Appendix
APPLICATION OF H-FUNCTION FOR OBTAINING AN ANALYTIC SOLUTION OF THE SPACE-AND-TIME FRACTIONAL INITIAL VALUE DIFFUSION PROBLEM

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ABSTRACT

In the present paper, we make an application of Fox's $H$-function to derive an analytic solution of the space-and-time fractional initial value diffusion problem defined in a bounded space domain $x \in (a, b)$. In this process, the Adomian decomposition techniques and the given initial conditions are taken into account. We also make some extensions of above solution through Lie group techniques.

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Keywords. Space-and-time fractional initial value diffusion equation, decomposition techniques, Fox's $H$-function, Lie group techniques.

1. Introduction. Nowadays connection between walk and fractional order dynamics is well known, see for instance, Adler et al. [1], Andries et al. [7] and Metzler and Klafter [24]. A number of constructive random walk models governed by fractional order differential equations in the one dimensional case were studied by Gills et al. [15], Chechkin et al. ([10], [11]), Gorenflo et al. ([16]-[18]). In these studies the governing equation depends on the parameters $\alpha \in (0,1)$ and $\beta \in (0,2]$.
and is given by the fractional order differential equation
\[ D_t^\alpha u(x,t) = D_x^\beta u(x,t), t > 0, x \in R \text{ (set of real numbers)} \tag{1.1} \]

Here \( D_t^\alpha u = (D_t^\alpha u)(t) \) and \( D_x^\beta u = (D_x^\beta u)(x), x \in R, t > 0 \) are the time-fractional derivative and the space-fractional derivative respectively defined in the Caputo sense ([8],[9]) in the equation (2.7) (see also, Kilbas, Srivastava and Trujillo [19]).

Oldham and Spanier [30] and Oldham [29] have studied some diffusion equations that contain first order derivative in space and half order derivative in time and apply them to analyze the electrochemical problems.

If we consider ordinary space and time derivatives in the equation (1.1) and then here set \( 1 \leq \alpha \) and \( \beta \leq 2 \), it becomes Cauchy problem due to Fujita [12] in which he has presented the existence and uniqueness of its solution. Further in ([13]-[14]) he has considered integro differential equations which exhibit heat diffusion and wave propagation properties. Mainardi [20-23] has presented an analytical investigation of the time-fractional diffusion and wave equations in which he has also provided a comprehensive review of research on the application of calculus in continuum and statistical mechanics including research on fractional diffusion-wave solutions. He has used Laplace transformation method and obtained the fundamental solutions of the basic Cauchy signaling problems and expressed them in terms of entire functions of Wright type [31].

Agrawal [4] has presented a general solution for a time-fractional diffusion-wave equation defined in a bounded space domain. He has applied finite sine transformation method to convert fractional diffusion-wave equation from a space domain to a wave number domain, then the Laplace transformation is used to reduce the resulting equation to an ordinary algebraic equation, finally, the inverse Laplace and the inverse sine transforms are used to obtain the desired solution. Again, in [5], he has used same techniques to obtain a general solution for a fourth-order fractional diffusion-wave problems.

Al-Khaled and Momani [6] have used the decomposition techniques to obtain an approximate solution for the generalized time-fractional diffusion-wave equations. Their results showed the transition from a pure diffusion process (For \( \alpha = 1 \)) to a pure wave process (For \( \beta = 2 \)).

In the present paper, we consider following space-and-time fractional diffusion equation.
\[ D_t^\alpha u(x,t) = D_x^\beta u(x,t), t > 0, x \in (a,b), \alpha \in (0,1) \text{ and } \beta \in (0,2). \tag{1.2} \]

Here \( D_t^\alpha u = (D_t^\alpha u)(t) \) and \( D_x^\beta u = (D_x^\beta u)(x), x \in R, t > 0 \) are the time-fractional
derivative and the space-fractional derivative respectively defined in the Caputo sense ([8],[9]) in the equation (2.7) (See also, Kilbas, Srivastava and Trujillo [19,p.90, section (2.4)]).

The initial condition is given by

\[ u(x,0) = f(x), x \in (a,b) . \]  

To solve the space-and-time fractional diffusion equation (1.2) with the initial condition \( H \)-function [31] defined by the Mellin-Barnes type contour integral formula

\[
H^{P,Q}_{R,S}(z) = H^{P,Q}_{R,S} \left( \sigma, A \bigg| \sigma, B \right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{P} \Gamma(\sigma_j - B_j s) \prod_{j=1}^{Q} \Gamma(1 - \rho_j + A_j s)}{\prod_{j=Q+1}^{R} \Gamma(1 - \sigma_j + B_j s) \prod_{j=Q+1}^{P} \Gamma(\rho_j - A_j s)} Z^s ds
\]  

(1.4)

The integral (1.4) converges absolutely if

\[
\Delta = \sum_{j=1}^{Q} A_j - \sum_{j=Q+1}^{R} A_j + \sum_{j=1}^{P} B_j - \sum_{j=P+1}^{S} B_j > 0
\]  

(1.5)

where, all \( A_j, 1 \leq j \leq Q \leq R \) and \( B_j, 1 \leq j \leq P \leq S \) are positive real.

The \( H \)-function is analytic in the sector \( |\arg(z)| < 1/2 \Delta \pi \) and the point \( z = 0 \) is tacitly excluded.

The gamma function \( \Gamma(z) \) is defined by

\[
\Gamma(\lambda) = \frac{\Gamma(\lambda + n)}{(\lambda)_n}, \text{Re}(\lambda) > -n; n \in N; \lambda \notin Z_0 := \{0,-1,-2,...\}
\]  

(1.7)

Here, the Pochhammer symbol \((\lambda)_n\) is given by

\[
(\lambda)_0 = 1 \text{ and } (\lambda)_n = \lambda(\lambda + 1)(\lambda + 2)...(\lambda + n - 1), n \in N.
\]  

(1.8)

Again, through Lie group techniques [25]-[27], we make the extensions of the obtained solution (4.2) of the space-and-time fractional diffusion equation (1.2) and with the initial condition (1.3).

2. Useful Formulae and the Properies. In this section, we present some formulae and properties of fractional integrals and fractional derivatives which are useful in our investigation (See, Kilbas, Srivastava and Trujillo [19]):
Riemann- Liouville Fractional Integral and Fractional Derivative.

