quad z \in U.$$

We have

$$\left| \frac{1 + (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)} - 1}{1 + (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)} - (2\beta - 1)} \right| = \frac{(1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)}}{2(\beta - 1) - (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)}}.$$
\[
\frac{(k+1)\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1}}{2(\beta-1) - \sum_{n=2}^{\infty} n\{(k+1)(n-1) - 2(\beta-1)\}|a_n||z|^{n-1}}
\leq
\frac{(k+1)\sum_{n=2}^{\infty} n(n-1)|a_n|}{2(\beta-1) - \sum_{n=2}^{\infty} n\{(k+1)(n-1) - 2(\beta-1)\}|a_n|}.
\]

The last expression is bounded above by 1 if
\[
\sum_{n=2}^{\infty} n(n+nk-k-\beta)|a_n| \leq \beta-1.
\]

But (6.2.1) is true by hypothesis. Hence
\[
\left| \frac{1 + (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)} - 1}{1 + (1 + ke^{i\phi}) \frac{zf''(z)}{f'(z)} - (2\beta-1)} \right| < 1, \quad z \in U,
\]
and the theorem is proved.

\[\square\]

**Remark 6.2.1.** The above theorem is true even if \(1 < \beta \leq \frac{3+k}{2}\).

In the special case, when \(k = 0\), Theorem 6.2.1 would correspond to following result given by Uralegaddi et. al. [115].

**Corollary 6.2.1.** Let \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\) be in S. If \(\sum_{n=2}^{\infty} n(n-\beta)|a_n| \leq \beta-1\), then
\[f \in L(\beta)\).

The proof of the following theorem would run parallel to that of Theorem 6.2.1, so we omit details involved.
Theorem 6.2.2. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be in \( S \). If \( \sum_{n=2}^{\infty} (n + nk - k - \beta) |a_n| \leq \beta - 1 \), then \( f \in k - S_p'(\beta) \).

Taking \( k = 0 \) in Theorem 6.2.2, we obtain following result given by Uralegaddi et al. [115].

Corollary 6.2.2. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be in \( S \). If \( \sum_{n=2}^{\infty} (n - \beta) |a_n| \leq \beta - 1 \), then \( f \in M(\beta) \).

Theorem 6.2.3. A function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is in \( k - PUCV'(\beta) \), if and only if

\[
\sum_{n=2}^{\infty} n(n + nk - k - \beta) |a_n| \leq \beta - 1 .
\]  

(6.2.2)

Proof. The if part, follows from Theorem 6.2.1. To prove the only if part, let \( f \in k - PUCV'(\beta) \), then by definition of \( k - PUCV'(\beta) \), we have

\[
\text{Re} \left\{ 1 + \left(1 + ke^{i\phi} \frac{zf'(z)}{f'(z)} \right) \right\} < \beta , \quad z \in U ,
\]

is equivalent to

\[
\text{Re} \left\{ \frac{\beta - 1 - \sum_{n=2}^{\infty} (n^2 - n \beta) |a_n| z^{n-1} - ke^{i\phi} \sum_{n=2}^{\infty} n(n-1)|a_n| z^{n-1}}{1 + \sum_{n=2}^{\infty} n|a_n| z^{n-1}} \right\} \geq 0 .
\]

The above condition must hold for all values of \( z, |z| = r < 1 \). Upon choosing the values of \( z \) on the positive real axis, where \( 0 \leq z = r < 1 \), we must have

\[
\text{Re} \left\{ \frac{\beta - 1 - \sum_{n=2}^{\infty} (n^2 - n \beta) |a_n| r^{n-1} - ke^{i\phi} \sum_{n=2}^{\infty} n(n-1)|a_n| r^{n-1}}{1 + \sum_{n=2}^{\infty} n|a_n| r^{n-1}} \right\} \geq 0 .
\]
Since \( \text{Re}(ke^{\theta}) \geq -|ke^{\theta}| = -k \), the above inequality reduces to

\[
(\beta - 1) - \sum_{n=2}^{\infty} n(n-\beta)|a_n|r^{n-1} - k\sum_{n=2}^{\infty} n(n-1)|a_n|r^{n-1}
\]

\[
1 + \sum_{n=2}^{\infty} n|a_n|r^{n-1}
\]

\[\geq 0.\]

Letting \( r \to 1 \), we have \( \sum_{n=2}^{\infty} n(n+nk-k-\beta)|a_n| \leq \beta - 1 \) and the proof is complete.  

\[\square\]

**Remark 6.2.2.** The above theorem is true even if \( 1 < \beta \leq \frac{3+k}{2} \).

**Corollary 6.2.3.** Let the function \( f(z) \) be defined by (6.1.8) belong to the class \( k-PUCV^{\ast}(\beta) \). Then

\[
|a_n| \leq \frac{(\beta - 1)}{n(n+nk-k-\beta)}.
\]  

(6.2.3)

Taking \( k = 0 \) in Theorem 6.2.3, we obtain the following result due to Uralegaddi et al. [115].

**Corollary 6.2.4.** A function \( f(z) = z + \sum_{n=2}^{\infty} |a_n|z^n \) is in \( U(\beta) \), if and only if

\[
\sum_{n=2}^{\infty} n(n-\beta)|a_n| \leq \beta - 1.
\]

**Theorem 6.2.4.** A function \( f(z) = z + \sum_{n=2}^{\infty} |a_n|z^n \) is in \( k-PS^{\ast}_p(\beta) \), if and only if

\[
\sum_{n=2}^{\infty} (n+nk-k-\beta)|a_n| \leq \beta - 1.
\]

**Proof.** The proof of this theorem is much akin to that of Theorem 6.2.3, therefore we omit details involved.  

\[\square\]
Corollary 6.2.5. Let the function $f(z)$ be defined by (6.1.8) belong to the class $k - PS_p^*(\beta)$. Then

$$|a_n| \leq \frac{(\beta-1)}{(n+nk-k-\beta)}.$$  \hspace{1cm} (6.2.4)

Taking $k = 0$ in Theorem 6.2.4, we obtain the following result due to Uralegaddi et al. [115].

Corollary 6.2.6. A function $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n$ is in $V(\beta)$, if and only if

$$\sum_{n=2}^{\infty} (n-\beta)|a_n| \leq \beta-1.$$  

Next, we determine the extreme points of the closed convex hulls of $k - PUCV^*(\beta)$ and $k - PS_p^*(\beta)$.

Theorem 6.2.5 (i). Let $f_1(z) = z$ and $f_n(z) = z + \frac{\beta-1}{n(n+nk-k-\beta)} z^n$, $(n = 2,3,\ldots)$.

Then $f \in k - PUCV^*(\beta)$, if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$  \hspace{1cm} (6.2.5)

where $\sum_{n=1}^{\infty} \lambda_n = 1$.

