CHAPTER 5
NEW SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS

Section 1

5.1 A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain $D$ is said to be harmonic in $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain we can write $f = h + g$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|, z \in D$. See Clunie and Sheil-Small [16].

Denote by $S_H$ the class of functions $f = h + g$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f'(0) - 1 = 0$. Then for $f = h + g \in S_H$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1.$$  \hspace{1cm} (5.1.1)

Note that $S_H$ reduces to class $S$ of normalized analytic univalent functions if the co-analytic part of its member is zero.

A function $f \in S_H$ is said to be harmonic starlike of order $\alpha$, $(0 \leq \alpha < 1)$, denoted by $S_H^*(\alpha)$, if

$$\frac{\partial}{\partial \theta} \left( \arg f(re^{i\theta}) \right) \geq \alpha, \quad |z| = r < 1,$$
and is said to be convex of order \( \alpha \) \( (0 \leq \alpha < 1) \) for \( |z| = r < 1 \), if

\[
\frac{\partial}{\partial \theta} \left( \arg \left( \frac{\partial}{\partial \theta} f(re^{\theta}) \right) \right) \geq \alpha ,
\]

or

\[
\text{Re} \left\{ \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right\} \geq \alpha ,
\]

and denoted by \( HK(\alpha) \). These classes \( S^*_H(\alpha) \) and \( HK(\alpha) \) have been extensively studied by Jahangiri [46]. The case for \( \alpha = 0 \) i.e. \( S^*_H(0) = S^*_H \) and \( HK(0) = HK \), respectively, the classes of starlike and convex functions in \( S_H \) are given in [104] and for \( \alpha = 0 = h = 1 \) in [101].

For \( (1 < \beta \leq 4/3) \) and \( z \in U \),

Let

\[
M_H(\beta) = \left\{ f \in S_H : \text{Re} \left( \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right) < \beta \right\},
\]

and

\[
L_H(\beta) = \left\{ f \in S_H : \text{Re} \left( 1 + \frac{z^2h''(z) + 2zg'(z) + z^2g''(z)}{zh'(z) - zg'(z)} \right) < \beta \right\}.
\]

Further, let \( V_H \) and \( U_H \) be the subclasses of \( S_H \) consisting of functions of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k - \sum_{k=4}^{\infty} b_k |z|^k ,
\]
and
\[ f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| z^k, \]  
(5.1.5)
respectively.

Let \( V_H^*(\alpha) \equiv S_H^*(\alpha) \cap V_H \), \( V_k(\alpha) \equiv HK(\alpha) \cap U_H \), \( V_H(\beta) \equiv M_H(\beta) \cap V_H \) and \( U_H(\beta) \equiv L_H(\beta) \cap U_H \). Further \( V_H^*(0) \equiv V_H^* \) and \( V_k(0) \equiv V_k \) are respectively the classes of starlike and convex functions in \( V_H \) and \( U_H \) respectively.

We note that for \( g = 0 \), the classes \( M_H(\beta) \equiv M(\beta), L_H(\beta) \equiv L(\beta), U_H(\beta) \equiv U(\beta) \) and \( V_H(\beta) \equiv V(\beta) \) have been extensively studied by Uralegaddi et al. [115].

In this chapter, coefficient inequalities, extreme points, distortion bounds, covering results, convolution and convex combinations are determined for the classes \( V_H(\beta) \) and \( U_H(\beta) \). Further, order of starlikeness and convexity are also obtained for these classes.

\section*{5.2 Main Results}

First, we give a sufficient coefficient bound for the class \( M_H(\beta) \).

\textbf{Theorem 5.2.1.} If \( f = h + g \in S_H \) be given by (5.1.1). If
\[ \sum_{k=2}^{\infty} \frac{k - \beta}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k + \beta}{\beta - 1} |b_k| \leq 1, \]  
(5.2.1)
then \( f \in M_H(\beta) \).

\textbf{Proof.} Let \( \sum_{k=2}^{\infty} \frac{k - \beta}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k + \beta}{\beta - 1} |b_k| \leq 1. \)

It suffices to show that
We have

\[
\frac{zh'(z) - zg'(z)}{h(z) + g(z)} - \frac{1}{1} < 1, \quad z \in U.
\]

The last expression is bounded above by 1, if

\[
\sum_{k=2}^{\infty} (k-1)|a_k||z|^{k-1} + \sum_{k=1}^{\infty} (k+1)|b_k||z|^{k-1} \\
\leq \frac{2(\beta - 1) - \sum_{k=2}^{\infty} (k - 2\beta + 1)|a_k||z|^{k-1} - \sum_{k=1}^{\infty} (k + 2\beta - 1)|b_k||z|^{k-1}}{2(\beta - 1) - \sum_{k=2}^{\infty} (k - 2\beta + 1)|a_k| - \sum_{k=1}^{\infty} (k + 2\beta - 1)|b_k|}.
\]

which is equivalent to

\[
\sum_{k=2}^{\infty} \frac{k-\beta}{\beta-1} a_k + \sum_{k=1}^{\infty} \frac{k+\beta}{\beta-1} b_k \leq 1.
\]

But (5.2.2) is true by hypothesis.
This completes the proof of Theorem 5.2.1.

\begin{theorem}
Let \( f = h + g \in S_\beta \) be given by (5.1.1). If
\[
\sum_{k=2}^{N} \frac{k(k-\beta)}{\beta - 1} |a_k| + \sum_{k=2}^{N} \frac{k(k+\beta)}{\beta - 1} |b_k| \leq 1,
\]
then \( f \in L_n(\beta) \).
\end{theorem}

\textbf{Proof.} The proof of this theorem is similar to that of Theorem 5.2.1, therefore we omit details involved.