Let \( \Omega = [a, b] \{ -\infty < a < b < \infty \} \) be a finite interval on the real axis \( \mathbb{R} \). The Riemann-Liouville fractional integral \( I_a^\alpha f \) of order \( \alpha \in C(\text{Re}(\alpha) > 0) \) is defined by

\[
(I_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad x > a, \text{Re}(\alpha) > 0
\]  

(2.1)

The Riemann-Liouville fractional derivative \( D_a^\alpha f \) of order \( \alpha \in C(\text{Re}(\alpha) \geq 0) \) is defined by

\[
(D_a^\alpha f)(x) = \left( \frac{d}{dx} \right)^n (I_a^{n-\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{ds} \right)^n \int_a^x f(t) dt \left( x-t \right)^{\alpha-n+1}, \quad x > a, n = \{\text{Re}(\alpha)\} + 1
\]

(2.2)

Property 1. If \( \text{Re}(\alpha) \geq 0 \) and \( \beta \in C(\text{Re}(\beta) > 0) \), then

\[
(I_a^\alpha (t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\beta+\alpha-1}, \text{Re}(\alpha) \geq 0.
\]

(2.3)

and

\[
(D_a^\alpha (t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (x-a)^{\beta-\alpha-1}, \text{Re}(\alpha) \geq 0.
\]

(2.4)

In particular, if \( \beta = 1 \) in (2.4) and \( \text{Re}(\alpha) \geq 0 \), then, the Riemann-Liouville fractional derivative \( D_a^\alpha f \) of a constant is given by

\[
(D_a^\alpha 1)(x) = \frac{1}{\Gamma(1-\alpha)} (x-a)^{-\alpha}.
\]

(2.5)

On the other hand, for \( j = 1, 2, ..., \{\text{Re}(\alpha)\} + 1 \)

\[
(D_a^\alpha (t-a)^{\alpha-j})(x) = 0.
\]

(2.6)

The connection between Caputo fractional derivative and the Riemann-Liouville fractional derivative is given by

\[
(D_a^\alpha f)(x) = \left( D_a^\alpha \left[ f(t) - \sum_{v=0}^{n-1} \frac{f^{(v)}(a)}{v!}(t-a)^v \right] \right)(x), \text{Re}(\alpha) \geq 0, n = \{\text{Re}(\alpha)\} + 1.
\]

(2.7)
Property 2. If \(\text{Re}(\alpha) > 0\) and \(\text{Re}(\beta) > 0\), then
\[
\left(D_{a}^{\alpha}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-1}, \text{Re}(\beta) > \alpha.
\] (2.8)

Lemma. Let \(\text{Re}(\alpha) > 0\) and let \(f(x) \in L, (a,b)\) and if \(\text{Re}(\alpha) \in N\), then
\[
(D_{a}^{\alpha}I_{a}^{\alpha}f)(x) = f(x)
\] (2.9)
and
\[
(I_{a}^{\alpha}D_{a}^{\alpha}f)(x) = f(t) - \sum_{v=0}^{n-1} \frac{f^{(v)}(a)}{v!}(t-a)^{v}, \text{Re}(\alpha) > 0, n = \{\text{Re}(\alpha)\} + 1.
\] (2.10)

3. Solution of the Problem and Analysis. In this section we solve our problem stated in (1.2) and (1.3) through making an application of the definitions and properties given in the section-2 and Adomian decomposition techniques ([2],[3]):

With the help of (2.7), the space-and-time fractional diffusion equation (1.2), with the initial condition (1.3), may be written as
\[
u(x,t) = \sum_{v=0}^{n-1} \frac{u^{(v)}(0)}{v!}(t) = (I_{t,0}^{\alpha}D_{x,0}^{\beta}u(x,t))
\] (3.1)
where \(I_{t,0}^{\alpha}f \equiv (I_{a}^{\alpha}f)(t)\) is the Riemann-Liouville fractional integral defined in (2.1).
The initial condition is given by
\[
u(x,0) = f(x).
\] (3.2)

Now, we consider the recursive formula in the form
\[
u_{m}(x,t) = u_{0}(x,t) + (I_{t,0}^{\alpha}D_{x,0}^{\beta}u_{m-1}(x,t))
\] (3.3)
where, \(u_{0}(x,t)\) is the initial function.

For obtaining the approximation solution of our problem from (3.1) we also write
\[
u_{m}(x,t) = \sum_{v=0}^{n-1} \frac{u^{(v)}(0)}{v!}(t) + (I_{t,0}^{\alpha}D_{x,0}^{\beta}u_{m-1}(x,t))
\] (3.4)

On comparing (3.3) and (3.4), we find
\[
u_{0}(x,t) = \sum_{v=0}^{n-1} \frac{u^{(v)}(0)}{v!}(t)
\] (3.5)
From (3.3), it is given that \( u_0(x,t) \) is the initial function and again with the help of initial condition given in (3.2) this function is the function of \( x \) only, hence due to Adomian decomposition techniques ([2],[3]) applied in (3.5), we get \( n=1 \) that is \( \nu=0 \).

Therefore, from (3.2) and (3.5), we have

\[
    u_0(x,t) = f(x) .
\]

(3.6)

Finally, we get the successive approximation formula in the form

\[
    u_m(x,t) = f(x) + \left( I_{1,0}^\alpha D_x^\nu u_{m-1}(x,t) \right) .
\]

(3.7)

Hence, with the help of (3.6) and (3.7) and then using (2.3), we find that

\[
    u_1(x,t) = f(x) + \left( \frac{t^\alpha}{\Gamma(\alpha+1)} D_x^\nu f(x) \right) .
\]

(3.8)

Similarly, we get

\[
    u_2(x,t) = \sum_{k=0}^2 \left( \frac{t^{ku}}{\Gamma(k\alpha+1)} D_x^{ku} f(x) \right) .
\]

(3.9)

\[
    u_m(x,t) = \sum_{k=0}^m \left( \frac{t^{ku}}{\Gamma(k\alpha+1)} D_x^{ku} f(x) \right) .
\]

(3.10)

Then, take \( m \to \infty \) in (3.10), finally, we find the general solution of our problem such that

\[
    u(x,t) = \sum_{k=0}^\infty \left( \frac{t^{ku}}{\Gamma(k\alpha+1)} D_x^{ku} f(x) \right) .
\]

(3.11)

Particularly, if we put \( a=0 \) in (3.11), it becomes the general solution for the space-and-time fractional diffusion equation already obtained due to Momani [28].

### 4. Application of Fox’s H-function for Obtaining Analytic Solution.

To find out an analytic solution involving various parameters of our problem stated in (1.2) and (1.3), we consider the initial function as the Fox’s H-function such that

\[
    f(x) = H_{\nu,S}^{p,q} \left( \frac{A}{\sigma,B} \right) (x-a)^\mu
\]

provided that \( h > 0, a \geq 0, \text{Re}(\mu) > 0, x \in (a,b) \) and all conditions given in (1.5) and (1.6) are followed.

Now making an appeal to (2.8), (3.11) and (4.1), we obtain an analytic solution of
our problem in the form

$$u(x,t) = \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} H_{R+1,S+1}^{p,Q+1} \left[ \left( \frac{\rho}{\sigma}, A \right), \left( 0, \mu \right) \middle| h(x-a)^\mu \right]$$  \hspace{1cm} (4.2)

provided that $h > 0, \alpha \geq 0, \Re(\mu) > 0, x \in (a,b), t > 0$ and all conditions given in (1.5) and (1.6) are followed.

5. Special Cases. The Fox's $H$-function is the analytic function and is the generalization of various hypergeometric functions given in the literature, therefore, on making some manipulations in the parameters of $H$-function in (4.2), we may obtain various solution of our problem as particular cases. For example, put $\mu = 1, P = 1, Q = 0, R = 0, S = 1, \sigma = 0$ and $B = 1$ in (4.2), we get

$$u(x,t) = \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} (-h(x-a))^{-\beta} E_{1,1;k+1}^{\beta} (-h(x-a))$$  \hspace{1cm} (5.1)

where, $E_{u,p}(z)$ is the Mittag Leffler function studied in the literature for example see Srivastava and Manocha [31], for $x \in (a,b), t > 0$ and all conditions given in (4.2) are followed.