(ii). Let $f_1(z) = z$ and $f_n(z) = z + \frac{\beta-1}{(n+nk-k-\beta)} z^n$, $(n = 2,3,\ldots)$.

Then $f \in k - PS_p^*(\beta)$, if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$  \hspace{1cm} (6.2.6)
where \( \sum_{n=1}^{\infty} \lambda_n = 1. \)

**Proof.** The proof of these results are straightforward, so we omit details involved. \( \square \)

The following theorem gives distortion bounds for functions in the classes \( k - PUCV^*(\beta) \) and \( k - PS^*_y(\beta) \), which yield a covering results for these classes.

**Theorem 6.2.6.** Let \( f \in k - PUCV^*(\beta) \). Then for \( |z| = r < 1 \)

\[
\frac{\beta - 1}{2(2+k-\beta)} r^2 \leq |f(z)| \leq r + \frac{\beta - 1}{2(2+k-\beta)} r^2,
\]

with equality for

\[
f(z) = z + \frac{\beta - 1}{2(2+k-\beta)} z^2.
\]

**Proof.** Since \( f(z) \in k - PUCV^*(\beta) \), in view of Theorem 6.2.3, we have

\[
2(2+2k-\beta) \sum_{n=1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} n(n+nk-k-\beta) |a_n| \leq \beta - 1,
\]

which evidently gives

\[
\sum_{n=2}^{\infty} |a_n| \leq \frac{(\beta - 1)}{2(2+k-\beta)}.
\]

Consequently, we have

\[
|f(z)| \leq r + \sum_{n=1}^{\infty} |a_n| r^n
\]

\[
\leq r + r^2 \sum_{n=2}^{\infty} |a_n|
\]

\[
\leq r + \frac{\beta - 1}{2(2+k-\beta)} r^2.
\]
Similarly \[ |f(z)| \geq r - \frac{\beta - 1}{2(2+k-\beta)} r^2. \]

Finally, we note that the equalities are attained for the functions \( f(z) \) defined by

\[ f(z) = z + \frac{\beta - 1}{2(2+k-\beta)} z^2, \quad \text{for } (z = \pm r). \]

This completes the proof of Theorem 6.2.6.

**Theorem 6.2.7.** Let \( f \in k - PS^*_{\rho}(\beta) \). Then for \( |z| = r < 1 \)

\[ r - \frac{\beta - 1}{2+k-\beta} r^2 \leq |f(z)| \leq r + \frac{\beta - 1}{2+k-\beta} r^2 \]

with equality for

\[ f(z) = z + \frac{\beta - 1}{2+k-\beta} z^2. \]

**Proof.** The proof of the above theorem is similar to that of Theorem 6.2.6, therefore we omit details involved.

In the special case when \( k = 0 \), Theorem 6.2.6 and 6.2.7 would correspond to the following results given by Uraleegaddi et al. [115].

**Corollary 6.2.7.** If \( f \in U(\beta) \) then for \( |z| = r < 1 \)

\[ r - \frac{\beta - 1}{2(2-\beta)} r^2 \leq |f(z)| \leq r + \frac{\beta - 1}{2(2-\beta)} r^2, \]

with equality for

\[ f(z) = z + \frac{\beta - 1}{2(2-\beta)} z^2. \]

**Corollary 6.2.8.** Let \( f \in V(\beta) \) then \( r - \frac{\beta - 1}{2-\beta} r^2 \leq |f(z)| \leq r + \frac{\beta - 1}{2-\beta} r^2 \), with equality for
\[ f(z) = z + \frac{\beta - 1}{2 - \beta} z^2, \quad (z = \pm r). \]

The following covering results follow from the left hand inequality in Theorem 6.2.6 and 6.2.7, respectively.

**Corollary 6.2.9 (i)** Let \( f \) of the form (6.1.8) be so that \( f \in k - PUCV^*(\beta) \). Then

\[
\left\{ \omega : |\omega| < \frac{5 + 2k - 3\beta}{2(2 + k - \beta)} \right\} \subset f(U).
\]

(ii) Let \( f \) of the form (6.1.8) be so that \( f \in k - PS^*_p(\beta) \). Then

\[
\left\{ \omega : |\omega| < \frac{3 + k - 2\beta}{2 + k - \beta} \right\} \subset f(U).
\]

**Remark 6.2.3.** If we put \( k = 0 \) in Corollary 6.2.9 (i) and (ii), we obtain the covering results given by Uralegaddi et al. [115].

### 6.3 Application of the Gaussian Hypergeometric Function

The Gaussian hypergeometric function \( f(z) = z F(a, b; c; z) , \ z \in U \) given by the series

\[
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n
\]

is the solution of the hypergeometric differential equation

\[
z(1-z) \omega''(z) + (c - (a + b + 1) z) \omega'(z) - ab \omega(z) = 0
\]

and has rich applications in various fields such as conformal mappings, quasiconformal theory, continued fractions and so on. For detailed study see [112] and references therein.
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Here $a, b, c$ are complex numbers such that $c \neq 0, -1, -2, \ldots$, $(a)_0 = 1$ for $a \neq 0$, and for each positive integer $n$, $(a)_n = (a + 1)(a + 2) \ldots (a + n - 1)$ is the pochhammer symbol.

For complex parameters $a, b, c$ ($c \neq 0, -1, -2, \ldots$), we define the function

$$\phi(z) = z F(a, b; c; z).$$

Corresponding to this function, we consider the convolution operator

$$\Omega = \Omega(a, b; c): A \to A$$

defined by $\Omega(a, b; c)f = f \ast \phi$ for any function $f$ in $A$.

Letting $\Omega(a, b; c)f = F(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} A_n z^n$ where $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$.

We observe that $\Omega(a, 1; a)f(z) = f(z) = f(z) \ast \frac{z}{1-z}$ is the identity mapping.

This convolution operator $\Omega$ were defined and studied by Hohlov [45].

Let $R^f(A, B), V(\beta), U(\beta), R(\beta)$ denote the subclasses of $S$ defined by various authors given below,

$$R^f(A, B) = \left\{ f \in A : \frac{f'(z)-1}{(A-B)f(z)-B(f'(z)-1)} < 1, \tau \in C \setminus \{0\}, -1 \leq B < A \leq 1, z \in U \right\},$$

(Dixit and Pal [17]),

$$V(\beta) = \left\{ f \in V : \text{Re} \frac{zf''(z)}{f'(z)} < \beta, z \in U, 1 < \beta \leq \frac{4}{3} \right\},$$

(Urallegaddi et al. [115]),

$$U(\beta) = \left\{ f \in V : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \beta, z \in U, 1 < \beta \leq \frac{3}{2} \right\},$$

(Urallegaddi et al. [115]) and

$$R(\beta) = \left\{ f \in V : \text{Re} f'(z) < \beta, z \in U, 1 < \beta \leq 2 \right\},$$

(Urallegaddi et al. [116]).

