Chapter 1

Introduction:

1.1 Linear and Nonlinear Statistical Models

Statistical models are designed to relate variable $Y$ known as response or dependent variable and variable $X$ termed as explanatory, regressor, predictor or independent variable. The variable $X$ is primarily used to predict or explain the behaviour of the variable $Y$. Often, values of $X$ are fixed or are observed without error contrary to $Y$ variable which is not controlled or preset. However, $X$-variables may be random. The relationship between $Y$ and $X$ may be expressed as

$$Y = f(X). \quad (1.1)$$

Knowledge regarding form of function $f$ enables to predict $Y$ for a given $X$. In practice, however, relation (1.1) can never hold exactly as data usually contain unexplained fluctuations, hence require inclusion of an error term also known as stochastic term in order to explain $Y$ completely. Thus a statistical model is represented as

$$Y = f(X) + u. \quad (1.2)$$

The error term $u$ is usually assumed to be independently and identically distributed random variable (i.i.d.r.v.) with mean zero and fixed variance $\sigma^2$.

The statistical models are broadly grouped into two categories:
(i) Linear models i.e. models in which all its parameters appear linearly. For example,

\[ Y = \theta_1 + \theta_2 X + u, \quad (1.3) \]

\[ Y = \theta_1 + \theta_2 X + \theta_3 X^2 + u. \quad (1.4) \]

\( \theta_1, \theta_2 \) and \( \theta_3 \) are known as parameters and are to be estimated from the model itself. The simple estimation procedures of parameters of the model, like Least Squares technique etc., provide a strong motivation to an investigator to opt for linear models. For details reference can be made of Johnston (1972), Saber (1977) and Draper and Smith (1981).

(ii) Nonlinear Models, i.e., the models in which at least one of its parameters appears nonlinearly.

Nonlinear models are also of wide importance in studies arising in the fields of agriculture, biology, business, chemistry, ecology, economics, engineering, forestry, physics, population studies, etc. It is a contemporary need that various statistical queries be expressed and solved with the help of some suitable nonlinear model.

The Nonlinear models are classified into following two groups:

(a) Nonlinear models which can be transformed into a form linear in its parameters. Such nonlinear models are also termed as intrinsically linear nonlinear models, for example

\[ Y = X^\beta . u \quad . \quad (1.5) \]

\( \beta \) is unknown parameter of the nonlinear model. The model can be transformed into a form linear in parameter by taking log of both
sides, we have

\[ \log Y = \beta \log X + \log u \]

\[ Y^* = \beta X^* + u^* \] (1.6)

where

\[ Y^* = \log Y \]

\[ X^* = \log X \]

\[ u^* = \log u. \]

Model (1.6) is now linear in parameters.

(b) Nonlinear models which can not be transformed into a form linear in parameters. Such models are also known as intrinsically nonlinear. For example:

\[ Y = e^{-\theta X} + u \] (1.7)

No transformation can bring (1.7) into a form in which its parameters appear linearly.

1.2 Estimation of Parameters in Nonlinear Statistical Models

For estimation of parameters in intrinsically nonlinear models, straightforward application of least squares method is not possible. However, there are several iterative procedures for the estimation of parameters.

There are three main methods for estimation of parameters in intrinsically nonlinear models. The first is the Linearization method with the help of Taylor’s series, the second is Steepest descent method and the third is Marquardt’s Compromise. All these methods are iterative in nature and computationally cumbersome, the computational intricacies increase progressively from first to third method. All the three
methods require prior intelligent guess values of parameters which in itself is a complicated task. The convergence of the solution is a big shortcoming of these methods. For the fuller account of these methods reference can be made of Bard (1974), Draper and Smith (1981), Gallant (1987) and Saber and Wild (1989). However, a brief outline of linearization method is presented below as the method will be further extended in the present study.

1.2.1 The linearization method with the help of Taylor’s series or Gauss-Newton Method

Let the nonlinear model be written as

\[ Y = f(x_1, x_2, ..., x_k, \theta_1, \theta_2, ..., \theta_p) + u_i \]  \hspace{1cm} (1.8)

\[ i = 1, 2, ..., n \]

\( Y_i \) is the \( i \)th observation on the dependant variable \( Y \).

\( x_1, x_2, ..., x_k \) represent \( i \)th observation on independent variables \( x_1, x_2, ..., x_k \).

\( \theta_1, \theta_2, ..., \theta_p \) are the parameters of the model (1.8). There are \( n \) observations on \( Y \) and each of \( X \) variables, \( u_i \)'s are i.i.d.r.v. and

\[ E u_i = 0 \quad \text{for all } i \]

\[ E (u_i u_j) = 0 \quad \text{for } i \neq j; \ i, j = 1, 2, ..., n \]

\[ = \sigma^2 \quad \text{for } i = j; \ i, j = 1, 2, ..., n \]  \hspace{1cm} (1.9)

\( u_i \)'s are also assumed to be normally distributed to felicitate the use of \( F \) - statistic for tests of significance.