6. Extensions. In this section, we make some extensions of our obtained solution (4.2) through applying some Lie group techniques of Miller [25]-[27]. So that we set the characteristic function related to $u(x,t)$ in the form

$$u^{\rho_j,\sigma_p}(x,t,y_j,z_p) = u(\rho_j, \sigma_p, x, t) y_j^{\rho_j} z_p^{\sigma_p} \hspace{1cm} (1 \leq j \leq Q, 1 \leq p \leq P)$$

$$= \sum_{k=0}^{\infty} \left( \frac{t^{k\alpha} y_1^{\rho_1} \cdots y_Q^{\rho_Q} z_1^{\sigma_1} \cdots z_P^{\sigma_P}}{\Gamma(k\alpha + 1)} \right) H_{R+1,S+1}^{p,Q+1} \left[ \left( \frac{\rho}{\sigma}, A \right), \left( 0, \mu \right) \middle| h(x-a)^\mu \right]$$  \hspace{1cm} (6.1)

provided that $h > 0, \alpha \geq 0, \Re(\mu) > 0, x \in (a,b), t > 0$ and all conditions given in (1.5) and (1.6) are followed.

Then, we introduce some Lie group operators such that

$$H_j = y_j^{-1} \left( \frac{(x-a)}{\mu} \frac{\partial}{\partial x} - \frac{y_j}{A_j} \frac{\partial}{\partial y_j} + \frac{1}{A_j} \right), 1 \leq j \leq Q, \hspace{1cm} (6.2)$$

and

$$L_p = z_p \left( \frac{(x-a)}{\mu} \frac{\partial}{\partial x} - \frac{z_p}{B_p} \frac{\partial}{\partial z_p} \right), 1 \leq p \leq P. \hspace{1cm} (6.3)$$
Thus on applying some Lie group techniques of Miller [25]-[27] we evaluate the actions of Lie group operators ((6.2) and (6.3) on the characteristic function (6.1) such that

$$H_j u^{\rho_j, \sigma_p}(x, t, y_j, z_p) = u^{\rho_j, \sigma_p}(x, t, y_j, z_p), 1 \leq j \leq Q,$$

(6.4)

and

$$L_p u^{\rho_j, \sigma_p}(x, t, y_j, z_p) = u^{\rho_j, \sigma_p + 1}(x, t, y_j, z_p), 1 \leq p \leq P,$$

(6.5)

respectively.

Then, on making an appeal to the multiplier representations for one parameter subgroups $e^{jH_j}, 1 \leq j \leq Q$ and $e^{\lambda_p}, 1 \leq p \leq P$, and the group of actions given in (6.4) and (6.5), we obtain following extensions

$$\left(1 - \frac{\lambda}{A_j y_j}\right)^{-1} u^{\rho_j, \sigma_p} \left\{ \left(1 - \frac{\lambda}{A_j y_j}\right)^{-A_j/\mu} + a \right\}, t, \left(1 - \frac{\lambda}{A_j y_j}\right) z_p,$$

(6.6)

and

$$u^{\rho_j, \sigma_p} \left\{ \left(1 + \frac{\xi z_p}{B_p}\right) + a \right\}, t, y_j, z_p \left(1 + \frac{\xi z_p}{B_p}\right),$$

$$= \sum_{s=0}^{m} \frac{(-\xi)^s}{s!(B_p)^s} u^{\rho_j, \sigma_p + s}(x, t, y_j, z_p), 1 \leq p \leq P,$$

(6.7)

respectively, and provided that $h > 0, a \geq 0, \text{Re}(\mu) > 0, x \in (a, b), t > 0$ and all conditions given in (1.5) and (1.6) are followed.

Finally, making the successive actions of above subgroups $e^{jH_j}, 1 \leq j \leq Q$, and $e^{\xi_p}, 1 \leq p \leq P$, to the characteristic function (6.1), we derive following extension formula

$$\left(1 - \frac{\lambda}{A_j y_j}\right)^{-1} u^{\rho_j, \sigma_p} \left\{ \left(1 - \frac{\lambda}{A_j y_j}\right)^{-A_j/\mu} + a \right\}, t, \left(1 - \frac{\lambda}{A_j y_j}\right) z_p \left(1 + \frac{\xi z_p}{B_p}\right)^{-1},$$

where

$$h > 0, a \geq 0, \text{Re}(\mu) > 0, x \in (a, b), t > 0$$

and all conditions given in (1.5) and (1.6) are followed.
provided that \( h > 0, a \geq 0, \text{Re}(\mu) > 0, x \in (a, b), t > 0 \) and all conditions given in (1.5) and (1.6) are followed.

Particularly, For \( k = 0, a = 0, \mu = 1 \) and \( \forall A_j = 1, 1 \leq j \leq S \) and \( \forall B_p = 1, 1 \leq p \leq R \), (6.8) gives various results for generalized functions and \( G \)-function studied by [26] and [27].

7. Physical Interpretations. The initial profile involves various parameters, on specialization it transforms to many hypergeometric structures occurring in the literature in a bounded space-domain \( x \in (a, b) \). The solution interprets the images between linear (when, \( \alpha = 1 \) and \( \beta = 1 \)) and heat conduction problems (when, \( \alpha = 1 \) and \( \beta = 2 \)) studied by various researchers.

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APPLICATION OF TWO-VERIALBE H-FUNCTION FOR OBTAINING
ANALYTIC SEQUENCE OF SOLUTIONS OF THREE-VARIABLE
SPACE-AND-TIME FRACTIONAL DIFFUSION PROBLEM

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Abstract
In the present paper, we make an application of two-variable H-function to derive analytic sequence of solutions of the three-variable space-and-time fractional diffusion problem defined in the bounded space-domains \( x \in (a, b) \) and \( y \in (c, d) \). In this process, the Adomian decomposition techniques are used and for that the given initial condition and the order of time derivative are taken into account to obtain the prescribed solution of the problem.

Keywords: Space-and-time fractional initial value diffusion-wave equation, Caputo fractional derivatives, Adomian decomposition techniques, two-variable H-function

2010 Mathematics Subject Classifications: 35S05, 35S10, 35G20, 26A33, 33C60, 22E60.

1. Introduction
Agrawal (\cite{2} and \cite{3}) has presented a general solution for a time fractional diffusion-wave equation defined in the bounded space domain through application of sine transform and together with Laplace transform Al-Khaled and Momani \cite{4} have used the decomposition techniques to obtain an approximate solution for the generalized time-fractional diffusion-wave equations. Their results showed the transition from a pure diffusion process to a pure wave process.