In the following theorem, it is proved that the condition 5.2.1 is also necessary for functions \( f = h + g \in V_\beta \) be given by (5.1.4).

\begin{theorem}
A function \( f(z) = z + \sum_{k=2}^{N} a_k z^k - \sum_{k=1}^{N} b_k \overline{z}^k \) is in \( V_\beta (\beta) \), if and only if
\[
\sum_{k=2}^{N} (k-\beta) |a_k| + \sum_{k=1}^{N} (k+\beta) |b_k| \leq \beta - 1.
\]
\end{theorem}

\textbf{Proof.} Since \( V_\beta (\beta) \subset M_\beta (\beta) \), we only need to prove the "only if" part of the theorem. For this we show that \( f \not\in V_\beta (\beta) \), if the condition (5.2.3) does not hold.

Note that a necessary and sufficient condition for \( f = h + g \) given by (5.1.4) is in \( V_\beta (\beta) \), if
\[
\text{Re}\left\{ \frac{zh'(z) - zg(z)}{h(z) + g(z)} \right\} < \beta
\]
is equivalent to
\[
\text{Re}\left\{ \frac{(\beta - 1)z - \sum_{k=2}^{N} (k-\beta) |a_k| z^k - \sum_{k=1}^{N} (k+\beta) |b_k| \overline{z}^k}{z + \sum_{k=2}^{N} |a_k| z^k - \sum_{k=1}^{N} |b_k| \overline{z}^k} \right\} \geq 0.
\]
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The above condition must hold for all values of \( z, \, |z| = r < 1 \). Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq \Re(z) = r < 1 \), we must have

\[
(\beta - 1) - \sum_{k=2}^{\infty} (k - \beta) |a_k| r^{k-1} - \sum_{k=1}^{\infty} (k + \beta) |b_k| r^{k-1}
\]

\[\geq 0. \tag{5.2.4}\]

If the condition (5.2.3) does not hold then the numerator of (5.2.4) is negative for \( r \) sufficiently close to 1. Thus there exists a \( z_0 = r_0 \in (0,1) \) for which the quotient in (5.2.4) is negative. This contradicts the required condition for \( f \in V_H(\beta) \) and so the proof is complete.

\[\square\]

Theorem 5.2.4. Let \( f = h + g \) be given by (5.1.5). Then \( f \in U_H(\beta) \), if and only if

\[
\sum_{k=2}^{\infty} k(k - \beta) |a_k| + \sum_{k=1}^{\infty} k(k + \beta) |b_k| \leq \beta - 1.
\]

\[\text{Proof.} \quad \text{The proof of this theorem is similar to that of Theorem 5.2.3. Therefore it is omitted.} \quad \square\]

Remark 5.2.1. The above theorem is true even if \( 1 < \beta \leq 3/2 \).

Next, we determine the extreme points of the closed convex hulls of \( V_H(\beta) \), denoted by clco \( V_H(\beta) \).

Theorem 5.2.5. \( f \in \text{clco} V_H(\beta) \), if and only if

\[
f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z)), \tag{5.2.5}\]

where \( h_1(z) = z, \quad h_k(z) = z + \frac{\beta - 1}{k - \beta} z^k, \quad (k = 2, 3, 4, \ldots) \).
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\[ g_k(z) = z - \frac{\beta - 1}{k + \beta} z^k, \quad (k = 1, 2, 3, \ldots), \quad \sum_{i=1}^{\infty} (x_k + y_k) = 1, \quad x_k \geq 0 \text{ and } y_k \geq 0. \]

In particular, the extreme points of \( V_H(\beta) \) are \( \{h_k\} \) and \( \{g_k\} \).

Proof. For functions \( f \) of the form (5.2.5), we have

\[ f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z)) \]

\[ = z + \sum_{k=2}^{\infty} \frac{\beta - 1}{k - \beta} x_k z^k - \sum_{k=1}^{\infty} \frac{\beta - 1}{k + \beta} y_k z^k. \]

Then

\[ \sum_{k=2}^{\infty} \frac{k - \beta}{\beta - 1} \left( \frac{\beta - 1}{k - \beta} x_k \right) + \sum_{k=1}^{\infty} \frac{k + \beta}{\beta - 1} \left( \frac{\beta - 1}{k + \beta} y_k \right) \]

\[ = \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k \]

\[ = 1 - x_1 \leq 1, \]

and so

\[ f \in clco V_H(\beta). \]

Conversely, suppose that \( f \in clco V_H(\beta) \).

Set \( x_k = \frac{k - \beta}{\beta - 1} |a_k|, \quad (k = 2, 3, 4, \ldots) \) and \( y_k = \frac{k + \beta}{\beta - 1} |b_k|, \quad (k = 1, 2, 3, \ldots) \).

