Chapter 6

JOINT-MOMENTS DUE TO DIRICHLET DENSITY OF THREE DIMENSIONAL SPACE AND THEIR APPLICATIONS IN SUMMABILITY OF HYPERGEOMETRIC FUNCTIONS
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JOINT-MOMENTS DUE TO DIRICHLET DENSITY
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APPLICATIONS IN SUMMABILITY OF
HYPERGEOMETRIC FUNCTIONS

In this Chapter, we consider the Exton’s joint-moments due to
Dirichlet density of three dimensional space and then, prove that
the series of integrals due to these joint-moments is summable.
Again, we make their applications to obtain the summation
formulae of the series consisting Saran’s triple hypergeometric
functions. Finally, we obtain the approximation formula of this
series and discuss some particular cases.

6.1 Introduction

Exton [4, p. 232] has defined the joint-moments $\mu_{m_1,\ldots,m_k}(x_1,\ldots,x_k)$ in
the form.

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Suppose that \((x_1, \ldots, x_k)\) is a \(k\)-dimensional random variable with density \(f(x_1, \ldots, x_k)\), then the joint-moments \(\mu'_{m_1 \ldots m_k}(x_1, \ldots, x_k)\), if they exist, are given by

\[
\mu'_{m_1 \ldots m_k}(x_1, \ldots, x_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{m_1} \cdots x_k^{m_k} f(x_1, \ldots, x_k) \, dx_1, \ldots, dx_k. \quad (6.1.1)
\]

Take \(k=3\) in Eqn. (6.1.1) and consider the Dirichlet density in three dimensional space as

\[
f(x, y, z) = \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma\left(a-\sum_{i=1}^{3}(\mu_i+\sigma_i)\right)} \left(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}\right)^{a-\sum_{i=1}^{3}(\mu_i+\sigma_i)-1} x^{\mu_1+\sigma_1-1} y^{\mu_2+\sigma_2-1} z^{\mu_3+\sigma_3-1} \quad (6.1.2)
\]

in the region \(x \geq 0, y \geq 0, z \geq 0\) and \(\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} \leq 1\), with the conditions that \(\mu_i>-\sigma_i; i = 1,2,3\) and \(a > \sum_{i=1}^{3}(\mu_i+\sigma_i)\) such that \(\sigma_i; i = 1,2,3\) are positive real and also \(\alpha, \beta\) and \(\gamma\) are positive real and otherwise, \(f\) is zero.

Then, the Exton’s joint-moments defined in Eqn. (6.1.1) due to Dirichlet density (6.1.2) become sequence of integrals such that

\[
\mu'_{m,p,q} = \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma\left(a-\sum_{i=1}^{3}(\mu_i+\sigma_i)\right)} \int \int \int x^{\mu_1+\sigma_1+m-1} y^{\mu_2+\sigma_2+p-1} z^{\mu_3+\sigma_3+q-1} \left(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}\right)^{a-\sum_{i=1}^{3}(\mu_i+\sigma_i)-1} \, dx \, dy \, dz \quad (6.1.3)
\]
for all \( m, p, q \in \mathbb{N}_0 (\mathbb{N}_0 = \mathbb{N} \cup \{0\}) \), (\( \mathbb{N} \) is the set of natural numbers) along with the conditions given in Eqn. (6.1.2).

In our investigations, we present following theorems.

**Theorem 1: [Generalized Bosanquet and Kestelman Theorem]**

Suppose that \((x_1, \ldots, x_k)\) is a \( k \)-dimensional random variable with density \( f(x_1, \ldots, x_k) \) and the sequence of functions \( g_n(x_1, \ldots, x_k) \) are measurable in the region \(-\infty < x_i < \infty, \ldots, -\infty < x_k < \infty (\forall n = 0, 1, 2, \ldots)\) and if

\[
\sum_{n=0}^{\infty} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, \ldots, x_k) g_n(x_1, \ldots, x_k) dx_1 \ldots dx_k \right| \leq \eta \tag{6.1.4}
\]

Then,

\[
\sum_{k=0}^{\infty} \left| g_n(x_1, \ldots, x_k) \right| \leq \eta \tag{6.1.5}
\]

for every \((x_1, \ldots, x_k) \in \mathbb{R}^k \) (\( \mathbb{R} \) is the set of real numbers) and \( \eta \) is absolutely constant.