Momani \cite{13} has derived the general solution for two variable space-and-time fractional diffusion-wave equation on applying Adomian decomposition techniques \cite{1} that equation is given by

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\[ D_t^\alpha u(x,t) = b^2 D_x^{2\beta} u(x,t), t > 0, 0 < \alpha \leq 2, 1 < \beta \leq 2 \] (1)

Subject to the boundary and initial conditions

\[ u(0,t) = h_1(t), u(l,t) = h_2(t), t \geq 0, \] (2)

\[ u(x,0) = f(x), 0 < x < l, \text{ for } 0 < \alpha \leq 1. \] (3)

\[ \frac{\partial u(x,0)}{\partial t} = g(x), 0 < x < l, \text{ for } 1 < \alpha \leq 2. \] (4)

where \( \alpha \) and \( \beta \) are parameters describing the order of time and space fractional derivatives, respectively, \( b \) denotes a constant coefficient, and \( u(x,t) \) is the field defined in the space domain \((0,l)\). The above terminology given in (1)-(4) has already been introduced by Mainardi ([9], [12]). Here, the equation (1) represents as to the space-and time-fractional diffusion and to the space and time fractional wave equation in the case \( \{0 < \alpha \leq 1, 1 < \beta \leq 2\} \) and \( \{1 < \alpha \leq 2, 1 < \beta \leq 2\} \), respectively.

Later on, Kumar, Pathan and Srivastava [8] have evaluated on analytic solution of two variable space and time fractional initial value problem on making an application of Fox's H-function (14) and (15).

In our present investigation, we introduce following three-variable space and time fractional diffusion problem.

\[ D_{t,0}^{\alpha,\gamma} u(x,y,t) = \left( D_{x,0}^{\alpha,\gamma} + D_{y,0}^{\alpha,\gamma} \right) u(x,y,t), t > 0 \]

where \( D_{t,0}^{\alpha,\gamma} u = \left( D_{0}^{\alpha,\gamma} u \right)(t) \) is time fractional derivative, \( t > 0 \) and \( D_{x,0}^{\alpha,\gamma} u = \left( D_{0}^{\alpha,\gamma} u \right)(x) \), \( D_{y,0}^{\alpha,\gamma} u = \left( D_{0}^{\alpha,\gamma} u \right)(y) \), \( x \in (a,b), y \in (c,d), \alpha \in (0,1] \) and \( \beta \in (1,2], \gamma \in (1,2] \) (5)

The initial condition is given by

\[ u(x,y,0) = f(x,y), x \in (a,b), y \in (c,d), 0 < \alpha \leq \gamma \leq 1 \text{ and } f \in R^2 \] (6)

and the boundary conditions are given by

\[ u(a,c,t) = u(b,d,t) = 0, t \geq 0 \] (7)

We solve above space and time fractional diffusion equation (5) with initial condition (6) and the boundary conditions given in (7) on introducing two-variable H-function [14] defined by

\[ H[x,y] = H \begin{bmatrix} x \\ y \end{bmatrix} \]
Application of two variable diffusion problem

\[ H_{0,n_1,m_2,n_2,m_3} = H_{P_1,P_2,P_3} \begin{bmatrix} \{a_j; \alpha_j, A_j\} & \{b_j; \beta_j, B_j\} \\ \{c_j; \gamma_j, C_j\} & \{d_j; \delta_j, D_j\} \end{bmatrix} \]

\[= \frac{1}{4\pi^2} \left\{ \phi_1(\xi, \eta) \phi_2(\xi) \phi_3(\eta) \right\} x^\xi y^\eta d\xi d\eta \]

Where

\[ \phi_1(\xi, \eta) = \prod_{j=1}^{n_1} \Gamma\left(1 - a_j + \alpha_j \xi + A_j \eta\right) \prod_{j=1}^{m_1} \Gamma\left(1 + a_j - \alpha_j \xi - A_j \eta\right) \]

\[ \phi_2(\xi) = \prod_{j=1}^{n_2} \Gamma\left(1 - c_j + \gamma_j \xi\right) \prod_{j=1}^{m_2} \Gamma\left(1 + c_j - \gamma_j \xi\right) \]

\[ \phi_3(\eta) = \prod_{j=1}^{n_3} \Gamma\left(1 - f_j + F_j \eta\right) \prod_{j=1}^{m_3} \Gamma\left(1 + f_j - F_j \eta\right) \]

Here, \( x \) and \( y \) are not equal to zero and an empty product is interpreted unity; the integers \( P_1, P_2, P_3, Q_1, Q_2, Q_3; N_1, N_2, N_3; M_2, M_3 \) are non-negative integers such that \( 0 \leq N_1 \leq P_1, 0 \leq N_2 \leq P_2, 0 \leq N_3 \leq P_3, 0 \leq Q_1, Q_2, Q_3 \) also all the \( A's, A's, B's, \gamma's, \delta's, E's \) and \( f's \) are assumed to be positive quantities for standardization purpose; the definition of the \( H \)-function of two variable given by (8)-(11), will, however, have a meaning even if some of these quantities are zero, the contour \( L_1 \) is in the \( \xi \)-plane and runs from \( -\infty \) to \( +\infty \) with loops, if necessary, to ensure that the poles of \( \Gamma\left(1 - c_j + \gamma_j \xi\right) \) \((j = 1, \ldots, m_2)\) lie to the right or the poles of \( \Gamma\left(1 - f_j + F_j \eta\right) \) \((j = 1, \ldots, m_3)\) to the left of the contour.

The contours \( L_2 \) is in the \( \eta \)-plane and runs from \( -\infty \) to \( +\infty \) with loops, if necessary, to ensure that the poles of \( \Gamma\left(1 + f_j - F_j \eta\right) \) \((j = 1, \ldots, m_3)\) lie to the right or the poles of \( \Gamma\left(1 - e_j + E_j \eta\right) \) \((j = 1, \ldots, n_3)\) and \( \Gamma\left(1 - e_j + E_j \eta\right) \) \((j = 1, \ldots, n_1)\) to the left of the contour. All poles are simple. The function \( H\left[\begin{array}{c} x \\ y \end{array}\right] \) is an analytic function of \( x \) and \( y \), if
The double integral defined by (8)-(11) converges absolutely under the conditions given by

\[ \left| \arg(x) \right| < -\frac{1}{2} \pi \quad \text{and} \quad \left| \arg(y) \right| < -\frac{1}{2} \pi \]

The gamma function \( \Gamma(\lambda) \) is defined by

\[ \Gamma(\lambda) = \frac{\Gamma(\lambda + n)}{(\lambda)_n}, \quad \Re(\lambda) > -n; n \in \mathbb{N}; \lambda \notin \mathbb{Z}_0 = \{0, -1, -2, \ldots\} \]

Here, the Pochhammer symbol \((\lambda)_n\) is given by

\[ (\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_n = \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1), n \in \mathbb{N} \]

2. Useful Formulae and the Properties

In this section, we present some formulae and the properties of fractional integrals and fractional derivatives which are useful in our investigation (see, Kilbas, Srivastava and Trujillo[7]).