Then note that by Theorem 5.2.3, \( 0 \leq x_k \leq 1, (k = 2, 3, 4, \ldots) \) and \( 0 \leq y_k \leq 1, (k = 1, 2, 3, \ldots) \).

We define \( x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k \), and note that by Theorem 5.2.3, \( x_1 \geq 0 \).

Consequently, we obtain \( f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z)) \) as required. \( \square \)
Theorem 5.2.6. \( f \in cleU_h(\beta) \), if and only if \( f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z)) \), where

\[
h_1(z) = z, \quad h_k(z) = z + \frac{\beta - 1}{k(k - \beta)} z^k, \quad (k = 2, 3, 4, \ldots) \text{ and } g_k(z) = z + \frac{\beta - 1}{k(k + \beta)} z^k
\]

\((k = 1, 2, 3, \ldots)\), \( \sum_{k=1}^{\infty} (x_k + y_k) = 1 \), \( x_k \geq 0 \) and \( y_k \geq 0 \). In particular, the extreme points of \( U_h(\beta) \) are \( \{h_k\} \) and \( \{g_k\} \).

Proof. The proof of this theorem is much akin to that of Theorem 5.2.5. Therefore it is omitted.

The following theorem gives the bounds for functions in \( V_h(\beta) \) and \( U_h(\beta) \) which yield a covering result for these classes.

Theorem 5.2.7. If \( f \in V_h(\beta) \) then

\[
|f(z)| \leq (1 + |b_1|) r + \left( \frac{\beta - 1}{2 - \beta} - \frac{\beta + 1}{2 - \beta} |b_1| \right) r^2, |z| = r < 1,
\]

and

\[
|f(z)| \geq (1 - |b_1|) r - \left( \frac{\beta - 1}{2 - \beta} - \frac{\beta + 1}{2 - \beta} |b_1| \right) r^2, |z| = r < 1.
\]

Proof. We only prove the right hand inequality. The proof for left hand inequality is similar and will be omitted.

Let \( f \in V_h(\beta) \). Taking the absolute value of \( f \) we have

\[
|f(z)| \leq (1 + |b_1|) r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k
\]

\[
\leq (1 + |b_1|) r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^2
\]
\[ \sum_{k=2}^{\infty} \left( \frac{k - \beta}{\beta - 1} |a_k| + \frac{k + \beta}{\beta - 1} |b_k| \right) r^2 \]
\[ \leq \left( 1 + |b_1| \right) r + \left( \frac{1 + \beta}{\beta - 1} |b_1| \right) r^2, |z| = r < 1 \]
\[ = \left( 1 + |b_1| \right) r + \left( \frac{\beta - 1}{2 - \beta} \frac{\beta + 1}{2 - \beta} |b_1| \right) r^2, |z| = r < 1. \]

**Theorem 5.2.8.** If \( f \in U_H(\beta) \), then

\[ |f(z)| \leq \left( 1 + |b_1| \right) r + \left( \frac{\beta - 1}{2 - \beta} \frac{\beta + 1}{2 - \beta} |b_1| \right) r^2, |z| = r < 1, \]

and

\[ |f(z)| \geq \left( 1 - |b_1| \right) r - \left( \frac{\beta - 1}{2 - \beta} \frac{\beta + 1}{2 - \beta} |b_1| \right) r^2, |z| = r < 1. \]

**Proof.** The proof of this theorem runs parallel to that of Theorem 5.2.7. Therefore we omit the details involved.

The following covering results follows from the left hand inequality in Theorem 5.2.7 and 5.2.8 for the classes \( V_H(\beta) \) and \( U_H(\beta) \), respectively.

**Corollary 5.2.1** (i) Let \( f \) of the form (5.1.4) be so that \( f \in V_H(\beta) \). Then

\[ \left\{ \omega : |\omega| < \frac{3 - 2\beta}{2 - \beta} + \frac{2\beta - 1}{2 - \beta} |b_1| \right\} \subset f(U). \]

(ii) Let \( f \) of the form (5.1.5) be so that \( f \in U_H(\beta) \). Then

\[ \left\{ \omega : |\omega| < \frac{5 - 3\beta}{2(2 - \beta)} + \frac{3(\beta - 1)}{2(2 - \beta)} |b_1| \right\} \subset f(U). \]

For our next theorem, we need to define the convolution of two harmonic functions.
For harmonic functions of the form
\[ f(z) = z + \sum_{k=2}^{\infty} a_k |z|^k - \sum_{k=1}^{\infty} |b_k| |z|^k \]
and
\[ F(z) = z + \sum_{k=2}^{\infty} A_k |z|^k - \sum_{k=1}^{\infty} |B_k| |z|^k, \]
we define the convolution of two harmonic functions \( f \) and \( F \) as
\[ (f * F)(z) = f(z) * F(z) = z + \sum_{k=2}^{\infty} a_k A_k |z|^k - \sum_{k=1}^{\infty} |b_k B_k| |z|^k. \] (5.2.6)

Using this definition, we show that the class \( V_H(\beta) \) is closed under convolution.