Throughout this section, we will frequently use the notations
\[ \Omega(f) = \Omega(a,b;c)f \quad \text{and} \quad D_{n+1} = \frac{(|a|)^{n-1}(|b|)^{n-1}}{(c)^{n-1}(1)^{n-1}}, \]

and a well-known formula

\[ F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma c}{\Gamma(c-a)\Gamma(c-b)}, \quad \text{Re}(c-a-b) > 0. \]

The main object of this chapter is to establish some important connections between the classes \( k-UCV^*(\beta) \), \( k-S^*_p(\beta) \), \( R^*(A,B) \), \( U(\beta) \), \( V(\beta) \) and \( R(\beta) \) by applying the convolution operator \( \Omega \).

### 6.4 Connections with Uniformly Convex Mappings

In order to establish connection between various subclasses of \( A \) and \( k-UCV^*(\beta) \) we need the following lemmas.

**Lemma 6.4.1.** If \( f \in R^*(A,B) \) is of form (6.1.1), then

\[ |a_n| \leq \frac{|A-B|}{n}, \quad (n \geq 2). \]  

(See [17]).

**Lemma 6.4.2.** If \( a,b,c > 0 \), then

\[ \sum_{n=2}^{\infty} (n-1)(a)^{(n-1)}(b)^{(n-1)}(c)^{(n-1)}(1)^{(n-1)} = \frac{ab}{c-a-b-1}F(a,b;c;1), \quad \text{if} \ c > a+b+1. \]  

(See [2])

**Theorem 6.4.1.** Let \( a,b \in C \setminus \{0\} \), \( c \in R \), \( c > |a| + |b| + 1 \). If for some \( k(0 \leq k < \infty) \),

\[ 1 < \beta \leq \frac{3+k}{2} \]  

and the condition

\[ QF(|a|,|b|;c;1) \leq \frac{(\beta-1)}{(A-B)|_1}(1-(A-B)|_1) \]

is satisfied, then

\[ \Omega \left( R^*(A,B) \right) \subset k-UCV^*(\beta), \quad \text{where} \quad Q = \frac{(k+1)|ab|}{c-|a|-|b|-1} - (\beta-1). \]
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Proof. Let $f \in R^i(A,B)$, where $f$ of the form (6.1.1). In view of Theorem 6.2.1, it is enough to show that $P \leq \beta - 1$, where

$$P = \sum_{n=2}^{\infty} n(n(k+1)-(k + \beta)) \left| \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} A_n \right|$$

$$\leq (A-B)|\tau| \sum_{n=2}^{\infty} ((k+1)(n-1)-(\beta-1)) \left| \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} \right|$$

$$= (A-B)|\tau| \left( \frac{k+1}{c-|a|-|b|-1} F(|a|,|b|;c;1) - (\beta-1) F(|a|,|b|;c;1) - 1 \right)$$

$$= (A-B)|\tau| \left( \frac{k+1}{c-|a|-|b|-1} - (\beta-1) \right) F(|a|,|b|;c;1) + (\beta-1).$$

Now $P \leq \beta - 1$, follows from the given condition.

Lemma 6.4.3. Let $f$ of form (6.1.8) and $f \in R(\beta)$ then $|a_n| \leq \frac{\beta - 1}{n}$. See [116].

Theorem 6.4.2. Let $a,b \in C \setminus \{0\}$, $c \in R$, $c > |a| + |b| + 1$. If for some $k (0 \leq k < \infty)$, $\beta_1 (1 \leq \beta_1 \leq 2)$ and $\beta_2 \left( 1 < \beta_2 \leq \frac{3+k}{2} \right)$ the condition $QF(|a|,|b|;c;1) \leq \frac{(\beta_2 - 1)(2-\beta_1)}{(\beta_1 - 1)}$ is satisfied, then $\Omega(R(\beta_1)) \subseteq k-UCV'(\beta_2)$, where $Q = \frac{(k+1)|ab|}{c-|a|-|b|-1} - (\beta_2 - 1)$.

Proof. Let $f \in R(\beta_1)$, where $f$ of the form (6.1.8). In view of Theorem 6.2.1, it is enough to show that $P \leq \beta_2 - 1$, where

$$P = \sum_{n=2}^{\infty} n(n(k+1)-(k + \beta_2)) \left| \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} A_n \right|$$
Theorem 6.4.3. Let $a, b \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{R}$ and $c > |a| + |b| + 1$ if for some $k \left(0 \leq k < \infty\right)$ and $\beta \left(1 < \beta \leq \frac{3+k}{2}\right)$ the condition $F(|a|, |b|; c; 1) \leq 2$ is satisfied, then $\Omega \left(k - \text{PUCV}^*(\beta)\right) \subseteq k - \text{UCV}^*(\beta)$.

Proof. Let $f \in k - \text{PUCV}^*(\beta)$, where $f$ is of the form (6.1.8). In view of Theorem 6.2.1, it is enough to show that $P \leq \beta - 1$, where

$$P = \sum_{n=2}^{\infty} n \left(n(k+1) - (k+\beta)\right) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} A_n$$

$$\leq (\beta - 1) \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(|c|)_{n-1}(1)_{n-1}} \text{[Using (6.2.3)]}$$

$$= (\beta - 1) \left[F(|a|, |b|; c; 1) - 1\right]$$

$$\leq (\beta - 1), \text{ by the given hypothesis.} \, \square$$

Theorem 6.4.4. Let $a, b \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{R}$ and $c > |a| + |b| + 1$ if for some $k \left(0 \leq k < \infty\right)$ and $\beta \left(1 < \beta \leq \frac{4+k}{3}\right)$ the condition $QF(|a|, |b|; c; 1) \leq 2$ is satisfied, then

$$\Omega \left(k - \text{PS}_{\beta}^*(\beta)\right) \subseteq k - \text{UCV}^*(\beta), \text{ where } Q = \frac{|ab|}{c-|a|-|b|-1} + 1.$$

Proof. Let $f \in k - \text{PS}_{\beta}^*(\beta)$, where $f$ is of the form (6.1.8). In view of Theorem 6.2.1, it is enough to show that $P \leq \beta - 1$, where
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$$P = \sum_{n=2}^{\infty} n (n(k+1)-(k+\beta)) \binom{a}{n-1} \binom{b}{n-1} A_n \binom{c}{n-1} \binom{l}{n-1}$$

$$\leq (\beta-1) \sum_{n=2}^{\infty} n \frac{(|a|)_{n-1} (|b|)_{n-1}}{(c)_{n-1} (l)_{n-1}} \quad \text{[Using (6.2.4)]}$$

$$= (\beta-1) \left( \sum_{n=2}^{\infty} (n-1) \frac{(|a|)_{n-1} (|b|)_{n-1}}{(c)_{n-1} (l)_{n-1}} + \frac{(|a|)_{n-1} (|b|)_{n-1}}{(c)_{n-1} (l)_{n-1}} \right)$$

$$= (\beta-1) \left[ \frac{|ab|}{c-|a|-|b|-1} \frac{F(|a|,|b|;c;1)+F(|a|,|b|;c;1)-1}{F(|a|,|b|;c;1)+F(|a|,|b|;c;1)-1} \right]$$

$$\leq (\beta-1), \text{ by the given hypothesis.} \quad \Box$$

### 6.5 Connections with Uniformly Starlike Mappings

In this section, we establish connections between various subclasses of $A$ and $k - S^*_p(\beta)$ by applying convolution operator $\Omega$.