For \([a, b]\) \((-\infty < a < b < \infty)\) be a finite interval and let \(AC[a, b]\) be the space of function \(f\) which are absolutely continuous on \([a, b]\) and the space \(AC[a, b]\) coincides with the space of primitives of Lebesgue summable functions, then

\[ f(x) \in AC[a, b] \Leftrightarrow f(x) = c + \int_a^x \phi(t) \, dt, (\phi(t) \in L(a, b)) \]
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and therefore an absolutely continuous function \( f(x) \) has a summable derivative \( f'(x) = \phi(t) \) almost everywhere on \([a,b]\). Thus from (16) there be
\[
\phi(t) = f'(t) \quad \text{and} \quad c = f(a)
\] (17)

For \( n \in N = \{1,2,3,\ldots\} \), \( AC'[a,b] \) be the space of complex-valued functions \( f(x) \) which have continuous derivative up to order \( m-1 \) on \([a,b]\) such that \( f^{m-1}(x) \in AC[a,b] \), then
\[
AC'[a,b] = \left\{ f[a,b] \to C \text{ and } \left(D^{m-1}f\right) \in AC[a,b] \right\} \quad \left(D = \frac{d}{dx}\right)
\] (18)

C being the set of complex numbers. In particular, \( AC'[a,b] = AC[a,b] \)

Let \( \Omega = [a,b] \} \} \to <a < b < \infty \} \) be a finite interval on the real axis \( R \). The Riemann-Liouville fractional integral \( I^\alpha_\alpha f \) of order \( \alpha \in C(Re(\alpha) > 0) \) is defined by, \( f \in AC[a,b] \), then
\[
(I^\alpha_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, x > a, Re(\alpha) > 0
\] (19)

The Riemann-Liouville fractional derivative \( D^\alpha_\alpha f \) of order \( \alpha \in C(Re(\alpha) \geq 0) \) is defined by
\[
(D^\alpha_\alpha f)(x) = \left(\frac{d}{dx}\right)^n \left(I^{n-\alpha}_\alpha f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)dt}{(x-t)^{n-\alpha}}, x > a, n = \{Re(\alpha)\} + 1.
\] (20)

Property 1: If \( Re(\alpha) \geq 0 \) and \( \beta \in C(Re(\beta) > 0) \), then
\[
(I^\alpha_\alpha (t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} (x-a)\alpha + \beta - 1, Re(\alpha) > 0.
\] (21)

and
\[
(D^\alpha_\alpha (t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (x-a)^{\beta - \alpha - 1}, Re(\alpha) \geq 0.
\] (22)

In particular, if \( \beta = 1 \) in (22) and \( Re(\alpha) \geq 0 \), then the Riemann-Liouville fractional derivative \( D^\alpha_\alpha f \) of a constant is given by
\[
(D^\alpha_\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} (x-a)^{-\alpha}
\] (23)

On the other hand, for \( j = 1,2,\ldots, \{Re(\alpha)\} + 1 \)
\[
(D^\alpha_\alpha (t-a)^{\alpha-j})(x) = 0
\] (24)

The connection between Caputo fractional derivative and the Riemann-Liouville fractional derivative is given by
\[ \left( D_\alpha^\alpha f \right)(x) = \left( D_\alpha^\alpha \left[ f(t) - \sum_{v=0}^{n-1} \frac{f^{(v)}(a)}{v!} (t-a)^v \right] \right)(x), \text{Re}(\alpha) \geq 0, n = [\text{Re}(\alpha)] + 1, f \in AC^n[a,b]. \]  

(25)

Property-2: If Re(\alpha) > 0 and Re(\beta) > 0, then

\[ \left( D_\alpha^a(t-a)^{\beta-1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (x-a)^{\beta-1}, \text{Re}(\beta) > [\text{Re}(\alpha)] + 1. \]  

(26)

Lemma 1: Let Re(\alpha) > 0 and let \( f(x) \in L_\infty(a, b) \) and Re(\alpha) \( \in \mathbb{N} \) then

\[ \left( D_\alpha^a \right)(x) = f(x) \]  

and

\[ \left( I_\alpha^a D_\alpha^a f \right)(x) = f(t) - \sum_{v=0}^{n-1} \frac{f^{(v)}(a)}{v!} (t-a)^v, \text{Re}(\alpha) > 0, n = [\text{Re}(\alpha)] + 1. \]  

(27)

(28)

3. Solution of the Problem and Analysis.

With the application of (28) in (5), we may write it in the form

\[ u(x, y, t) = \sum_{v=0}^{n-1} \frac{u^{(v)}(x, y, 0)}{v!} t^v + \left( I_0^a D_x^\beta + I_0^\alpha D_y^\gamma \right) u(x, y, t) \]  

(29)

Now, to find out the sequence of solutions, we consider \( u(x, y, t) \) in the series form

\[ u(x, y, t) = \sum_{m=0}^{\infty} u_m(x, y, t), x, y \in \mathbb{R}, t \geq 0 \]  

(30)

From (29) and (30), we also write

\[ u(x, y, t) = u_0(x, y, t) + \sum_{m=1}^{\infty} u_m(x, y, t) \]  

(31)

\[ \sum_{v=0}^{n-1} \frac{u^{(v)}(x, y, 0)}{v!} t^v + \sum_{m=1}^{\infty} \left( I_0^a D_x^\beta + I_0^\alpha D_y^\gamma \right) u_{m-1}(x, y, t), x \in (a, b), y \in (c, d), t \geq 0 \]

Where \( u_0(x, y, t) \) initial function.

Then, from (31) we make the decomposition of the series solution and write that

\[ u_0(x, y, t) = \sum_{v=0}^{n-1} \frac{u^{(v)}(x, y, 0)}{v!} t^v, \text{for } m = 0 \]  

(32)
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and

\[ u_m(x, y, t) = \left( \frac{D_{x,\alpha}^* + D_{y,\gamma}^*}{\Gamma(\alpha + 1)} \right)^m u_{m-1}(x, y, t), \text{ for } m \geq 1 \]  

(33)

Here, in (32) we have that \( u_0(x, y, t) \) is the initial function and again with the help initial condition given in (6), this function is the function of \( x \) and \( y \) only, and we have also \( 0 < \alpha \leq 1 \), hence due to Adomian decomposition techniques ([5],[6]) applied in (32), we get \( n = 1 \) that is \( v = 0 \).

There form (32) we find

\[ u_0(x, y, t) = f(x, y), x \in (a, b), y \in (c, d), f \in R^2 \]  

(34)

Further on using (21) and (33), we get the successive approximation formula in the form

\[ u_m(x, y, t) = \left( \frac{D_{x,\alpha}^* + D_{y,\gamma}^*}{\Gamma(\alpha + 1)} \right)^m u_{m-1}(x, y, t), \text{ for } m \geq 1 \]  

(35)

Hence with the help of (34) and (35) and we get the sequence of solutions such that

\[ u_1(x, y, t) = \left( \frac{D_{x,\alpha}^* + D_{y,\gamma}^*}{\Gamma(\alpha + 1)} \right) f(x, y) \]  

(36)

\[ u_2(x, y, t) = \sum_{k=1}^{2} \left( \frac{D_{x,\alpha}^* + D_{y,\gamma}^*}{\Gamma(\alpha + 1)} \right) f(x, y) \]  

(37)

\[ u_m(x, y, t) = \sum_{k=1}^{m} \left( \frac{D_{x,\alpha}^* + D_{y,\gamma}^*}{\Gamma(\alpha + 1)} \right) f(x, y) \]  

(38)

Then, making an use of (34) to (38), finally we find the general solution of our problem such that

\[ u(x, y, t) = \sum_{k=0}^{m} \left( \frac{D_{x,\alpha}^* + D_{y,\gamma}^*}{\Gamma(\alpha + 1)} \right) f(x, y) \]  

(39)

Particularly, when \( \gamma = 2 \) and \( y \) becomes constant in (3.11), it is converted into solution of two-dimensional fractional diffusion equation of Kumar, Pathan and Srivastava [8].