The expression (1.8) can be abbreviated as
\[ Y_1 = f(X_1, \theta) + u_1 \]  

where  
\[ X_1 = (X_{11}, X_{21}, \ldots, X_{k1})' \]
\[ \theta = (\theta_1, \theta_2, \ldots, \theta_p)' \]

The deterministic component of (1.10) is
\[ Y = f(X_1, \theta) \]  

The function (1.11) will be used in Taylor's series for linearization purpose. Let \( \theta_0 = (\theta_{10}, \theta_{20}, \ldots, \theta_{p0})' \) be the initial guess values of the parameters \( \theta_1, \theta_2, \ldots, \theta_p \). These values are obtained from whatever source available. These initial values are expected to be hopefully improved on successive iterations. Let us expand the function (1.11) about \( \theta_0 \) with the help of Taylor's expansion and curtail expansion at the first derivatives, we obtain, approximately when \( \theta \) is near to \( \theta_0 \),
\[ f(X_1, \theta) = f(X_1, \theta_0) + \sum_{h=1}^{p} \left[ \frac{\delta f(X_1, \theta)}{\delta \theta_h} \right]_{\theta=\theta_0} (\theta - \theta_0) \quad (1.12) \]

If we write
\[ f_1^0 = f(X_1, \theta_0), \]
\[ \beta_h^0 = (\theta_h - \theta_{h0}), \]
\[ Z_{h1}^0 = \left[ \frac{\delta f(X_1, \theta)}{\delta \theta_h} \right]_{\theta=\theta_0} \]

(1.13)

The equation (1.10), approximately, reduces to
\[ Y_1 - f_1^0 = \sum_{h=1}^{p} \beta_h^0 Z_{h1}^0 + u_1 \]

(1.14)

equation (1.14) is the linear form of (1.10) to the selected order of approximation. The level of approximation will depend on closeness of \( \theta_0 \) to \( \theta \). The least squares technique can now be applied to expression (1.14) to yield least squares estimation of \( \beta_h^0 \). If we write expression (1.14) for its n observations, then it
can be written in matrix notation as
\[ y^0 = Z^0 \beta^0 + u \]
where,
\[ Z^0 = \begin{bmatrix} Z_{11}^0 & Z_{12}^0 & \cdots & Z_{1p}^0 \\ Z_{21}^0 & Z_{22}^0 & \cdots & Z_{2p}^0 \\ \vdots & \vdots & \ddots & \vdots \\ Z_{n1}^0 & Z_{n2}^0 & \cdots & Z_{np}^0 \end{bmatrix} = (Z_{h1}^0)_{n \times p} \]  
\[ b^0 = \begin{bmatrix} b^0_1 \\ b^0_2 \\ \vdots \\ b^0_p \end{bmatrix} \quad \text{and} \quad y^0 = \begin{bmatrix} Y_1 - f_1^0 \\ Y_2 - f_2^0 \\ \vdots \\ Y_n - f_n^0 \end{bmatrix} = Y - f^0 \]

The least squares estimates of \( \beta^0 = (\beta_1^0, \ldots, \beta_p^0) \)' is
\[ b^0 = (Z^0'Z^0)^{-1} Z^0'y^0 \]

The vector \( b^0 \) minimizes the sum of squares
\[ SS(\theta) = \sum_{i=1}^{n} (Y_i - f(X_i, \theta_0) - \sum_{h=1}^{p} \beta^0_h Z_{hi}^0)^2 \]

Let us write \( b^0_h = \theta_{h1} - \theta_{h0} \), then \( \theta_{h1}, h = 1, 2, \ldots, p \), can be thought of as an improved estimate of \( \theta \). The estimate \( \theta_{h1} \) is given same role to play as \( \theta_{h0} \) and we go through the same process as shown above from equations (1.12) to (1.18). This will lead to another estimate, \( \theta_{h2} \) expected to be further improved estimate of \( \theta \) and so on. This iterative process is continued till solution converges. That is, in successive iterations \( j \) and \((j+1)\),
\[ ||(\theta_{h(j+1)} - \theta_{hj})/\theta_{hj}|| < \delta \quad , \quad h=1, 2, \ldots, p \]  
where \( \delta \) is a specific small amount like .000001. At each stage of iteration, residual mean square, \( SS(\theta) \), is also calculated and
change in its value is observed. As the solution converges there will be no further change in SS(θ). Theoretically the method converges (Hartley, 1961). But in linearization method some times convergence may be very slow involving very large number of iterations. Some times solutions oscillate in reversing direction with increasing as well as decreasing sum of squares with solution converging eventually. There are cases where solution does not converge at all and even diverge resulting in increase in sum of squares SS(θ) without bound. Hartley (1961) suggested modifications to the method to improve the speed of convergence.