**Proof:** Since \( f(x_1, \ldots, x_k) \) is a density function, therefore

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, \ldots, x_k) dx_1 \ldots dx_k = 1 \tag{6.1.6}
\]

Now, make an appeal to the Eqns. (6.1.4) and (6.1.6), we get

\[
\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, \ldots, x_k) \left\{ \sum_{n=0}^{\infty} g_n(x_1, \ldots, x_k) - \eta \right\} dx_1 \ldots dx_k \right| \leq 0. \tag{6.1.7}
\]

Again, \( f(x_1, \ldots, x_k) \neq 0, \forall x_i \in (-\infty, \infty), (i = 1, \ldots, k) \) in Eqn. (6.1.7), so that it gives the result (6.1.5).

**Corollary 1:** In the three dimensional space \((x, y, z)\), if in the region \( 0 \leq x \leq \alpha, 0 \leq y \leq \beta, 0 \leq z \leq \gamma \) the probability density function
is \( f(x,y,z) \) and otherwise \( f = 0 \), also in this region the the sequence of functions \( g_n(x_1,\ldots,x_k) \) is measurable and

\[
\sum_{n=0}^{\infty} \left| \int_0^1 \int_0^1 \int_0^1 g_n(x,y,z) f(x,y,z) \, dx \, dy \, dz \right| \leq \eta
\]  

(6.1.8)

where, \( \alpha, \beta, \gamma \) are positive real numbers.

Then,

\[
\sum_{n=0}^{\infty} |g_n(x,y,z)| \leq \eta
\]  

(6.1.9)

for every \( (x,y,z) \in \mathbb{R}^3 \) and \( \eta \) is absolutely constant.

This is another proof of the theorem due to Kumar and Yadav [7].

**Theorem 2:** If \( b_1, b_2, b_3, c_1 \) and \( c_2 \in \mathbb{C} \) (the set of complex numbers) such that \( \Re(b_i) > 0 \) \( (\forall i = 1,2,3) \), \( \Re(c_i) > \Re(b_i) \)

and \( \Re(c_2) > \Re(b_2 + b_3) \) and \( h_1, h_2, h_3 \) are real numbers; \( \alpha, \beta, \gamma \) are positive real numbers, then the triple series, consisting joint moments defined in Eqn. (6.1.3),

\[
\sum_{m,p,q=0}^{\infty} \frac{(a)_{m+p+q} (b_1)_m (b_2)_p (b_3)_q}{(c_1)_m (c_2)_p q} \mu_{m,p,q}' \frac{h_1^m h_2^p h_3^q}{m! p! q!}
\]  

(6.1.10)

is summable for \( |h_1 \alpha| < 1, |h_2 \beta| < 1, |h_3 \gamma| < 1, \) \( (a) > \sum_{i=1}^{3} (\mu_i + \sigma_i) \)

and \( \mu_i > -\sigma_i, \sigma_i > 0, \forall i = 1,2,3 \) and also it is equal to

\[
\sum_{i=1}^{3} \binom{\mu_i + \sigma_i}{b_1; c_i; h_1 \alpha} \binom{\mu_2 + \sigma_2, \mu_3 + \sigma_3}{b_2, b_3; c_2; h_2 \beta, h_3 \gamma}
\]  

(6.1.11)
Proof: Making an appeal to the Eqn. (6.1.3) and the series given in Eqn. (6.1.10), and then defining the Saran’s triple hypergeometric function $F_G(\cdot)$ (see Saran [10], Exton [4]) such that

$$F_G(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= \sum_{m, p, q=0}^{\infty} \frac{(\alpha_1)_m (\beta_1)_m (\beta_2)_p (\beta_3)_q x^m y^p z^q}{(\gamma_1)_m (\gamma_2)_p q! m! p! q!}$$

$$|x| < r, \ |y| < s, \ |z| < t, \ r + s = 1 = r + t,$$  \hspace{1cm} (6.1.12)

we get the integral

$$\frac{\Gamma(a)(\alpha)^{(\mu_1 + \sigma_1)}(\beta)^{(\mu_2 + \sigma_2)}(\gamma)^{(\mu_3 + \sigma_3)}}{\Gamma(\mu_1 + \sigma_1)\Gamma(\mu_2 + \sigma_2)\Gamma(\mu_3 + \sigma_3)\Gamma(a - \sum_{i=1}^{3}(\mu_i + \sigma_i))}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{\mu_1 + \sigma_1 - 1} y^{\mu_2 + \sigma_2 - 1} z^{\mu_3 + \sigma_3 - 1} (1 - x\alpha^{-1} - y\beta^{-1} - z\gamma^{-1})^{a - \sum_{i=1}^{3}(\mu_i + \sigma_i) - 1}$$  \hspace{1cm} (6.1.13)