4. Application of \( H \)-function of two variable for Obtaining Analytic Sequence of Solutions

To find out analytic sequence of solution of our problem (5) along with initial and boundary conditions (6) and (7), respectively, we consider the initial function as the \( H \)-function of two variable defined by (8)-(11) such that

\[ f(x, y) = H_{0, m; m_2, m_3; n_1, n_2} \left[ P_1(x), P_2(x) \right] \]  

(40)
Provided that $h_1, h_2 > 0, a, c \geq 0, \text{Re}(\mu_1) > 0, \text{Re}(\mu_2) > 0, x \in (a, b), y \in (c, d)$ and the conditions due to (12) and (13) are followed.

Then, making an use of (39) and (40) we obtain analytic sequence of solution of our problem (5) - (7) in the form

$$
H_0, q_1, q_2, p_1, p_2, q_3, \mu_1, \mu_2)
$$

Provided that $h_1, h_2 > 0, a, c \geq 0, \text{Re}(\mu_1) > 0, \text{Re}(\mu_2) > 0, x \in (a, b), y \in (c, d)$ and the conditions due to (12) and (13) are followed.

Finally, we evaluate the analytic solution involving various parameter of our problem (5) with initial and boundary conditions (6) and (7), respectively, in the form

$$
\sum_{k=0}^{\infty} \frac{k \alpha}{\Gamma(k \alpha + 1)} H^0, q_1, q_2, p_1, p_2, q_3, \mu_1, \mu_2)
$$

Provided that $h_1, h_2 > 0, a, c \geq 0, \text{Re}(\mu_1) > 0, \text{Re}(\mu_2) > 0, x \in (a, b), y \in (c, d), t > 0$ and the conditions due to (12) and (13) are followed.

5. Special Cases

We set $n_1 = p_1 = q_1 = 0$ in (4.3), then, particularly, we get the solution in the form

$$
\sum_{k=0}^{\infty} \frac{k \alpha}{\Gamma(k \alpha + 1)} H^0, q_1, q_2, p_1, p_2, q_3, \mu_1, \mu_2)
$$

Provided that $h_1, h_2 > 0, a, c \geq 0, \text{Re}(\mu_1) > 0, \text{Re}(\mu_2) > 0, x \in (a, b), y \in (c, d), t > 0$ and the conditions due to (12) and (13) are followed.
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+ \left( \frac{H_{m_1,n_1+1}}{p_2+1 q_2+1} \right) \left[ h_1(x-a) \mu_1 \left( \begin{array}{c} \ell_j, \gamma_j \end{array} \right)_{p_2+1}, (0, \mu_1) \right]

\times \sum_{k=0}^{\infty} \left( \frac{t^{\frac{k\alpha}{(k\alpha+1)}}}{\Gamma(k\alpha+1)} \frac{H_{m_2,n_2+1}}{p_2+1 q_2+1} \right) \left[ h_2(y-c) \mu_2 \left( \begin{array}{c} \ell_j, \gamma_j \end{array} \right)_{q_2+1}, (0, \mu_2) \right]

Provided that \( h_1, h_2 > 0, a, c \geq 0, \text{Re}(\mu_1) > 0, \text{Re}(\mu_2) > 0, x \in (a, b), y \in (c, d), t > 0 \) and the conditions due to (12) and (13) are followed.

\[ u(x, y, t) = \frac{H^{m_1,n_1}}{p_2+1} \left[ h_1(y-c) \mu_2 \left( \begin{array}{c} \ell_j, \gamma_j \end{array} \right)_{p_2+1}, (0, \mu_2) \right] \times \sum_{k=0}^{\infty} \left( \frac{t^{\frac{k\alpha}{(k\alpha+1)}}}{\Gamma(k\alpha+1)} \frac{H_{m_2,n_2+1}}{p_2+1 q_2+1} \right) \left[ h_2(x-a) \mu_1 \left( \begin{array}{c} \ell_j, \gamma_j \end{array} \right)_{p_2+1}, (0, \mu_2) \right]

Provided that \( h_1, h_2 > 0, a, c \geq 0, \text{Re}(\mu_1) > 0, \text{Re}(\mu_2) > 0, x \in (a, b), y \in (c, d), t > 0 \) and the conditions due to (12) and (13) are followed.

6. Physical Interpretations

When we set \( \alpha = 1, \beta = 1 and \gamma = 1 \) in the equation (5), then this equation with initial conditions (6) and the boundary conditions (7) gives a linear combination. Further when we set \( \alpha = 1, \beta = 1 and \gamma = 1 \) in the equation (5), then this equation with initial conditions (6) and the boundary conditions (7) becomes a heat equation of two variables. Again we set \( \alpha = 1, \beta = 1 and \gamma = 1 \) in the equation (5), then this equation with initial conditions (6) and the boundary conditions (7) becomes a wave equation of two variables. Hence, when \( \alpha, \beta \text{ and } \gamma \) are in fraction and have the values between \( 0 < \alpha \leq 1, 1 < \beta \leq 2 \text{ and } 1 < \gamma \leq 2 \), respectively, then the equation (5), with initial conditions (6) and the boundary conditions (7), has approximate analytic sequence of solutions of in the form of (41) which give various analytic structure between linear combination and heat equation in two variables. On specializing the parameters and takings the values of \( \alpha, \beta \text{ and } \gamma \) in fraction in the equation (41) we may plot various graphs by MATHEMATIKA, MATLAB etc. SOFTWARES and may understand and analyze the position of the unknown function \( u \) between linear equation in two variables.
References


SOLUTION OF ELECTRIC CIRCUIT PROBLEM IN H-FUNCTION OF TWO VARIABLES AND NUMERICAL ANALYSIS

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ABSTRACT

In the present paper, we make an application of Fox's H-function in an electric circuit problem consisting of a resistance $R$, an inductance $L$, a condenser of capacity $C$, and a source of electromotive force $E_0 p(t)$, where $E_0$ is constant and $p(t)$ is known function of time $t$. The charge $q(t)$ on the plates of condenser at any time $t$ is obtained in the series involving $H$-function of two variables. Some interesting results through fractional calculus are also analysed.

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1. Introduction and Preliminaries.

The generalized beta function is given by (see, Mathai, Saxena and Haubold [5])

$$
\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} dt = (b-a)^{\alpha+\beta-1} B(\alpha, \beta) \quad (\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0; a \neq b).
$$

(1.1)

and

$$
B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.
$$

(1.2)

We use the binomial functions in the form

$$
(ut + v) = (au + v)^{\gamma} \sum_{l=0}^{\infty} \frac{(-\gamma)_l}{l!} \left( \frac{(t-a)u}{au + v} \right)^l
$$

(1.3)

where, the pochhammer symbol is $(\lambda)_l = \frac{\Gamma(\lambda + l)}{\Gamma(\lambda)}$, (see, Rainville [9]).