**Theorem 5.2.9.** For \( 1 < \alpha \leq \beta \leq 4/3 \), let \( f \in V_H(\alpha) \) and \( F \in V_H(\beta) \). Then
\[ f * F \in V_H(\alpha) \subseteq V_H(\beta). \]

**Proof.** Let \( f(z) = z + \sum_{k=2}^{\infty} a_k |z|^k - \sum_{k=1}^{\infty} |b_k| |z|^k \) be in \( V_H(\alpha) \)
and
\[ F(z) = z + \sum_{k=2}^{\infty} A_k |z|^k - \sum_{k=1}^{\infty} |B_k| |z|^k \) be in \( V_H(\beta) \).

Then the convolution \( f * F \) is given by (5.2.6). We wish to show that the coefficients of \( f * F \) satisfy the required condition given in Theorem 5.2.3. For \( F(z) \in V_H(\beta) \), we note that \( |A_k| \leq 1 \) and \( |B_k| \leq 1 \). Now, for the convolution function \( f * F \), we obtain
\[
\sum_{k=2}^{\infty} \frac{k-\alpha}{\alpha-1} |a_k A_k| + \sum_{k=1}^{\infty} \frac{k+\alpha}{\alpha-1} |b_k B_k| \leq \sum_{k=2}^{\infty} \frac{k-\alpha}{\alpha-1} |a_k| + \sum_{k=1}^{\infty} \frac{k+\alpha}{\alpha-1} |b_k| .
\]
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\[ \leq 1. \quad \text{(Since } f \in V_h(\alpha)) \]

Therefore \( f \ast F \in V_h(\alpha) \subseteq V_h(\beta) \).

**Theorem 5.2.10.** For \( 1 < \alpha \leq \beta \leq 3/2 \) let \( f \in U_h(\alpha) \) and \( F \in U_h(\beta) \).

Then \( f \ast F \in U_h(\alpha) \subseteq U_h(\beta) \).

**Proof.** The proof of this theorem is similar to that of Theorem 5.2.9. Therefore, we omit details involved.

Now we show that \( V_h(\beta) \) is closed under convex combinations of its members.

**Theorem 5.2.11.** The class \( V_h(\beta) \) is closed under convex combination.

**Proof.** For \( i = 1, 2, 3, \ldots \), let \( f_i(z) \in V_h(\beta) \), where \( f_i(z) \) is given by

\[ f_i(z) = z + \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=2}^{\infty} |b_k| \overline{z}^k. \]

Then by Theorem 5.2.3,

\[ \sum_{k=2}^{\infty} \frac{k - \beta}{\beta - 1} |a_k| + \sum_{k=2}^{\infty} \frac{k + \beta}{\beta - 1} |b_k| \leq 1. \quad (5.2.7) \]

For \( \sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1 \), the convex combination of \( f_i \) may be written as

\[ \sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_k| \right) z^k - \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_k| \right) \overline{z}^k. \]

Now, using (5.2.7), we have

\[ \sum_{k=2}^{\infty} \frac{k - \beta}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i |a_k| \right) + \sum_{k=2}^{\infty} \frac{k + \beta}{\beta - 1} \left( \sum_{i=1}^{\infty} t_i |b_k| \right) \]

\[ = \sum_{i=1}^{\infty} t_i \left( \sum_{k=2}^{\infty} \frac{k - \beta}{\beta - 1} |a_k| + \sum_{k=2}^{\infty} \frac{k + \beta}{\beta - 1} |b_k| \right) \]

\[ \leq t_i = 1. \]
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This is the condition required by Theorem 5.2.3 and so $\sum_{i=1}^{\infty} t_i f_i(z) \in V_h(\beta)$. □

**Theorem 5.2.12.** The class $U_h(\beta)$ is closed under convex combination.

**Proof.** The proof of this theorem is much akin to that of Theorem 5.2.11. Therefore it is omitted. □

### 5.3 Order of Starlikeness and Convexity

**Theorem 5.3.1.** If $f \in V_h(\beta)$ then $f \in V_h^\ast \left( \frac{4-3\beta}{3-2\beta} \right)$.

**Proof.** Since

$$\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{k+\alpha}{1-\alpha} |b_k| \leq 1,$$

is a sufficient condition for $f \in S_h$ to be in $f \in S_h^\ast (\alpha)$. In view of Theorem 5.2.3, we must have

$$\sum_{k=2}^{\infty} \frac{k-\beta}{\beta-1} |a_k| + \sum_{k=1}^{\infty} \frac{k+\beta}{\beta-1} |b_k| \leq 1$$

implies

$$\sum_{k=2}^{\infty} \frac{k-\beta}{\beta-1} \frac{4-3\beta}{3-2\beta} |a_k| + \sum_{k=1}^{\infty} \frac{k+\beta}{\beta-1} \frac{4-3\beta}{3-2\beta} |b_k| \leq 1.$$

It suffices to show that

$$\frac{k-\beta}{\beta-1} \geq \frac{k-\frac{4-3\beta}{3-2\beta}}{1-\frac{4-3\beta}{3-2\beta}}, \quad (k = 2, 3, 4, \ldots),$$

and
\[
\frac{k + \beta}{\beta - 1} \geq \frac{k + \frac{4 - 3\beta}{3 - 2\beta}}{1 - \frac{4 - 3\beta}{3 - 2\beta}}, \quad (k = 1, 2, 3, 4 \ldots \ldots),
\]

or

\[(\beta - 1)(k - 2) \geq 0, \quad (k = 2, 3 \ldots \ldots),\]

and

\[(\beta - 1)(k + 2) \geq 0, \quad (k = 1, 2, 3 \ldots \ldots),\]

which is true and the theorem is proved.