$$F_G(a, a, a, b_1, b_2, b_3; c_1, c_2, c_3; h_1x, h_2y, h_3z)$$

Then, in the integrand of the integral (6.1.13), define Euler-type integral formula of triple hypergeometric functions $F_G(\cdot)$ due to Exton [4] (see also Tiwari [15, p. 37]) and again make some manipulations to get

$$\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(c_1 - b_1)\Gamma(b_2)\Gamma(c_2 - b_2 - b_3)\Gamma(c_3 - b_2 - b_3)}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^{a_1 - 1} v^{a_2 - 1} w^{a_3 - 1}$$

$$\times (1 - u)^{a_1 - h_1 - 1} (1 - uh_1\alpha)^{-(\mu_1 + \sigma_1)} (1 - v - w)^{a_2 - b_2 - b_3 - 1}$$

$$\times (1 - wh_2\beta)^{-(\mu_2 + \sigma_2)} (1 - wh_3\gamma)^{-(\mu_3 + \sigma_3)} dudvdw$$  \hspace{1cm} (6.1.14)
provided that $0 \leq u \leq 1$, $v \geq 0$, $w \geq 0$ and $v + w \leq 1$. Finally, in the result (6.1.14) make an appeal to Euler-type integrals of Gaussian hypergeometric function of $\binom{2}{F_1}(.)$ and Appell’s hypergeometric function $F_1(.)$ (see Srivastava and Manocha [14], Srivastava and Karlsson [13], Mathai and Haubold [8], Exton [4]) we obtain the result (6.1.11). Further, the bi-product (6.1.11) converges when; $|h_1\alpha|<1$, $|h_2\beta|<1$ and $|h_3\gamma|<1$; hence for these conditions the series given in Eqn. (6.1.10) is summable.

6.2 Applications in Summability of the Series Involving Saran’s Triple Hypergeometric Functions

In this section, we make an application of the Theorem-1 and Theorem-2 presented in section-1 to evaluate the convergent summation formulae of the series involving Saran’s triple hypergeometric functions $F_G(.)$ defined by Eqn. (6.1.12).

Theorem 3: For the conditions given in the Eqn. (6.1.2) and in the theorem-2 and $|\eta|<1$, the series

$$\sum_{n=0}^{\infty} \frac{(b_1)_n \theta^n}{n!} F_G(a,a,a;b_1+n,b_2,b_3;c_1,c_2,c_2;h_1x,h_2y,h_3z)$$

is summable and is equal to

$$(1-\tau)^{-b} F_2 \left[ b_1, \eta_1 + \sigma_1; c_1; \frac{h_\alpha}{1-\tau} \right] F_3 \left[ \mu_2 + \sigma_2, \mu_3 + \sigma_3, b_2, b_3; c_2; h_2\beta, h_3\gamma \right]$$
**Proof:** In the Eqn. (6.1.11) replace $b_i$ by $b_i + n$ and again multiply it by $\frac{(b_i)_n t^n}{n!}$, where $|t| < 1$ and then sum $n$ from 0 to $\infty$. Again, use the result due to Chaundy [1] (see also Miller [9], Srivastava and Manocha [14, p. 351]), to get the result

$$\left(1 - t \right)^{-b} \binom{b_1}{n} \binom{b_2}{n} \binom{b_3}{n} \binom{b_4}{n} \binom{h\alpha}{1-t} F_3 \left[ \frac{\mu_2 + \sigma_2, \mu_3 + \sigma_3, b_2, b_3; c_2; h_2, h_3 \gamma}{1-t} \right] (6.2.3)$$