The integral due to Prudnikov et.al. [7]

$$
\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} (ut + v)^{\gamma} dt = (b-a)^{\alpha+\beta-1} (au + v)^{\gamma} B(\alpha, \beta) \cdot_{2} F_{1} \left[ \alpha - \gamma; \alpha + \beta; -\frac{(b-a)u}{au + v} \right]
$$

(1.4)
is also used in our derivation of the solutions of electric circuit problem, where the following conditions are followed

\[
\left\{ R(a) > 0; R(\beta) > 0; \arg \left( \frac{bu + v}{au + v} \right) \leq \pi - \epsilon (0 < \epsilon < \pi); b = a \right\},
\]

the \( pF_q \) denotes, as usual, a generalized hypergeometric function with \( p \) numerator and \( q \) denominator parameters, and the arguments condition emerges from the analytic continuation of the Gaussian hypergeometric function \( _2F_1 \) occurring on the right-hand side of (1.4)

**Riemann-Liouville Fractional Integrals.** Let \( f(x) \in L(a,b) \) such that \( L(a,b) \) consists of Lebesgue measurable real or complex valued function \( f(x) \) (see Mathai, Saxena and Haubold [5]).

\[
L(a,b) = \left\{ f : \| f \| = \int_a^b |f(t)| dt < +\infty \right\}, a \in C, R(a) > 0.
\]

Then \( a^\alpha \int^x f(t) dt, x > a, \) (1.5)

which is Riemann-Liouville integral of order \( \alpha \).

**H-Function of Two Variables.** The \( H \)-functions of two variables is defined by (see Srivastava et al. (1982), p.82-83; [11] also Srivastava and Panda [12]) (also see Mathai, Saxena and Haubold (p.61; [5])

\[
H \left[ \begin{array}{c} x  \\ y \end{array} \right] = H_{\alpha_1, \alpha_2, \ldots, \nu_1, \nu_2}^{\gamma_1, \gamma_2, \ldots, \nu_1, \nu_2} \left[ \begin{array}{c} (a_1; \alpha_1, \nu_1)_{\gamma_1, \nu_1}; (c_1, \gamma_1)_{\nu_1, \nu_3}; (e_1, E_1)_{\nu_1, \nu_3} \\ y \right] \left( b_j; \beta_j, \nu_1 \right)_{\gamma_j, \nu_j}; (d_j, \delta_j)_{\nu_j, \nu_3}; (f_j, F_j)_{\nu_j, \nu_3} \right]
\]

(1.6)

\[
= \frac{1}{4\pi^2} \int_{L_1} \int_{L_2} \phi(s,t) \phi_1(s) \phi_2(t) x^s y^t ds dt,
\]

(1.7)

where \( x \) and \( y \) are not equal to zero, for convenience the parameters \( (a_i, \alpha_i; A_{\nu_i}) \) and \( (c_i, \gamma_i)_{\nu_i, \nu_3} \) will abbreviate the sequence of the parameters \( (a_1, \alpha_1; A_{\nu_1}), \ldots, (a_{\nu_i}, \alpha_{\nu_i}; A_{\nu_i}) \) and \( (c_1, \gamma_1), \ldots, (c_{\nu_i}, \gamma_{\nu_i}) \) respectively and similar meanings hold for the other parameters \( (b_j; \beta_j, \nu_{\gamma_j}) \) and \( (d_j, \delta_j)_{\nu_j, \nu_3}, etcetra. \)

Here,
\[
\phi(s, t) = \frac{\prod_{i=1}^{n_1} \Gamma(1 - \alpha_i + \alpha, s + A_i t)}{\prod_{i=1}^{n_1} \Gamma(\alpha_i - \alpha_i s - A_i t) \prod_{j=1}^{n_2} \Gamma(1 - \beta_j + \beta, s + B_j t)}.
\]

\[
\phi_1(s) = \frac{\prod_{j=1}^{n_3} \Gamma(c_j - \delta_j s) \prod_{i=1}^{n_7} \Gamma(1 - c_i + \gamma_i s)}{\prod_{j=1}^{n_3} \Gamma(1 - c_j + \delta_j s) \prod_{i=1}^{n_7} \Gamma(c_i - \gamma_i s)}.
\]

\[
\phi_2(t) = \frac{\prod_{j=1}^{n_4} \Gamma(f_j - F_j t) \prod_{i=1}^{n_5} \Gamma(1 - e_i - E_i t)}{\prod_{j=1}^{n_4} \Gamma(1 - f_j + F_j t) \prod_{i=1}^{n_5} \Gamma(e_i - E_i t)}.
\]

It is assumed that all poles of the integrand are simple, an empty product is interpreted as unity further, we suppose that all the parameter \(\alpha_i, b_j, c_i, d_j, e_i\) and \(f_i\) be complex and associated coefficients \(\alpha_i, A_i, B_j, \beta_j, \gamma_i, \delta_j, E_i\) and \(F_j\) be real and positive for the standardization purposes such that

\[
\rho_1 = \sum_{i=1}^{n_1} \alpha_i + \sum_{i=1}^{n_2} \gamma_i - \sum_{j=1}^{n_3} \beta_j - \sum_{j=1}^{n_4} \delta_j \leq 0,
\]

\[
\rho_2 = \sum_{i=1}^{n_1} A_i + \sum_{i=1}^{n_2} E_i - \sum_{j=1}^{n_3} B_j - \sum_{j=1}^{n_4} F_j \leq 0,
\]

\[
\Omega_1 = -\sum_{i=n_1+1}^{n_1} \alpha_i - \sum_{j=1}^{n_3} \beta_j + \sum_{j=1}^{n_4} \delta_j - \sum_{j=m_2+1}^{m_2} \delta_j - \sum_{i=n_1+1}^{n_2} \gamma_i - \sum_{j=m_4+1}^{m_4} \gamma_i > 0,
\]

\[
\Omega_2 = -\sum_{i=n_1+1}^{n_1} A_i - \sum_{j=1}^{n_3} B_j + \sum_{j=1}^{n_4} F_j - \sum_{j=m_2+1}^{m_2} F_j - \sum_{i=n_1+1}^{n_2} E_i - \sum_{i=n_1+1}^{n_2} E_i > 0.
\]

It can be seen that the contour integral (1.6) converges absolutely under the conditions (1.11-1.14) and defines an analytic function of two complex variables \(x\) and \(y\) inside the region

\[
|\arg(x)| < \pi \Omega_1 / 2 \text{ and } |\arg(y)| < \pi \Omega_2 / 2.
\]

The point \(x=0\) and \(y=0\) being tacitly excluded.