\textbf{Corollary 5.3.1.} \(V_h(\beta) \subset V_h(4/3) \subset V_h^*\).

\textbf{Corollary 5.3.2.} If \(f \in U_h(\beta)\) then \(f \in V_h^*\left(\frac{4 - 3\beta}{3 - 2\beta}\right)\).

\textbf{Corollary 5.3.3.} \(U_h(\beta) \subset U_h(4/3) \subset V_h\).

\textbf{Theorem 5.3.2.} If \(f \in U_h(\beta)\) then \(f \in V_h^*\left(\frac{2}{3 - \beta}\right)\).

\textbf{Proof.} The proof is similar to that of Theorem 5.3.1, so it is omitted.

\textbf{Corollary 5.3.4.} \(U_h(4/3) \subset V_h(6/5)\).

\textbf{Corollary 5.3.5.} \(U_h(4/3) \subset V_h^*(2/3)\).

\section*{5.4 A Family of Class Preserving Integral Operator}

Let \(f(z) = h(z) + \overline{g(z)}\) be defined by (5.1.1), then \(F(z)\) defined by the relation

\[F(z) = \frac{c + 1}{z^c} \int \overline{t^{c-1}h(t)}dt + \frac{c + 1}{z^c} \int t^{c-1}g(t)dt, \quad (c > -1).\]  \hfill (5.4.1)
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**Theorem 5.4.1.** Let \( f(z) = h(z) + g(z) \in S_H \) be given by (5.1.4) and \( f \in V_H(\beta) \) then \( F(z) \) be defined by (5.4.1) also belong to \( V_H(\beta) \).

**Proof.** Let \( f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| z^{-k} \) be in \( V_H(\beta) \), then by Theorem 5.2.3, we have

\[
\sum_{k=2}^{\infty} \frac{k-\beta}{\beta-1} |a_k| + \sum_{k=1}^{\infty} \frac{k+\beta}{\beta-1} |b_k| \leq 1.
\]

By definition of \( F(z) \), we have

\[
F(z) = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k} |a_k| z^k - \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k| z^{-k}.
\]

Now

\[
\sum_{k=2}^{\infty} \frac{k-\beta}{\beta-1} \left( \frac{c+1}{c+k} |a_k| \right) + \sum_{k=1}^{\infty} \frac{k+\beta}{\beta-1} \left( \frac{c+1}{c+k} |b_k| \right) \\
\leq \sum_{k=2}^{\infty} \frac{k-\beta}{\beta-1} |a_k| + \sum_{k=1}^{\infty} \frac{k+\beta}{\beta-1} |b_k| \\
\leq 1.
\]

Thus \( F(z) \in V_H(\beta) \).

**Theorem 5.4.2.** Let \( f(z) = h(z) + g(z) \in S_H \) be given by (5.1.5) and \( f \in U_H(\beta) \) then \( F(z) \) be defined by (5.4.1) also belong to \( U_H(\beta) \).

**Proof.** The proof of this theorem is much akin to that of Theorem 5.4.1. Therefore it is omitted.
Section 2

5.5 A function $f$ of the form (5.1.1) is harmonic starlike for $|z| = r < 1$, if

$$\frac{\partial}{\partial \theta} (\arg f(z)) = \Re \left\{ \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right\} > 0, \quad |z| = r < 1.$$  

Silverman [101] proved that the coefficient conditions $\sum_{k=2}^{\infty} k \left( |a_k| + |b_k| \right) \leq 1$ and $\sum_{k=2}^{\infty} k^2 \left( |a_k| + |b_k| \right) \leq 1$ are sufficient conditions for functions $f = h + g$ to be harmonic starlike and harmonic convex functions, respectively.

Denote by $V_H$ the subclass of $S_H$ consisting of functions of the form $f = h + g$, where

$$h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k, |h_1| < 1. \quad (5.5.1)$$

Recently, Yalcin et al. [118] studied the class $HP(\alpha)$, $(0 \leq \alpha < 1)$ the subclass of $S_H$ satisfying the condition

$$\Re \{ h'(z) + g'(z) \} > \alpha. \quad (5.5.2)$$

Further, let $V_H P(\alpha)$ be the subclass of $V_H$ consisting of functions of the form (5.5.1) which satisfy the condition (5.5.2).

Let $R_H (\beta)(1 < \beta \leq 2)$, denote the subclass of $V_H$ satisfying the condition

$$\Re \{ h'(z) + g'(z) \} < \beta. \quad (5.5.3)$$

We note that the class $R_H (\beta)$ reduces to the class $R(\beta)$ if co-analytic part of $f$ is zero i.e. $g \equiv 0$ studied by Uralegaddi et al. [115].
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We need the following Lemma due to Theorem 2.1 of [118].