Further, make an appeal to the Eqn. (6.1.13), to get

$$\frac{\Gamma(a)(a + \sigma_1)(a + \sigma_2)(a + \sigma_3)}{\Gamma(\mu_1 + \sigma_1)\Gamma(\mu_2 + \sigma_2)\Gamma(\mu_3 + \sigma_3)\Gamma\left(a - \sum_{i=1}^{3} (\mu_i + \sigma_i)\right)}$$

$$\sum_{n=0}^{\infty} \frac{(b_i)_n t^n}{n!} \int \int \int x^{\mu_1 + \sigma_1 - 1} y^{\mu_2 + \sigma_2 - 1} z^{\mu_3 + \sigma_3 - 1} \left(1 - x\alpha^{-1} - y\beta^{-1} - z\gamma^{-1}\right)^{a - \sum_{i=1}^{3} (\mu_i + \sigma_i) - 1} F_G \left(a, a, a, b_i + n, b_2, b_3; c_1, c_2, c_2; h_2, h_3, h_2, h_3 z\right) dx dy dz (6.2.4)$$

Then, making an appeal to the Theorem-2, the Eqns. (6.2.3) and (6.2.4), we get

$$\frac{\Gamma(a)(a + \sigma_1)(a + \sigma_2)(a + \sigma_3)}{\Gamma(\mu_1 + \sigma_1)\Gamma(\mu_2 + \sigma_2)\Gamma(\mu_3 + \sigma_3)\Gamma\left(a - \sum_{i=1}^{3} (\mu_i + \sigma_i)\right)}$$

$$\sum_{n=0}^{\infty} \frac{(b_i)_n t^n}{n!} \int \int \int x^{\mu_1 + \sigma_1 - 1} y^{\mu_2 + \gamma + \gamma - 1} z^{\mu_3 + \sigma_3 - 1} \left(1 - x\alpha^{-1} - y\beta^{-1} - z\gamma^{-1}\right)^{a - \sum_{i=1}^{3} (\mu_i + \sigma_i) - 1} F_G \left(a, a, a, b_i + n, b_2, b_3; c_1, c_2, c_2; h_2, h_3, h_2, h_3 z\right) dx dy dz$$

$$= \left(1 - t \right)^{-b} \binom{b_1}{n} \binom{b_2}{n} \binom{b_3}{n} \binom{h\alpha}{1-t} F_3 \left( b_1, \mu_1 + \sigma_1; c_1; h_2, h_3 \gamma \right) (6.2.5)$$

Finally, make an appeal to the Theorem-1, with Dirichlet density
(6.1.2) in the Eqn. (6.2.5) to get the result (6.2.1) with Eqn. (6.2.2) of theorem-3. Again, the result (6.2.2) is convergent for \( |\lambda| < 1, \ |h_1\alpha| < 1, \ |h_2\beta| < 1, \ |h_3\gamma| < 1, \) hence the series involving Saran’s triple hypergeometric functions \( F_c(.) \) given in Eqn. (6.2.1) is summable.

**Theorem 4:** For the given conditions of the Eqn. (6.1.2) and in the theorem-2 and \( \frac{|\omega|}{T} < 1, \ \frac{|\omega(1-h_1\alpha)|}{T} < 1, \ \frac{|h_1\alpha T}{T+\omega(1-h_1\alpha)} + |T+\omega| < 1, \) the series \[ \sum_{n=-\infty}^{\infty} \frac{\Gamma(b_1 + n)\Gamma(b_1' + n)\Gamma(c_1')}{n!\Gamma(b_1)\Gamma(b_1')\Gamma(c_1') + n} \] 
\[ \begin{array}{c} F_2 \left[ b_1 + n, b_1' + n, 1 + b_1 - c_1 + n; \right] \\
+ \frac{h_1\alpha T}{T + \omega(1-h_1\alpha)}, T + \omega \right] \\
F_3 \left[ \mu_2 + \sigma_2, \mu_3 + \sigma_3, b_2, b_3; c_2; h_2\beta, h_3\gamma \right] \] 
(6.2.6)
is summable and is equal to
\[ \frac{1}{\left( 1 + \frac{\omega}{T} \right)^{b_1 + \mu_1 + \sigma_1 - c_1}} \left\{ 1 + \frac{\omega(1-h_1\alpha)}{T} \right\}^{-(\mu_1 + \sigma_1)} \]
\[ F_2 \left[ b_1, \mu_1 + \sigma_1, b_1'; c_1, c_1'; \frac{h_1\alpha T}{T + \omega(1-h_1\alpha)}, T + \omega \right] \]
\[ F_3 \left[ \mu_2 + \sigma_2, \mu_3 + \sigma_3, b_2, b_3; c_2; h_2\beta, h_3\gamma \right] \] 
(6.2.7)