**2. Main Problem.** We consider an electric circuit problem consisting of a resistance \(R\), an inductance \(L\), a condenser of capacity \(C\) and a source of electromotive force \(E_0 p(t)\), where \(E_0\) is constant and \(p(t)\) is known function of time, \(t\). The charge
$q(t)$ on the plate of condenser at any time $t$, satisfies the following second order differential equation

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E_0 p(t). \quad (2.1)$$

The solution of this differential equation (2.1) subject to initial condition $q=Q$, $i=\frac{dq}{dt} = I_0(t=0)$ is given by Sneddon [10]

$$q(t) = J(t) + \frac{E_0}{wL} \exp\{ -Rt / 2L \} \int_0^t \! p(\xi) \exp\{ R\xi / 2L \sin w(t - \xi) \} d\xi, \quad (2.2)$$

where for convenience

$$J(t) = \exp\{ -Rt / 2L \} \left[ Q \cos w t + \frac{I_1}{w} \sin wt \right] \quad \text{and}$$

$$I_1 = I + \frac{RQ}{2L} \quad \text{and} \quad \frac{1}{LC} - \frac{R^2}{4L^2} = w^2 > 0. \quad (2.3)$$

The equations (2.2) may be written as

$$q(t) = J(t) + \frac{E_0}{wL} \exp\{ -Rt / 2L \} \int_0^t \! p(t - \xi) \exp\{ R(\xi - t) / 2L \} \sin w\xi d\xi. \quad (2.4)$$

where $J(t)$ is given in (2.3).

Now to find out the analytic solution of electric circuit problem in $H$-function of two variables we suppose that

$$p(t - \xi) = \xi^{a-1} (t - \xi)^{\beta-1} (c\xi + d)^{\gamma} H_{p,q}^{m,n} \left[ k(c\xi + d)^{-\delta} \left| a_p, A_p \right| \left| b_q, B_q \right| \right], \quad (2.5)$$

where $\min\{ \Re(\alpha), \Re(\beta) \} > 0, c, d, \alpha, \beta, \gamma \in C, t > 0, |ct/d| < 1, t > 0$

$$|ct/d| < 1, |\arg( (d + ct) / d) | < \pi; \delta > 0, |\arg(k)| < \pi\phi/2,$$

and $\phi$ is given by

$$\phi = \sum_{j=1}^n A_j + \sum_{j=m+1}^n A_j + \sum_{j=1}^n B_j - \sum_{j=m+1}^n B_j > 0. \quad (2.6)$$

Then using (2.5) in (2.4) and making an appeal to the formulae (1.1)-(1.4), we get

$$q(t) = J(t) + \frac{E_0}{WL} \exp\{ -Rt / (2L) \} t^{a+b-1} \Gamma(\beta) \sum_{r,s=0}^{\infty} \left( \beta \right)_r \left( (Rt / 2L)^{s} (wt)^{2s+1} \right) r!(2s+1)!$$
provided that all conditions given in (2.6) are satisfied.

3. Numerical Analysis. To analyse our above analytic solution, we appeal to fractional calculus (1.5) in our result (2.4) and find that

\[ q(t) = J(t) + \frac{E_0}{wL} \exp\{-Rt / 2L\} \sum_{r=0}^{\infty} (w)^{2r+1} t^{2r+2} F(\xi) \exp\{R\xi / 2L\}. \]

(3.1)

Here we get many sequence of fractional integral \( _0I_{t}^{2r+2} \frac{p(\xi) e^{2L}}{R} \) (\( r = 0,1,2,\ldots \)).

Further to find the numerical solutions of above problem (2.1), we consider \( p(t)=1 \) in (3.1) and use the formula due to Mathai, Saxena and Honbold [5]

\[ \left( {_aI_x^a} \left[ e^{ix} (x-a)^{\beta-1} \right]\right) x = e^{ix} \frac{\Gamma(\beta)}{\Gamma(a+\beta)} (x-a)^{\alpha+\beta-1} \right] F(\beta, \alpha+\beta; \lambda x - \lambda a) \]

(3.2)

we get

\[ q(t) = J(t) + \frac{E_0}{wL} \exp\{-Rt / 2L\} \sum_{r=0}^{\infty} (w)^{2r+1} \frac{1}{\Gamma(2r+3)} t^{2r+2} \right] F(1; 2r + 3; R / 2L). \]

(3.3)

Again for obtaining another numerical solution of above problem (2.1), we consider \( p(t)=1 \) in (3.1) and use the result

\[ \left[ {_aI_x^a} \left( e^{ix} \right) \right] x = e^{ix} (x-a)^{\alpha} E_{1,\alpha+1} (\lambda x - \lambda a). \]

(3.4)

where \( x > a, \alpha, \lambda \in C, R(a) > 0, \)

and \( E_{1,\alpha+1}(\cdot) \) is the Mittag-Leffler function. (see Mathai, Saxena and Haubold [5]) we obtain

\[ q(t) = J(t) + \frac{E_0}{wL} \exp\{-Rt / 2L\} \sum_{r=0}^{\infty} (w)^{2r+1} t^{2r+2} E_{1,2r+3} (Rt / 2L). \]

(3.5)

Further to find out the graphical results, we make an appeal to the formula due to Diethlem [1]

\[ E_{1,r}(x) = \frac{1}{x^{r-1}} \left( \exp(x) - \sum_{k=0}^{r-2} x^k / k! \right). \]

(3.6)

For any complex number \( x \) and natural number \( r \), (3.5) and (3.6), give
\[ q(t) = J(t) + \frac{E_0}{wL} \exp\left\{-\frac{Rt}{2L}\right\} \sum_{r=0}^{\infty} w^{2r+1} t^{2r+2} \left(\frac{Rt}{2L}\right)^{-(2r+2)} \left\{ \exp\left(\frac{Rt}{2L}\right) - \sum_{k=0}^{2r+1} \frac{1}{k!} \left(\frac{Rt}{2L}\right)^k \right\}. \quad (3.7) \]

From (3.7), we compute

\[ Q(t) = q(t) - J(t) = \frac{E_0}{wL} \exp\left\{-\frac{Rt}{2L}\right\} \sum_{r=0}^{\infty} w^{2r+1} t^{2r+2} \left(\frac{Rt}{2L}\right)^{-(2r+2)} \left\{ \exp\left(\frac{Rt}{2L}\right) - \sum_{k=0}^{2r+1} \frac{1}{k!} \left(\frac{Rt}{2L}\right)^k \right\}. \quad (3.8) \]

4. Numerical Example. In this section we consider the ratio of \( R/L \) in view of formula (3.8) and discuss three conditions \( R/L = 1, R/L > 1, \) and \( R/L < 1 \) and express the results through the following figures:

**Fig. 1**

**Fig. 2**
We put \( r=0, r=1 \) and \( E_0=5, w=50\text{Hz} \). and consider the cases \( R=L, R>L, R<L \) in (3.8) to compute \( Q(t) \), and we draw the Fig.1, Fig.2 respectively.

5. Discussion and Conclusions. From figure 1 and 2, we conclude that

1. When the resistance \( R \) and inductance \( L \) in the circuit are equal then the charge on the plate of the condenser will increase according to time increment.
2. When the resistance \( R \) is greater than inductance \( L \) of the circuit then the charge on the plate of the condenser will increase slowly according to time increment.
3. When the resistance \( R \) is less than inductance \( L \) of the circuit then the charge lying on the plate of the condenser, rises after time \( t \).

REFERENCES

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