**Lemma 5.5.1.** Let \( f = h + g \in V_H \) be given by (5.5.1) and

\[
\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq 1-\alpha, \quad (0 \leq \alpha < 1), \text{ then } f \in V_H P(\alpha).
\]

### 5.6 Main Results

**Theorem 5.6.1.** Let \( f = h + g \) be given by (5.5.1). Then \( f \in R_H (\beta) \), if and only if

\[
\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1. \tag{5.6.1}
\]

**Proof.** Let \( \sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1. \)

It suffices to prove that

\[
\frac{h'(z) + g'(z) - 1}{h'(z) + g'(z) - (2\beta - 1)} < 1, \quad z \in U.
\]

We have

\[
\left| \frac{h'(z) + g'(z) - 1}{h'(z) + g'(z) - (2\beta - 1)} \right| 
\]

\[
= \left| \frac{\sum_{k=2}^{\infty} k|a_k| z^{k-1} + \sum_{k=1}^{\infty} k|b_k| z^{k-1}}{\sum_{k=2}^{\infty} k|a_k| z^{k-1} + \sum_{k=1}^{\infty} k|b_k| z^{k-1} - 2(\beta - 1)} \right| 
\]

\[
\leq \frac{\sum_{k=2}^{\infty} k|a_k| z^{k-1} + \sum_{k=1}^{\infty} k|b_k| z^{k-1}}{2(\beta - 1) - \sum_{k=2}^{\infty} k|a_k| z^{k-1} - \sum_{k=1}^{\infty} k|b_k| z^{k-1}} 
\]

\[
\leq \frac{\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k|}{2(\beta - 1) - \sum_{k=2}^{\infty} k|a_k| - \sum_{k=1}^{\infty} k|b_k|}.
\]
which is bounded above by 1 if
\[
\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq 2(\beta - 1) - \sum_{k=2}^{\infty} k|a_k| - \sum_{k=1}^{\infty} k|b_k|
\]
or
\[
\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1.
\] (5.6.2)

But (5.6.2) is true by hypothesis so \( f \in R_h(\beta) \).

Conversely, suppose that \( \text{Re}\{h'(z) + g'(z)\} < \beta \)
\[
\text{Re}\left\{1 + \sum_{k=2}^{\infty} k|a_k|z^{k-1} + \sum_{k=1}^{\infty} k|b_k|z^{k-1}\right\} < \beta, \quad z \in U.
\]
The above condition must hold for all values of \( z, |z| = r < 1 \). Upon choosing the values of \( z \) to be real and let \( z \to 1^- \), we get
\[
1 + \sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta
\]
or
\[
\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1.
\]

Next, we determine bounds for the class \( R_h(\beta) \).

**Theorem 5.6.2.** If \( f \in R_h(\beta) \), then
\[
|f(z)| \leq (1 + |b_1|)r + \frac{1}{2}(\beta - 1 - |b_1|)r^2, |z| = r < 1
\]
and
\[
|f(z)| \geq (1 - |b_1|)r - \frac{1}{2}(\beta - 1 - |b_1|)r^2, |z| = r < 1.
\]
Proof. Let $f \in R_{\beta}(\beta)$. Taking the absolute value of $f$ we have

$$|f(z)| \leq (1 + |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^k$$

$$\leq (1 + |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^2$$

$$\leq (1 + |b_1|)r + \frac{1}{2} \sum_{k=2}^{\infty} k(|a_k| + |b_k|)r^2$$

$$\leq (1 + |b_1|)r + \frac{1}{2} (\beta - 1 - |b_1|)r^2,$$

and

$$|f(z)| \geq (1 - |b_1|)r - \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^k$$

$$\geq (1 - |b_1|)r - \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^2$$

$$\geq (1 - |b_1|)r - \frac{1}{2} \sum_{k=2}^{\infty} k(|a_k| + |b_k|)r^2$$

$$\geq (1 - |b_1|)r - \frac{1}{2} (\beta - 1 - |b_1|)r^2.$$

The functions

$$f(z) = z + |b_1|z + \frac{1}{2} (\beta - 1 - |b_1|)z^2$$

and

$$f(z) = z + |b_1|z + \frac{1}{2} (\beta - 1 - |b_1|)z^2,$$

for $|b_1| \leq \beta - 1$, show that the bounds given in Theorem 5.6.2 are sharp. □

The following result follows from the left hand inequality in Theorem 5.6.2.
Corollary 5.6.1. If \( f \in R_{H}(\beta) \), then
\[
\left\{ \omega : |\omega| < \frac{1}{2} (\beta - |\beta|) \right\} \subseteq f(U).
\]

Next, we determine the extreme points of the closed convex hulls of \( R_{H}(\beta) \), denoted by \( \text{clco } R_{H}(\beta) \).

Theorem 5.6.3. \( f \in \text{clco} R_{H}(\beta) \), if and only if
\[
f(z) = \sum_{k=1}^{\infty} \left( \lambda_{k} h_{k} + \gamma_{k} g_{k} \right)
\]
where \( h_{1}(z) = z \), \( h_{k}(z) = z + \frac{\beta-1}{k} z^{k} \), \( (k = 2, 3, 4, \ldots) \), \( g_{k}(z) = z + \frac{\beta-1}{k} z^{k} \), \( (k = 1, 2, 3, \ldots) \) and \( \sum_{k=1}^{\infty} (\lambda_{k} + \gamma_{k}) = 1 \), \( \lambda_{k} \geq 0 \) and \( \gamma_{k} \geq 0 \). In particular the extreme points of \( R_{H}(\beta) \) are \( \{h_{k}\} \) and \( \{g_{k}\} \).