For the relationship between classes $R^r(A,B)$ and $k - S^*_p(\beta)$, we need the following result.

**Lemma 6.5.1.** Let $a, b \in C \setminus \{0\}, a \neq 1, b \neq 1, c \in (0,1) \cup (1, \infty)$ and $c > \max \{0,|a|+|b|-1\}$.

Then

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{(|a|)_{n-1} (|b|)_{n-1}}{(c)_{n-1} (l)_{n-1}} = \frac{c-|a|-|b|}{(|a|-1)(|b|-1)} F(|a|,|b|;c;1) - \frac{c-1}{(|a|-1)(|b|-1)}.$$  

**Proof.** We can write

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{(|a|)_{n-1} (|b|)_{n-1}}{(c)_{n-1} (l)_{n-1}} = \frac{c-1}{(|a|-1)(|b|-1)} \sum_{n=1}^{\infty} \frac{(|a|-1)^n}{(c-1)_n} \frac{(|b|-1)^n}{(l-1)_n}$$
\[
\frac{(c-1)}{(|a|-1)(|b|-1)} [F(|a|-1,|b|-1; c-1;1)-1]
\]
and the result immediately follows.

**Theorem 6.5.1.** Let \(a, b \in C \setminus \{0\}, \ c \in R, \ c > \max \{0,|a|+|b|-1\}, \ (|a| \neq 1,|b| \neq 1\). If for some \(k(0 \leq k < \infty), \ \beta \left( 1 < \beta \leq \frac{4+k}{3} \right)\) and the inequality

\[
QF(|a||b|;c;1) \leq (\beta-1) \left[ \frac{(1-(A-B)|r|)}{(A-B)|r|} \right] - \frac{(c-1)(k+\beta)}{(|a|-1)(|b|-1)}
\]

is satisfied, then \(\Omega(R'(A,B)) \subset k - S_p^*(\beta)\), where \(Q = \left( (k+1)- \frac{(c-|a|-|b|)(k+\beta)}{(|a|-1)(|b|-1)} \right)\).

**Proof.** Let \(f \in R'(A,B)\), where \(f\) of the form (6.1.1). In view of Theorem 6.2.2, it is enough to show that \(P_i \leq \beta-1\), where

\[
P_i = \sum_{n=2}^{\infty} \left( n(k+1)-(k+\beta) \right) \left[\frac{A_n}{(a_{n-1} b_{n-1})}(c_{n-1} l_{n-1}) \right]
\]

\[
\leq (A-B)|r| \sum_{n=2}^{\infty} \left( (k+1)-(k+\beta) \frac{(|a|_{n-1}(|b|_{n-1})}{(|c|_{n-1}(|l|_{n-1})} \right)
\]

\[
= (A-B)|r| \left[ \left( (k+1)- \frac{(c-|a|-|b|)(k+\beta)}{(|a|-1)(|b|-1)} \right) F(|a||b|;c;1) + \frac{(c-1)(k+\beta)}{(|a|-1)(|b|-1)} + (\beta-1) \right].
\]

Thus \(P_i \leq \beta-1\), follows from the given condition.

**Theorem 6.5.2.** Let \(a, b \in C \setminus \{0\}, \ c \in R, \ c > \max \{0,|a|+|b|-1\}, \ (|a| \neq 1,|b| \neq 1\). If for some \(k(0 \leq k < \infty), \ \beta \left( 1 < \beta \leq 2 \right)\) and \(\beta \left( 1 < \beta \leq \frac{4+k}{3} \right)\) the inequality
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QF([a],[b];c;1) ≤ \frac{(β_2 - 1)(2 - β_1)}{(β_1 - 1)} - \frac{(k + β_2)(c - 1)}{(|a| - 1)(|b| - 1)} is satisfied, then

Ω(R(β_1)) ⊆ k - S_p^*(β_2), where Q = \left[ (k + 1) - \frac{(k + β_2)(c - |a| - |b|)}{(|a| - 1)(|b| - 1)} \right].

**Proof.** Let f ∈ R(β_1), where f of the form (6.1.8). In view of Theorem 6.2.2, it is enough to show that P_1 ≤ β_2 - 1, where

\begin{align*}
P_1 &= \sum_{n=2}^{∞} \left( n(k+1)-(k+β_2) \right) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} A_n \\
&\leq (β_1 - 1) \sum_{n=2}^{∞} \left( (k+1)-\frac{(k+β_2)}{n} \right) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= (β_1 - 1) \left[ (k+1)-\frac{(k+β_2)(c-|a|-|b|)}{(|a| - 1)(|b| - 1)} \right] F(|a|,|b|;c;1) + \frac{(k+β_2)(c-1)}{(|a| - 1)(|b| - 1)}(β_2 - 1) \\
&\leq (β_2 - 1), \text{ by the given hypothesis.}
\end{align*}

**Theorem 6.5.3.** Let a,b ∈ C \{0\}, c ∈ R if for some k \(0 ≤ k < ∞\) and β \(1 < β ≤ \frac{4+k}{3}\) the condition \(F(|a|,|b|;c;1) ≤ 2\) is satisfied, then \(Ω(k - PS_p^*(β)) ⊆ k - S_p^*(β)\).

**Proof.** Let f ∈ k - PS_p^*(β), where f of the form (6.1.8). In view of Theorem 6.2.2, it is enough to show that P_1 ≤ β - 1, where

\begin{align*}
P_1 &= \sum_{n=2}^{∞} \left( n+nk-k-β \right) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} A_n \\
&\leq (β-1) \sum_{n=2}^{∞} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \text{ (Using 6.2.4)}
\end{align*}
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\[(\beta - 1) \left[ F(|a|,|b|;c;1) - 1 \right] \]

\[\leq (\beta - 1), \text{ by the given hypothesis.} \]

**Theorem 6.5.4.** Let \(a, b \in C \setminus \{0\}, \ c \in R \) and \( c > \max \{0,|a|+|b|-1\}, (|a| \neq 1, |b| \neq 1) \) for some \( k(0 \leq k < \infty) \), and \( \beta \left( 1 < \beta \leq \frac{4+k}{3} \right) \) the condition

\[QF(|a|,|b|;c;1) \leq 2 + \frac{(c-1)}{(|a|-1)(|b|-1)} \]

is satisfied, then \( \Omega(k - \text{PUCV}'(\beta)) \subset k - S'_\nu(\beta) \),

where \( Q = \frac{c - |a| - |b|}{(|a|-1)(|b|-1)} \).