**Proof:** In the result (6.1.11) replace \( b_1 \) by \( b_1 + n \) and then multiply it by
\[ \frac{\Gamma(b_1 + n)\Gamma(b_1' + n)\Gamma(c_1')T^n}{n!\Gamma(b_1)\Gamma(b_1')\Gamma(c_1') + n} \] 
\[ \begin{array}{c} F_2 \left[ b_1 + n, b_1' + n, 1 + b_1 - c_1 + n; \right] \\
+ \frac{h_1\alpha T}{T + \omega(1-h_1\alpha)}, T + \omega \right] \\
F_3 \left[ \mu_2 + \sigma_2, \mu_3 + \sigma_3, b_2, b_3; c_2; h_2\beta, h_3\gamma \right] \] 
and again, sum \( n \) from \(-\infty\) to \( \infty \) to get
\[ \frac{\Gamma(b_1 + n)\Gamma(b_1' + n)\Gamma(c_1')T^n}{n!\Gamma(b_1)\Gamma(b_1')\Gamma(c_1') + n} \] 
\[ \begin{array}{c} F_1 \left[ b_1 + n, \mu_1 + \sigma_1; \right] \\
+ \frac{h_1\alpha}{c_1;} \right] \]
Now in the series (6.2.8) under the conditions

\[
\frac{\omega}{T} < 1, \quad \frac{\omega(1-h_\alpha)}{T} < 1, \quad \frac{h_\alpha T}{T + \omega(1-h_\alpha)} + |T + \omega| < 1, \quad \text{apply the result due to Srivastava and Manocha [14, p. 348] and then make an appeal to Theorem-1 and proof of the Theorem-2, to get the result (6.2.6) with (6.2.7).}
\]

Again, the function given in Eqn. (6.2.7) converges when \( \frac{\omega}{T} < 1, \frac{\omega(1-h_\alpha)}{T} < 1, \frac{h_\alpha T}{T + \omega(1-h_\alpha)} + |T + \omega| < 1, |h_\alpha| < 1, |h_2\beta| < 1, |h_3\gamma| < 1 \) and hence, the series involving Saran’s triple hypergeometric function \( F_3(.) \) given in Eqn. (6.2.6) is summable.

**Theorem 5:** For any \( \lambda \in \mathbb{C}, \ |\lambda| < 1 \) and all given conditions of the Eqn. (6.1.2) and Theorem 2, the series

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_3(a, a, a, \rho-n, b_2, \sigma-n, c_2, c_2, h_3x, h_3y, h_3z) \tag{6.2.9}
\]

is summable and is equal to

\[
(1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu_1 + \sigma_1)_n (\mu_3 + \sigma_3)_n}{n! (c_1)_n (c_2)_n} \left( \frac{h_1 h_3 \alpha \gamma}{(t-1)^2} \right)^n
\]
\[
F_1 \left( \mu_1 + \sigma_1 + n, \lambda + n, \rho; c_1 + n; \frac{h_1 \alpha t}{t - 1} \right) \\
F_S \left[ b_2, \mu_3 + \sigma_3 + n, \mu_3 + \mu_2 + \sigma_3 + n, \mu_2 + \sigma_2, \lambda + n; c_2 + n, c_2 + n, c_2 + n; \frac{h_2 \beta}{t - 1}, \frac{h_3 \gamma}{t - 1} \right]
\]

(6.2.10)

**Proof:** In the result (6.1.11) replace \( b_1 \) by \( \rho - n \) and \( b_3 \) by \( \sigma - n \) and again multiply it by \( \frac{(\lambda)_n t^n}{n!} \) and then sum \( n \) from 0 to \( \infty \) to get

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!} F_1 \left[ \mu_1 + \sigma_1, \rho - n; c_1; \frac{h_1 \alpha}{1} \right] \\
F_3 \left[ \mu_2 + \sigma_2, \mu_3 + \sigma_3, b_2, \sigma - n; c_2; \frac{h_2 \beta}{h_3 \gamma} \right]
\]

(6.2.11)