Proof. For functions \( f \) of the form (5.6.3), we have
\[
f(z) = \sum_{k=1}^{\infty} \left( \lambda_{k} h_{k} + \gamma_{k} g_{k} \right)
= z + \sum_{k=2}^{\infty} \left( \frac{\beta-1}{k} \right) \lambda_{k} z^{k} + \sum_{k=1}^{\infty} \left( \frac{\beta-1}{k} \right) \gamma_{k} z^{k}.
\]
Then
\[
\sum_{k=2}^{\infty} \frac{k}{\beta-1} \left( \frac{\beta-1}{k} \right) \lambda_{k} + \sum_{k=1}^{\infty} \frac{k}{\beta-1} \left( \frac{\beta-1}{k} \right) \gamma_{k}
= \sum_{k=2}^{\infty} \lambda_{k} + \sum_{k=1}^{\infty} \gamma_{k}
= 1 - \lambda_{1} \leq 1,
\]
and so \( f \in \text{clco} R_{H}(\beta) \).
Conversely, suppose that \( f \in clcoR_H(\beta) \).

Set \( \lambda_k = \frac{k}{\beta-1}|a_k|, \ (k = 2, 3, 4,...) \) and \( \gamma_k = \frac{k}{\beta-1}|b_k|, \ (k = 1, 2, 3,...) \). Then note that by Theorem 5.6.1, \( 0 \leq \lambda_k \leq 1, \ (k = 2, 3, 4,...) \) and \( 0 \leq \gamma_k \leq 1, \ (k = 1, 2, 3,...) \). We define 

\[
\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \gamma_k
\]

and note that by Theorem 5.6.1, \( \lambda_1 \geq 0 \).

Consequently, we obtain 
\[
f(z) = \sum_{k=1}^{\infty} (\lambda_k h_k + \gamma_k g_k)
\]
as required. 

\( \Box \)

**Theorem 5.6.4.** If \( f \in R_H(\beta) \) then \( f \in V_H P(2 - \beta) \).

**Proof.** In view of Theorem 5.6.1 and Lemma 5.5.1, we must prove 

\[
\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k| \leq \beta - 1 \implies \sum_{k=2}^{\infty} \frac{k}{\beta-1}|a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta-1}|b_k| \leq 1 - (2 - \beta) = \beta - 1,
\]

which is obvious. 

Next, we give the interrelation between the class \( R_H(\beta) \) and \( S^*_H \), where \( S^*_H \) is the class of harmonic starlike function in \( U \).

**Theorem 5.6.5.** \( R_H(\beta) \subseteq S^*_H \), where \( 1 < \beta \leq 2 \).

**Proof.** Let \( f \in R_H(\beta) \).

Thus we have

\[
\sum_{k=2}^{\infty} \frac{k}{\beta-1}|a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta-1}|b_k| \leq 1.
\]  

(5.6.4)

Now 

\[
\sum_{k=2}^{\infty} k|a_k| + \sum_{k=1}^{\infty} k|b_k|
\]
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\[ \sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k| \]

\[ \leq 1. \quad \text{[Using (5.6.4)]} \]

Thus \( f \in S^{*}_{\mu} \).

This completes the proof of Theorem 5.6.5.

**Theorem 5.6.6.** Each function in the class \( R_{\mu}(\beta) \) maps a disks \( U \), where

\[ r < \inf \left\{ \frac{1}{k \left( \beta - 1 - |b_k| \right)} \right\}^{1/2} \]

onto convex domains for \( \beta > 1 + |b_k| \).

**Proof.** Let \( f \in R_{\mu}(\beta) \) and let \( r \), be fixed is that \( 0 < r < 1 \), then \( r^{-1} f(rz) \in R_{\mu}(\beta) \) and we have

\[ \sum_{k=2}^{\infty} k^2 (|a_k| + |b_k|) r^{k-1} = \sum_{k=2}^{\infty} k (|a_k| + |b_k|) (kr^{k-1}) \]

\[ \leq \sum_{k=2}^{\infty} k (|a_k| + |b_k|) \]

\[ \leq \beta - 1 - |b_k| \leq 1, \]

provided

\[ kr^{k-1} \leq \frac{1}{\beta - 1 - |b_k|} \]

or

\[ r < \inf \left\{ \frac{1}{k \left( \beta - 1 - |b_k| \right)} \right\}^{1/2}. \]

For our next theorem, we need to define the convolution of two harmonic functions.

For harmonic functions of the form

\[ f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \bar{z}^k \]
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and

\[ F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| z^k \]

we define the convolution of two harmonic functions \( f \) and \( F \) as

\[ (f * F)(z) = f(z) * F(z) = z + \sum_{k=2}^{\infty} |a_k A_k| z^k + \sum_{k=1}^{\infty} |b_k B_k| z^k. \quad (5.6.5) \]

Using this definition, we show that the class \( R_H(\beta) \) is closed under convolution.