**Proof.** Let \( f \in k - \text{PUCV}'(\beta) \), where \( f \) is of the form (6.1.8). In view of Theorem 6.2.2, it is enough to show that \( P_1 \leq \beta - 1 \), where

\[P_1 = \sum_{n=2}^{\infty} (n + nk - k - \beta) \left| \frac{(a)_{n-1} \frac{b}{(c)_{n-1} (1)_{n-1}} A_n}{n} \right| \]

\[\leq (\beta - 1) \sum_{n=2}^{\infty} 1 \frac{(|a|)_{n-1} (|b|)_{n-1}}{(|a|-1)(|b|-1)} \frac{(c-1)}{(|a|-1)(|b|-1)} \]

\[= (\beta - 1) \left[ \frac{(c - |a| - |b|)}{(|a|-1)(|b|-1)} F(|a|,|b|;c;1) - \frac{(c-1)}{(|a|-1)(|b|-1)} \right] \]

\[\leq (\beta - 1), \text{ by the given hypothesis.} \]
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Section 2

6.6 Let \( A \) denote the class of functions \( f \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disc \( U \).

Hohlov [45] introduced the convolution operator \( H(a, b, c): A \to A \) defined by

\[
H(a, b, c)f(z) = z F(a, b; c; z)* f(z)
\]

where the symbol "\( * \)" stands for the convolution of two functions and \( F(a, b; c; z) \) is a well-known Gaussian hypergeometric function. Here \( a, b, c \) are complex numbers such that \( c \neq 0, -1, -2, \ldots \). A hypergeometric function \( F(a, b; c; z) \) is analytic in \( U \) and plays an important role in Geometric Function Theory. See, for example, the works by Ahuja [3], Carleson and Shaffer [14], Miller and Mocanu [66], Owa and Srivastava [75], Ponnusamy and Rønning [82], Ruscheweyh and Singh [90] and Swaminathan [112].

Let \( H \) be the family of all harmonic functions of the form \( f = h + \bar{g} \), where

\[
h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad |B_n| < 1, \quad z \in U,
\]

are in the class \( A \). For complex parameters \( a_1, b_1, c_1, a_2, b_2, c_2, (c_1, c_2 \neq 0, -1, -2, \ldots) \), we define the functions

\[
\varphi_1(z) = z F(a_1, b_1; c_1; z) \quad \text{and} \quad \varphi_2(z) = z F(a_2, b_2; c_2; z).
\]

Corresponding to these functions, we consider the following convolution operator

\[
\Omega \equiv \Omega \left( \begin{array}{ccc}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
\end{array} \right) : H \to H
\]

defined by
for any function \( f = h + g \) in \( H \).

Letting

\[
\Omega \left( \begin{array}{ccc}
 a_1 & b_1 & c_1 \\
 a_2 & b_2 & c_2 \\
 \end{array} \right) f(z) = H(z) + G(z),
\]

we have

\[
H(z) = z + \sum_{n=2}^{\infty} \left( \frac{a_1}{c_1} \right)_{n-1} \left( \frac{b_1}{c_1} \right)_{n-1} A_n z^n, \quad G(z) = \sum_{n=1}^{\infty} \left( \frac{a_2}{c_2} \right)_{n-1} \left( \frac{b_2}{c_2} \right)_{n-1} B_n z^n.
\]

(6.6.3)

We observe that

\[
\Omega \left( \begin{array}{ccc}
 a_1 & 1 & a_1 \\
 a_2 & 1 & a_2 \\
 \end{array} \right) f(z) = f(z) = f(z) * \left( \frac{z}{1-z} + \frac{\overline{z}}{1-\overline{z}} \right)
\]

is the identity mapping. This convolution operator \( \Omega \) were defined and studied by the author in [2].

Denote by \( S_H \) the subclass of \( H \) that are univalent and sense-preserving in \( U \). Note that

\[
\frac{f - B f}{1 - |B_i|^2} \in S_H, \quad \text{whenever} \quad f \in S_H.
\]

Thus we may our restrict our attention to the subclass \( S_H^0 \) of \( S_H \) defined by

\[
S_H^0 = \{ f = h + g \in S_H : g'(0) = B_i = 0 \}.
\]

The classes \( S_H^0 \) and \( S_H \) were first studied in [16]. Also, we let \( K_H^0, S_H^{*0} \) and \( C_H^0 \) denote the subclasses of \( S_H^0 \) of harmonic functions which are, respectively, convex, starlike and close-to-convex in \( U \). For definitions and properties of these classes, one may refer to [1], [16] or [25].
For $0 \leq \gamma < 1$, let

$$N_H(\gamma) = \left\{ f \in H : \Re \frac{f'(z)}{z'} \geq \gamma, z = re^{i\theta} \in U \right\},$$

$$G_H(\gamma) = \left\{ f \in H : \Re \left( (1 + e^{i\theta}) \frac{zf'(z)}{f(z)} - e^{i\theta} \right) \geq \gamma, \alpha \in R, z \in U \right\},$$

where

$$z' = \frac{\partial}{\partial \theta} (z = re^{i\theta}), \quad f'(z) = \frac{\partial}{\partial \theta} f(re^{i\theta}).$$

Define $TN_H(\gamma) = N_H(\gamma) \cap T$ and $TG_H(\gamma) = G_H(\gamma) \cap T$,

where $T$ consists of the functions $f = h + \bar{g}$ in $S_h$ so that $h$ and $g$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |A_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |B_n| z^n. \quad (6.6.4)$$

The classes $N_H(\gamma)$ and $G_H(\gamma)$ were initially introduced and studied, respectively, in ([5], [87]). A function $f$ in $G_H(\gamma)$ is called Goodman-Rønning type harmonic univalent function in $U$.

Throughout this section, we will frequently use the notations

$$\Omega(f) = \Omega \left( a_1, b_1, c_1 \right),$$

$$D_{n-1} = \frac{\langle a_1 \rangle_{n-1} \langle b_1 \rangle_{n-1}}{\langle c_1 \rangle_{n-1} \langle 1 \rangle_{n-1}}, \quad E_{n-1} = \frac{\langle a_2 \rangle_{n-1} \langle b_2 \rangle_{n-1}}{\langle c_2 \rangle_{n-1} \langle 1 \rangle_{n-1}},$$

and a well-known formula

$$F(a, b; c; 1) = \frac{\Gamma(c - a - b) \Gamma(c)}{\Gamma(c - a) \Gamma(c - b)}, \quad \Re(c - a - b) > 0.$$
The main object of this section is to establish some important connections between the classes \( K^0_H, S^*^0_H, C^0_H, N_H(\gamma) \) and \( G_H(\gamma) \) by applying the convolution operator \( \Omega \).