Now, in the series (6.2.11), make an appeal to the relations of Erdélyi et al. [3, section (2.5), Eq. (1.2)], Srivastava [12, p. 71, Eq. (3.4)] and Srivastava and Manocha [14, p. 149, Eq. (4.1)] and after making some manipulations, we generalize it by

\[
(1 - t)^{-2} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left( \frac{(\mu_1 + \sigma_1)_n (\mu_3 + \sigma_3)_n}{(c_1)_n (c_2)_n} \right) \left( \frac{h_1 h_2 \alpha \gamma t}{(t - 1)^2} \right)^n \\
F_1 \left[ \mu_1 + \sigma_1 + n, \lambda + n, \rho; \gamma_1 + n; \frac{h_1 \alpha t}{t - 1} \right] \\
F_S \left[ b_2, \mu_3 + \sigma_3 + n, \mu_3 + \mu_2 + \sigma_3 + n, \mu_2 + \sigma_2, \lambda + n; c_2 + n, c_2 + n, c_2 + n; \frac{h_2 \beta}{h_3 \gamma}, \frac{h_3 \gamma}{t - 1} \right]
\]

(6.2.12)

Then, in the series (6.2.12) use Theorem-1 and proof of the Theorem-2, to obtain the summation formula

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!} F_G \left( a, a, \rho - n, b_2, \sigma - n; c_1, c_2, c_2; h_x, h_y, h_z \right)
\]
The right hand side of the Eqn. (6.2.13) exists for $|t| < 1$, $|h_1\alpha| < 1$, $|h_2\beta| < 1$, $|h_3\gamma| < 1$. Hence in left hand side of the Eqn. (6.2.13), the series involving Saran’s triple hypergeometric function $F_{G}(.)$ is summable.

6.3 Approximation Formula

In this section, we evaluate the approximation formula for sum of the series involving Saran’s triple hypergeometric function $F_{G}(.)$.

We recall the Theorem Joshi and Arya [6] and consider the conditions.

\[
c_1 > b_1 > c_1 - (\mu_1 + \sigma_1) > 0, \quad (\mu_1 + \sigma_1) > 0, \quad 0 < \frac{|h_1\alpha|}{1-t} < 1,
\]

\[
c_2 > b_2 > c_2 - (\mu_2 + \sigma_2) > 0, \quad (\mu_2 + \sigma_2) > 0, \quad 0 < |h_2\beta| < 1
\]

and

\[
c_3 > b_3 > c_3 - (\mu_3 + \sigma_3) > 0, \quad (\mu_3 + \gamma_3) > 0, \quad 0 < |h_3\gamma| < 1,
\]

we get

\[
\left| {}_2F_1\left( b_1, \mu_1 + \sigma_1; c_1; \frac{h_1\alpha}{1-t} \right) \right| < \frac{\Gamma(b_1 + \mu_1 + \sigma_1 - c_1)\Gamma(c_1)}{\Gamma(b_1)\Gamma(\mu_1 + \sigma_1)} \left(1 - \frac{h_1\alpha}{1-t} \right)^{c_1 - h_1(\mu_1 + \sigma_1)}
\]

(6.3.1)

and

\[
\left| {}_3F_2\left[ \mu_2 + \sigma_2, \mu_3 + \sigma_3, b_2, b_3, c_2; h_2\beta, h_3\gamma \right] \right| < \frac{\Gamma(b_2 + \mu_2 + \sigma_2 - c_2)\Gamma(b_3 + \mu_3 + \sigma_3 - c_2)\Gamma(c_2)^2}{\Gamma(b_2)\Gamma(b_3)\Gamma(\mu_2 + \sigma_2)\Gamma(\mu_3 + \sigma_3)}
\]
\[(1 - h_2 \beta)^{-h_2 - \mu_1 - \sigma_2} (1 - h_3 \gamma)^{-h_3 - \mu_3 - \sigma_3} F_1\left[\frac{c_2 - 1}{c_2 - 1/2} \left(\frac{c_2 + 1}{2} - h_2 \beta \gamma\right)\right] (6.3.2)\]

Then, making an appeal to results (6.3.1)-(6.3.2), we present following theorem.