**Theorem 5.6.7.** For \( 1 < \beta \leq \alpha \leq 2 \), let \( f \in R_H(\alpha) \), and \( F \in R_H(\beta) \). Then

\[ f * F \in R_H(\beta) \subseteq R_H(\alpha). \]

**Proof.** Let \( f(z) = z + \sum_{k=2}^{\infty} |a_k A_k| z^k + \sum_{k=1}^{\infty} |b_k B_k| z^k \) be in \( R_H(\beta) \)

and

\[ F(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + \sum_{k=1}^{\infty} |B_k| z^k \text{ be in } R_H(\alpha). \]

Then the convolution \( f * F \) is given by (5.6.5). We wish to show that the coefficients of \( f * F \) satisfy the required condition given in Theorem 5.6.1. For \( F(z) \in R_H(\alpha) \), we note that \( |A_k| \leq 1 \) and \( |B_k| \leq 1 \). Now, for the convolution function \( f * F \), we obtain

\[ \sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k A_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k B_k| \]

\[ \leq \sum_{k=2}^{\infty} \frac{k}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} |b_k| \]

\[ \leq 1. \quad (\text{Since } f \in R_H(\beta)) \]

Therefore \( f * F \in R_H(\beta) \subseteq R_H(\alpha) \).

Next, we show that \( R_H(\beta) \) is closed under convex combinations of its members.
Theorem 5.6.8. The class $R_H(\beta)$ is closed under convex combination.

Proof For $i = 1, 2, 3, \ldots$, let $f_i(z) \in R_H(\beta)$, where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{k=2}^{\infty} a_k |z|^k + \sum_{k=1}^{\infty} b_k |z|^k.$$

Then by Theorem 5.6.1, we have

$$\sum_{k=2}^{\infty} \frac{k}{\beta-1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta-1} |b_k| \leq 1.$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of $f_i$ may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{i=1}^{\infty} \left( \sum_{k=2}^{\infty} t_i a_k |z|^k + \sum_{k=1}^{\infty} t_i b_k |z|^k \right).$$

Then by Theorem 5.6.1, we have

$$\sum_{i=1}^{\infty} t_i \left( \sum_{k=2}^{\infty} \frac{k}{\beta-1} |a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta-1} |b_k| \right) \leq \sum_{i=1}^{\infty} t_i = 1.$$

Therefore $\sum_{i=1}^{\infty} t_i f_i(z) \in R_H(\beta)$.

The $\delta$-neighborhood of $f$ is the set

$$N_\delta(f) = \left\{ F : F(z) = z + \sum_{k=2}^{\infty} A_k |z|^k + \sum_{k=1}^{\infty} B_k |z|^k \text{ and } k \left( |a_k - A_k| + |b_k - B_k| \right) \leq \delta \right\}.$$

See ([8], [89]).
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Theorem 5.6.9. Let \( f \in R_{\beta}(\beta) \) and \( \delta \leq 2 - \beta \). If \( F \in N_{\delta}(f) \), then \( F \) is harmonic starlike function.

Proof. Let \( F(z) = z + \sum_{k=1}^{\infty} |A_k|z^k + \sum_{k=1}^{\infty} |B_k|z^k \) belongs to \( N_{\delta}(f) \). We have

\[
\sum_{k=2}^{\infty} k|A_k| + \sum_{k=1}^{\infty} k|B_k| \leq \sum_{k=2}^{\infty} k(|a_k| + |b_k|) + \sum_{k=2}^{\infty} k(|a_k| + |b_k|) + |h_1 - B_1| + |h_1| \\
\leq \delta + \beta - 1 \leq 1.
\]

Hence, \( F(z) \) is harmonic starlike function.

Next, we discuss a class preserving integral operator for this class.

Let \( f(z) = h(z) + g(z) \) be defined by (5.1.1) then \( F(z) \) defined by the relation

\[
F(z) = \frac{c+1}{z^c} \int t^{-1}h(t)dt + \frac{c+1}{z^c} \int t^{-1}g(t)dt, \quad (c > -1).
\] (5.6.6)

Theorem 5.6.10. Let \( f(z) = h(z) + g(z) \in S_{\beta} \) be given by (5.5.1) and \( f(z) \in R_{\beta}(\beta) \) then \( F(z) \) be defined by (5.6.6) also belong to \( R_{\beta}(\beta) \).

Proof. Let \( f(z) = z + \sum_{k=1}^{\infty} |a_k|z^k + \sum_{k=1}^{\infty} |b_k|z^k \) be in \( R_{\beta}(\beta) \) then by Theorem 5.6.1, we have

\[
\sum_{k=2}^{\infty} \frac{k}{\beta - 1}|a_k| + \sum_{k=1}^{\infty} \frac{k}{\beta - 1}|b_k| \leq 1.
\] (5.6.7)

By definition of \( F(z) \), we have

\[
F(z) = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k}|a_k|z^k + \sum_{k=1}^{\infty} \frac{c+1}{c+k}|b_k|z^k.
\]

Now

\[
\sum_{k=2}^{\infty} \frac{k}{\beta - 1} \left( \frac{c+1}{c+k} |a_k| \right) + \sum_{k=1}^{\infty} \frac{k}{\beta - 1} \left( \frac{c+1}{c+k} |b_k| \right)
\]
Thus $F(z) \in R_{\beta}(\beta)$. \hfill \Box