### 6.7 Connections with Goodman-Rønning-type Harmonic Univalent Functions

In order to establish connections between harmonic convex functions, we need the following results in the form of Lemmas given in [16], [87] and [2], respectively.

**Lemma 6.7.1.** If \( f = h + \overline{g} \in K^0_H \) where \( h \) and \( g \) are given by (6.6.2) with \( B_1 = 0 \), then

\[
|A_n| \leq \frac{n+1}{2}, \quad |B_n| \leq \frac{n-1}{2}.
\]

**Lemma 6.7.2.** Let \( f = h + \overline{g} \) be given by (6.6.2). If

\[
\sum_{n=2}^{\infty} (2n - 1 - \gamma)' |A_n| + \sum_{n=1}^{\infty} (2n + 1 + \gamma)' |B_n| \leq 1 - \gamma,
\]

then \( f \) is sense-preserving, Goodman-Rønning type harmonic univalent functions in \( U \) and \( f \in G_H(\gamma) \).

**Remark 6.7.1.** In [87], it is also shown that \( f = h + \overline{g} \) given by (6.6.4) is in the family \( TG_H(\gamma) \), if and only if the coefficient condition (6.7.1) holds. Moreover, if \( f \in TG_H(\gamma) \), then

\[
|A_n| \leq \frac{1-\gamma}{2n-1-\gamma}, \quad n \geq 2 \quad \text{and} \quad |B_n| \leq \frac{1-\gamma}{2n+1+\gamma}, \quad n \geq 1.
\]

**Lemma 6.7.3.** If \( a,b,c > 0 \), then

\[
(i) \sum_{n=2}^{\infty} (n-1) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(d)_{n-1}} = \frac{ab}{c-a-b-1} F(a,b;c;1), \quad \text{if } c > a + b + 1.
\]
(ii) $\sum_{n=2}^{\infty} (n-1)^2 \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} = \frac{(a)_{2} (b)_{2}}{(c-a-b-2)_{2}} + \frac{ab}{c-a-b-1} F(a,b;c;1)$, if $c > a+b+2$.

(iii) $\sum_{n=2}^{\infty} (n-1)^3 \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} = \frac{(a)_{3} (b)_{3}}{(c-a-b-3)_{3}} + \frac{3(a)_{2} (b)_{2}}{(c-a-b-2)_{2}} + \frac{ab}{c-a-b-1} F(a,b;c;1),$

if $c > a+b+3$.

**Theorem 6.7.1.** Let $a_j, b_j \in C \setminus \{0\}$, $c_j \in R$ and $c_j > |a_j| + |b_j| + 2$ for $j = 1,2$. If for some $\gamma (0 \leq \gamma < 1)$, the inequality $Q_1 F(|a_1|, |b_1|; c_1) + R_1 F(|a_2|, |b_2|; c_2) \leq 4(1-\gamma)$ is satisfied, then $\Omega(K^0_H) \subset G_H(\gamma)$, where

\[
Q_1 = \frac{2(|a_1|, |b_1|)}{(c_1 - |a_1| - |b_1| - 2)_{2}} + (7-\gamma) \frac{|a_1 b_1|}{(c_1 - |a_1| - |b_1| - 1)} + 2(1-\gamma)
\]

and

\[
R_1 = \frac{2(|a_2|, |b_2|)}{(c_2 - |a_2| - |b_2| - 2)_{2}} + (5+\gamma) \frac{|a_2 b_2|}{(c_2 - |a_2| - |b_2| - 1)}.
\]

**Proof.** Let $f = h + \bar{g} \in K^0_H$ with $h$ and $g$ of the form (6.6.2) with $B_1 = 0$. We need to show that $\Omega(f) = H + \bar{G} \in G_H(\gamma)$, where $H$ and $G$ defined by (6.6.3) are analytic functions in $U$.

In view of Lemma 6.7.2, we need to prove that

$$P_1 \leq 1-\gamma,$$

where

\[
P_1 = \sum_{n=2}^{\infty} (2n-1-\gamma) \left| \frac{(a_1)_{n-1} (b_1)_{n-1}}{(c_1)_{n-1} (1)_{n-1}} A_n \right| + \sum_{n=2}^{\infty} (2n+1+\gamma) \left| \frac{(a_2)_{n-1} (b_2)_{n-1}}{(c_2)_{n-1} (1)_{n-1}} B_n \right|.
\]

In view of Lemma 6.7.1 and 6.7.3, it follows that

$$P_1 \leq \frac{1}{2} \sum_{n=2}^{\infty} (n+1)(2n-1-\gamma) D_{n-1} + \frac{1}{2} \sum_{n=2}^{\infty} (n-1)(2n+1+\gamma) E_{n-1}$$
\[
\begin{align*}
&= \frac{1}{2} \sum_{n=2}^{\infty} \left[ 2(n-1)^2 + (5-\gamma)(n-1) + 2(1-\gamma) \right] D_{n-1} + \frac{1}{2} \sum_{n=2}^{\infty} \left[ 2(n-1)^2 + (3+\gamma)(n-1) \right] E_{n-1} \\
&= \frac{1}{2} Q_1 F(a_1,|b_1|;c_1;i) + \frac{1}{2} R_1 F(a_2,|b_2|;c_2;i) - (1-\gamma).
\end{align*}
\]

Now \( P_i \leq 1 - \gamma \), follows from the given condition. \( \square \)

Analogous to Theorem 6.7.1, we next find connections of the classes \( S_{\mu}^{*-0} \), \( C_{\mu}^{0} \) with \( G_{\mu}(\gamma) \). However, we first need the following result which may be found in ([1], [16]) or [114].