**Theorem 6:**

If, \(2 \geq c_2 \geq \max \{1, 2b_2, 2b_3, 2(\mu_2 + \sigma_2) - 1\}\), \(0 < \mu_3 + \sigma_3 \leq 1\)

\[c_1 > b_1 > c_1 - (\mu_1 + \sigma_1) > 0, (\mu_1 + \sigma_1) > 0, 0 < \left|\frac{h_1 \alpha}{1 - t}\right| < 1, 0 < t < 1,\]

then for \(a > \sum_{i=1}^{3} (\mu_i + \sigma_i), 0 \leq x \leq \alpha, 0 \leq y \leq \beta, 0 \leq z \leq \gamma, h_1, h_2, h_3\) are positive real such that \(0 < h_1 x < 1, 0 < h_2 y < 1, 0 < h_3 z < 1\), there is an inequality

\[\sum_{n=0}^{\infty} \left|\frac{(b_1)^n}{n!} f_G(a, a, a, b_1 + n, b_2, b_3; c_1, c_2, c_3; h_1 x, h_2 y, h_3 z)\right| < \]

\[\left|\frac{\Gamma(h_1 + \mu_1 + \sigma_1 - c_1) \Gamma(c_1)}{\Gamma(h_1) \Gamma(\mu_1 + \sigma_1)} \left|\frac{\Gamma(b_2 + \mu_2 + \sigma_2 - c_2) \Gamma(b_2 + \mu_2 + \sigma_2 - c_2) \Gamma(c_2)}{\Gamma(b_2) \Gamma(b_2 + \mu_2 + \sigma_2) \Gamma(b_2 + \mu_2 + \sigma_2) \Gamma(c_2)}\right| \right|\]

\[\left|\frac{(1 - t - h_1 \alpha)^{\mu_1 - \sigma_1}}{1 - \frac{2(c_2 - 1)}{c_2} + \frac{2(c_2 - 1)}{c_2} \left[1 + \frac{c_2 (c_2 + 1) h_2 h_2 \beta \gamma}{2(c_2 - 1)}\right]^{-1}}\right| (6.3.3)\]

**Proof:** On applying results (6.3.1)-(6.3.2), along with Joshi and Arya's result [6, Eq. (3.8)], we are easily led to result (6.3.3).

**Example:** For the set of values
\[ b_1 = 1.1, \mu_1 + \sigma_1 = 0.6, c_1 = 1.3; \quad b_2 = 0.9, \mu_2 + \sigma_2 = 0.6, c_2 = 1.4; \]
\[ b_3 = 0.9, \mu_3 + \sigma_3 = 0.7, \quad t = 0.1, \quad h_1 = h_2 = h_3 = \frac{1}{10}, \quad \alpha = 6, \beta = 7, \gamma = 5; \]
\[ 0 \leq x \leq 6, \quad 0 \leq y \leq 7, \quad 0 \leq z \leq 5, \quad a > 1.9; \quad (6.3.4) \]

and making an appeal to the Theorem-6, we get the inequality
\[ \sum_{n=0}^{\infty} \frac{(0.1)^n}{n!} (1.1)^n F_G \left( a, a, a, 1.1 + n, 0.9, 0.9; 1.3, 1.4, 1.4; \frac{1}{10}, \frac{1}{10}, \frac{1}{10} \right) < 8.0562 \]
\[ (6.3.5) \]

### 6.4 Special Cases

Before going to obtain the particular cases we present following theorem.

**Theorem 7: [Generalized Lebesgue’s monotone convergence theorem for Dirichlet measure]**

If in three dimensional space \( \mathbb{R}^3 \), such that \( x \geq 0, \ y \geq 0, \) and
\[ \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} \leq 1, \]
with the conditions that \( \mu_i > -\sigma_i; \ i = 1, 2, 3 \)
\[ a > \sum_{i=1}^{3} (\mu_i + \sigma_i), \]
all \( \sigma_i (i = 1, 2, 3), \ \alpha, \beta, \gamma \) are positive real, the Dirichlet measure is defined by
\[ dF = \frac{\Gamma(a) (\alpha)^{-(\mu_1 + \sigma_1)} (\beta)^{-(\mu_2 + \sigma_2)} (\gamma)^{-(\mu_3 + \sigma_3)}}{\Gamma(\mu_1 + \sigma_1) \Gamma(\mu_2 + \sigma_2) \Gamma(\mu_3 + \sigma_3) \Gamma \left( a - \sum_{i=1}^{3} (\mu_i + \sigma_i) \right)} \]
\[ \left( 1 - x\alpha^{-1} - y\beta^{-1} - z\gamma^{-1} \right)^{a - \sum_{i=1}^{3} (\mu_i + \sigma_i)} x^{\mu_1 + \sigma_1 - 1} y^{\mu_2 + \sigma_2 - 1} z^{\mu_3 + \sigma_3 - 1} \]
\[ (6.4.1) \]

and \( \{g_n(x,y,z)\}, \ (x,y,z) \in \mathbb{R}^3 \) be a sequence of non-negative measurable three variable functions such that
\[
\sum_{n=0}^{\infty} g_n(x, y, z) \leq \eta. \quad (6.4.2)
\]