**Lemma 6.7.4.** If \( f = h + \bar{g} \in S_{\mu}^{*-0} \) or \( C_{\mu}^{0} \) with \( h \) and \( g \) as given by (6.7.2) with \( B_i = 0 \), then

\[
|A_n| \leq \frac{(2n+1)(n+1)}{6}, \quad |B_n| \leq \frac{(2n-1)(n-1)}{6}.
\]

**Theorem 6.7.2.** Let \( a_j, b_j \in C \setminus \{0\} \), \( c_j \in R \) and \( c_j > |a_j| + |b_j| + 3 \) for \( j = 1,2 \). If for some \( \gamma (0 \leq \gamma < 1) \), the inequality

\[
Q_2 F(a_1,|b_1|;c_1;i) + R_2 F(a_2,|b_2|;c_2;i) \leq 12(1-\gamma)
\]

is satisfied, then

\[\Omega(S_{\mu}^{*-0}) \subset G_{\mu}(\gamma) \quad \text{and} \quad \Omega(C_{\mu}^{0}) \subset G_{\mu}(\gamma),\]

where

\[
Q_2 = 4 \left( \frac{a_1}{c_1 - |a_1| - |b_1| - 3} \right) + 2(14-\gamma) \left( \frac{a_1}{c_1 - |a_1| - |b_1| - 1} \right) + 3(13-3\gamma) \left( \frac{|a_1b_1|}{c_1 - |a_1| - |b_1| - 1} \right) + 6(1-\gamma)
\]

and

\[
R_2 = 4 \left( \frac{a_2}{c_2 - |a_2| - |b_2| - 3} \right) + 2(10+\gamma) \left( \frac{a_2}{c_2 - |a_2| - |b_2| - 1} \right) + 3(5+\gamma) \left( \frac{|a_2b_2|}{c_2 - |a_2| - |b_2| - 1} \right).
\]
Proof. Let \( f = h + \overline{g} \in S_n^0 (C_n^0) \) where \( h \) and \( g \) are given by (6.6.2) with \( B_1 = 0 \). We need to prove that \( \Omega(f) = H + \overline{G} \in G_n (\gamma) \), where \( H \) and \( G \) given by (6.6.3) are analytic functions in \( U \).

In view of Lemma 6.6.2, we need to prove that

\[
P_1 \leq 1 - \gamma,
\]

where

\[
P_1 = \sum_{n=2}^{\infty} (2n-1-\gamma) \left| \frac{(a_1)_{n-1} (b_1)_{n-1} A_n}{(c_1)_{n-1} (1)_{n-1}} \right| + \sum_{n=2}^{\infty} (2n+1+\gamma) \left| \frac{(a_2)_{n-1} (b_2)_{n-1} B_n}{(c_2)_{n-1} (1)_{n-1}} \right|.
\]

In view of Lemma 6.7.3 and 6.7.4, it follows that

\[
P_1 \leq \frac{1}{6} \sum_{n=2}^{\infty} [(2n+1)(n+1)(2n-1-\gamma)] D_{n-1} + \frac{1}{6} \sum_{n=2}^{\infty} [(2n-1)(n-1)(2n+1+\gamma)] E_{n-1}
\]

\[
= \frac{1}{6} \sum_{n=2}^{\infty} \left[ 4(n-1)^3 + 2(8-\gamma)(n-1)^2 + (19-7\gamma)(n-1) + 6(1-\gamma) \right] D_{n-1}
\]

\[
+ \frac{1}{6} \sum_{n=2}^{\infty} \left[ 4(n-1)^3 + 2(4+\gamma)(n-1)^2 + (3+\gamma)(n-1) \right] E_{n-1}
\]

\[
= \frac{1}{6} Q_{1} F \left( |a_1|, |b_1|, c_1; 1 \right) + \frac{1}{6} R_{2} F \left( |a_2|, |b_2|, c_2; 1 \right) - (1-\gamma).
\]

Now \( P_1 \leq 1 - \gamma \) follows from the given condition. \( \square \)

In order to determine connection between \( TN_n (\beta) \) and \( G_n (\gamma) \), we need the following results in Lemma 6.7.5 [5] and Lemma 6.7.6 [3].

Lemma 6.7.5. Let \( f = h + \overline{g} \) where \( h \) and \( g \) are given by (6.6.4), and suppose \( 0 \leq \beta < 1 \). Then

\[
f \in TN_n (\beta) \Leftrightarrow \sum_{n=2}^{\infty} n |A_n| + \sum_{n=2}^{\infty} n |B_n| \leq 1 - \beta.
\]
Remark 6.7.2. If $f \in TN_{\beta}(\beta)$,

$$|A_n| \leq \frac{1-\beta}{n}, \quad n \geq 2 \quad \text{and} \quad |B_n| \leq \frac{1-\beta}{n}, \quad n \geq 1.$$ 

Lemma 6.7.6. Let $a, b \in C \setminus \{0\}, \quad a \neq 1, b \neq 1, \quad c \in (0,1) \cup (1,\infty) \quad \text{and} \quad c > \max \{0,|a|+|b|-1\}$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{|a|}{n-1} \right)^{|b|} = \frac{(c-|a|-|b|)}{(|a|-1)(|b|-1)} F\left(|a|,|b|,c;1\right) - \frac{(c-1)}{(|a|-1)(|b|-1)}.$$ 

Theorem 6.7.3. Let $a_j, b_j \in C \setminus \{0\}, \quad \{|a_j| \neq 1, |b_j| \neq 1\}, \quad c_j \in R \quad \text{and} \quad c_j > \max \{0,|a_j|+|b_j|-1\}$ for $j=1,2$. If for some $\beta(0 \leq \beta < 1)$ and $\gamma(0 \leq \gamma < 1)$, the inequality

$$Q_3 F\left(|a_1|,|b_1|;c_1;1\right) + R_3 F\left(|a_2|,|b_2|;c_2;1\right) \leq \frac{(1-\gamma)(2-\beta)}{(1-\beta)} - (1+\gamma) \left[ \frac{(c_1-1)}{(|a_1|-1)(|b_1|-1)} - \frac{(c_2-1)}{(|a_2|-1)(|b_2|-1)} \right]$$

is satisfied, then

$$\Omega(TN_{\beta}(\beta)) \subset G_{\beta}(\gamma),$$

where

$$Q_3 = 2 - (1+\gamma) \frac{|c_1| - |a_1| - |b_1|}{(|a_1|-1)(|b_1|-1)} \quad \text{and} \quad R_3 = 2 + (1+\gamma) \frac{|c_2| - |a_2| - |b_2|}{(|a_2|-1)(|b_2|-1)}.$$

Proof. Let $f = h + \tilde{g} \in TN_{\beta}(\beta)$ where $h$ and $g$ are given by (6.6.4). In view of Lemma 6.6.2, it is enough to show that $P_2 \leq 1-\gamma$, where

$$P_2 = \sum_{n=2}^{\infty} \left(2n-1-\gamma \right) \left(\frac{a_1}{n-1} \right)^{|b_1|} A_n + \sum_{n=1}^{\infty} \left(2n+1+\gamma \right) \left(\frac{a_2}{n-1} \right)^{|b_2|} B_n.$$ 

Using Remark 6.7.2 and Lemma 6.7.6, it follows that

$$P_2 \leq (1-\beta) \left[ \sum_{n=2}^{\infty} \left(2 - \frac{(1+\gamma)}{n} \right) D_{n-1} + \sum_{n=1}^{\infty} \left(2 + \frac{(1+\gamma)}{n} \right) E_{n-1} \right].$$
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\[(1 - \beta) \left[ Q_2 F \left( |a_1|, |b_1|; c_1; 1 \right) + R_2 F \left( |a_2|, |b_2|; c_2; 1 \right) - (1 - \gamma) + \frac{(1 + \gamma)(c_1 - 1)}{(|a_1| - 1)(|b_1| - 1)} - \frac{(1 + \gamma)(c_2 - 1)}{(|a_2| - 1)(|b_2| - 1)} \right] \leq 1 - \gamma ,

by the given hypothesis.

In the next theorem, we establish connections between \( TG_H (\gamma) \) and \( G_H (\gamma) \).

**Theorem 6.7.4.** Let \( a_j, b_j \in C \setminus \{0\}, c_j \in R \) and \( c_j > |a_j| + |b_j| \) for \( j = 1, 2 \). If for some \( \gamma (0 \leq \gamma < 1) \), the inequality

\[ F \left( |a_1|, |b_1|; c_1; 1 \right) + F \left( |a_2|, |b_2|; c_2; 1 \right) \leq 2 \]

is satisfied, then

\[ \Omega \left( TG_H (\gamma) \right) \subset G_H (\gamma) . \]

**Proof.** Making use of Lemma 6.7.2 and the definition of \( P_2 \) in Theorem 6.7.3, we only need to prove that \( P_2 \leq 1 - \gamma \).

Using Remark 6.7.1, it follows that

\[ P_2 = \sum_{n=2}^{\infty} (2n - 1 - \gamma) \left( \frac{(a_1)_{n-1} (b_1)_{n-1}}{(c_1)_{n-1} (l)_{n-1}} A_n + \sum_{n=1}^{\infty} (2n + 1 + \gamma) \left( \frac{(a_2)_{n-1} (b_2)_{n-1}}{(c_2)_{n-1} (l)_{n-1}} B_n \right) \right) \]

\[ \leq (1 - \gamma) \left[ \sum_{n=2}^{\infty} D_{n-1} + \sum_{n=1}^{\infty} E_{n-1} \right] \]

\[ = (1 - \gamma) \left[ F \left( |a_1|, |b_1|; c_1; 1 \right) - 1 + F \left( |a_2|, |b_2|; c_2; 1 \right) \right] \]

\[ \leq 1 - \gamma , \]

by the given condition and this completes the proof.

In the next result, we establish connections between \( TG_H (\gamma) \) and \( G_H (\gamma) \) by diluting the restrictions on the complex coefficients of Theorem 6.7.4.
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**Theorem 6.7.5.** If \( a_i, b_i > -1, \ a_i b_i < 0, \ c_i > \max \{0, a_i + b_i\}, \ a_2, b_2 \in C \setminus \{0\}, \ c_2 > |a_2| + |b_2|, \) then a sufficient condition for \( \Omega(TG_H(y)) \subset G_H(y) \) is that

\[
F(a_i, b_i; c_i; 1) - F(|a_2|, |b_2|; c_2; 1) \geq 0,
\]

for any \( \gamma(0 \leq \gamma < 1) \).

**Proof.** Let \( f = h + \bar{g} \in TG_H(y) \) with \( h \) and \( g \) in (6.6.4). Then

\[
\Omega(f) = z - \sum_{n=2}^{\infty} \frac{(a_i)_{n-1} (b_i)_{n-1}}{(c_i)_{n-1} (1)_{n-1}} |A_n| z^n + \sum_{n=1}^{\infty} \frac{(a_2)_{n-1} (b_2)_{n-1}}{(c_2)_{n-1} (1)_{n-1}} |B_n| z^n.
\]

(6.7.2)

This function can be rewritten as

\[
\Omega(f) = z + \frac{a_1 b_1}{c_1} \sum_{n=2}^{\infty} \frac{(a_i + 1)_{n-2} (b_i + 1)_{n-2}}{(c_i + 1)_{n-2} (1)_{n-1}} |A_n| z^n + \sum_{n=1}^{\infty} \frac{(a_2)_{n-1} (b_2)_{n-1}}{(c_2)_{n-1} (1)_{n-1}} |B_n| z^n.
\]

In view of Lemma 6.7.2, we need to prove that \( P_3 \leq 1 - \gamma \),

where

\[
P_3 = \sum_{n=2}^{\infty} (2n - 1 - \gamma) \frac{|a_i b_i| (a_i + 1)_{n-2} (b_i + 1)_{n-2}}{(c_i + 1)_{n-2} (1)_{n-1}} |A_n| + \sum_{n=1}^{\infty} (2n + 1 + \gamma) \frac{(a_2)_{n-1} (b_2)_{n-1}}{(c_2)_{n-1} (1)_{n-1}} |B_n|
\]

\[
\leq (1 - \gamma) \left[ \frac{|a_i b_i|}{a_i b_i} \sum_{n=2}^{\infty} D_{n-1} + \sum_{n=1}^{\infty} E_{n-1} \right]
\]

\[
\leq (1 - \gamma) [-F(a_i, b_i; c_i; 1) + 1 + F(|a_2|, |b_2|; c_2; 1)]
\]

\[
\leq (1 - \gamma),
\]

by the given condition. \( \square \)

In next theorem, we present conditions on the parameters \( a_i, a_2, b_i, b_2 c_i, c_2 \) and obtain a characterization for operator \( \Omega \) which maps \( TG_H(y) \) onto itself.
Theorem 6.7.6. Let \( a_j, b_j > 0 \), \( c_j > a_j + b_j \) for \( j = 1,2 \) and \( \gamma (0 \leq \gamma < 1) \). Then \( \Omega (T_G_H (\gamma)) \subset T_G_H (\gamma) \), if and only if, \( F(a_1, b_1; c_1; 1)+F(a_2, b_2; c_2; 1) \leq 2 \).

Proof. Let \( f = h + \tilde{g} \in T_G_H (\gamma) \) with \( h \) and \( g \) in (6.6.4). In view of Remark 6.7.1, we only need to prove that \( \Omega (f) \) given by (6.7.2) is in \( T_G_H (\gamma) \), if and only if \( P_2 \leq 1 - \gamma \),

where

\[
P_2 = \sum_{n=2}^{\infty} \left( 2n-1-\gamma \right) \frac{(a_1)_{n-1} (b_1)_{n-1}}{(c_1)_{n-1} (1)_{n-1}} A_n + \sum_{n=1}^{\infty} \left( 2n+1+\gamma \right) \frac{(a_2)_{n-1} (b_2)_{n-1} B_n}{(c_2)_{n-1} (1)_{n-1}}.
\]

Using the coefficient estimates stated in Remark 6.7.1, we obtain

\[
P_2 \leq (1-\gamma) \left[ \sum_{n=2}^{\infty} D_n + \sum_{n=1}^{\infty} E_n \right]
\]

\[
\leq (1-\gamma) \left[ F(a_1, b_1; c_1; 1)-1 + F(a_2, b_2; c_2; 1) \right]
\]

\[
\leq (1-\gamma),
\]

by the given condition. \( \square \)