Then,
\[
\sum_{n=0}^{\infty} \int g_n(x, y, z) \, dF \leq \eta
\]
\[
(6.4.3)
\]

**Proof:** Making an appeal to the Eqns. (6.4.1) and (6.4.2) in left hand side of Eqn. (6.4.3) and then in consequence of Corollary 1, we find right hand side of Eqn. (6.4.3).

Set \(\alpha = \beta = \gamma = 1\) and \(h_1 = 0\) in Theorem 2, particularly, to get Dirichlet integral transformation of Appell’s double hypergeometric function \(F_1(.)\) or hypergeometric integral due to Dirichlet integral derived by Exton ([5, p. 123], for \(n = 2\)) in the form

\[
\frac{\Gamma(a)}{\Gamma(\mu_1 + \sigma_1)\Gamma(\mu_2 + \sigma_2)\Gamma(\mu_3 + \sigma_3)\Gamma\left(a - \sum_{i=1}^{3}(\mu_i + \sigma_i)\right)}
\]

\[
\int x^{\mu_1 + \sigma_1 - 1}y^{\mu_2 + \sigma_2 - 1}z^{\mu_3 + \sigma_3 - 1}(1-x-y-z)^{a-\sum_{i=1}^{3}(\mu_i + \sigma_i)-1} F_1(a, b_2, b_3; c_2; h_2y, h_3z) \, dx\, dy\, dz
\]

\[
= F_3[\mu_2 + \sigma_2, \mu_3 + \sigma_3, b_2, b_3; c_2; h_2, h_3] \quad (6.4.4)
\]

Setting \(h_2 = h_3 = 0\) in Theorem 4 and make an appeal to Theorem 7 particularly, we get generating relation of Srivastava and Manocha ([14, p. 348, Eq. (23)]) in the form

\[
\sum_{n=0}^{\infty} \frac{\Gamma(b_1 + n) \Gamma(b'_1 + n) \Gamma(c'_1)}{n! \Gamma(b_1) \Gamma(b'_1) \Gamma(c'_1 + n)} \binom{b_1 + n, b'_1 + n, 1 + b_1 - c_1 + n;}{n + 1, c'_1 + n; \omega} F_2[\mu_1 + \sigma_1, b_1 + n; c_1; h_1 \alpha] T^n
\]
Setting $h_2 = 0$ in Theorem 5 and in consequence of theorem 7 specially, we obtain the generating relation due to Srivastava ([11], for $p = q = r = s = 1$),

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!} F_1 \left[ \mu_1 + \sigma_1; n; h_1 \alpha \right] F_1 \left[ \mu_3 + \sigma_3; n; c_1, c_2; h_3 \gamma \right]$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu_1 + \sigma_1)_n (\mu_3 + \sigma_3)_n}{n! (c_1)_n (c_2)_n} \left( \frac{h_1 \alpha t}{t-1} \right)^n$$

$$F_1 \left[ \mu_1 + \sigma_1 + n, \lambda + n, \rho; c_1 + n; h_1 \alpha, \frac{h_1 \alpha t}{t-1} \right]$$

$$F_1 \left[ \mu_3 + \sigma_3 + n, \lambda + n, \sigma; c_2 + n; h_3 \gamma, \frac{h_3 \gamma t}{t-1} \right]$$

In Eqn. (6.4.6), after replacing $h_1 \alpha$ by $\frac{h_1 \alpha}{\mu_1 + \sigma_1}$ and $h_3 \gamma$ by $\frac{h_3 \gamma}{\mu_3 + \sigma_3}$ and then set $\mu_1 + \sigma_1 \to \infty$ and $\mu_3 + \sigma_3 \to \infty$, particularly, we get the result of Erdélyi ([2, p. 344, Eq. (15)]) (see also Srivastava and Manocha [14, p. 133, Eq. (12)]).

Again, letting $h_1 \to 0$ in both sides of the Eqn. (6.4.6), we get the generating function of Srivastava and Manocha ([14, p. 150-151, Eq. (44)]).
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