1.2.2 Steepest descent method

The steepest descent method is usually less favoured than the linearization method due to serious computational complexities. Although the method converges theoretically (Curry, 1944), the convergence may be very slow resulting into large number of iterations (Spang, 1962). In the year 1847, A.L. Cauchy set forth the method and was subsequently used by Temple (1939) and Levenburg (1944). The method centralizes to find the minimum of error sum of squares for the nonlinear model by an iterative procedure. The error sum of squares S(θ) for the nonlinear model (1.11) is

\[ S(θ) = \sum_{i=1}^{n} (Y_i - f(X_i, θ))^2 \]  

(1.21)

The crux of the method is to move from an initial guess point \( θ_0 \), along the vector with components

\[ -\frac{δS(θ)}{δθ_1}, -\frac{δS(θ)}{δθ_2}, ..., -\frac{δS(θ)}{δθ_p} \]  

(1.22)
whose magnitudes alter rapidly on moving forward. This technique has got tremendous importance in Design of Experiments for determination of optimum conditions such as obtaining best operating conditions for a process by maximising yield or parity of item or minimising cost. For further details reference can be made of Davies (1956), Fletcher (1980) and Gill et. al. (1981). Beginning from one particular region of parameter space, several observations are made by choosing n combinations of values of $\theta_1$, $\theta_2$, ..., $\theta_p$ and $S(\theta)$ is calculated for those combinations. Usually, a two level factorial design pattern is chosen for the observations. Assume Calculated $S(\theta)$ as dependent variable and combinations of values of $\theta_1$, $\theta_2$, ..., $\theta_p$ as independent variable, a model of the form

$$\text{"observed } S(\theta)" = \beta_0 + \sum_{h=1}^{p} \beta_h (\theta_h - \theta_h^-) / s_h^2 + u$$  \hspace{1cm} (1.23)

is fitted by least squares method. $\theta_h^-$ denotes the mean of the values, $\theta_{h,i}$, $i=1,2,\ldots,n$, of $\theta_h$ used in the runs and $s_h^2$ is an adjusting factor selected in a manner that

$$\sum_{i=1}^{n} (\theta_{h,i} - \theta_h^-)^2 / s_h^2 = \text{Constant}$$  \hspace{1cm} (1.24)

The estimated coefficients

$$-b_1, -b_2, \ldots, -b_p$$  \hspace{1cm} (1.25)

indicate the direction of steepest descent.

This implies that if linear approximation of nonlinear function (1.11) is correct, the maximum decrease in $S(\theta)$ will be obtained by shifting in direction which contains points such that

$$(\theta_h - \theta_h^-) / s_h \alpha - b_h$$  \hspace{1cm} (1.26)
\[
\theta_{h} - \frac{\theta_{h}^2}{s_{h}} = \lambda b_{h} \quad (1.27)
\]
\[
\theta_{h} = \theta_{h}^2 - \lambda b_{h} s_{h} \quad (1.28)
\]

where \( \lambda > 0 \) and is a constant.

For a number of values given to \( \lambda \), the path of steepest descent is followed as long as \( S(\theta) \) decreases. When \( S(\theta) \) does not decrease, another experimental design is tried and the process is continued till it converges to the estimate of \( \theta \) i.e. \( \hat{\theta} \) which minimizes \( S(\theta) \).

### 1.2.3 Marquardt’s Compromise

The Marquardt’s Compromise method adopts a compromise between the Linearization method by Taylor’s series and the Steepest descent method. It adopts best qualities of both the methods ignoring their sort comings. The method was first developed by Marquardt (1963) based on the work of Levenberg (1944). Improvement to it was also suggested by Smith and Shanno (1971). In fact none of the three methods can be categorized as best for all nonlinear models. A method may have preference over other in a particular situation. However, in Marquardt’s Compromise procedure, solution converges and process of convergence does not slow down as in case of Steepest decent method. Again, method is iterative and computationally complicated.

### 1.2.4 Transformation of nonlinear model into form linear in its parameters

The various complexities and computational
intricacies of the estimation procedures of parameters in intrinsically nonlinear models motivate a user to adopt nonlinear models which can be transformed into a form which is linear in its parameters, even to a situation where an intrinsically nonlinear model is a proper choice. However, transforming a nonlinear model into a model linear in its parameters does not solve all the problems, on the contrary, it gives rise to several other complexities. The various techniques of estimation of parameters in linear models can well be applied to transformed model which is linear in its parameters but the properties of estimators are valid to the parameters of the transformed model and may not be valid for parameters of the original model. In several cases investigator is forced to assume a particular type of error term in an arbitrary manner for the feasibility of the transformation, such as, assumption of a multiplicative error term in the model (1.5), whereas on theoretical considerations an additive error term may be a proper choice. In order to get rid of computational intricacies, one even opts for ad hoc procedure like method of partial sums, three selected points etc. even though the estimators possess poor statistical properties.

1.3 The Modified Exponential or Monomolecular Growth Function
The function with deterministic component
\[ Y = \alpha + \beta \rho^x, \quad 0 < \rho < 1 \] (1.29)
is the most useful intrinsically nonlinear model, whenever the phenomena exhibits asymptotic behaviour. The form (1.29) is
popularly known as Modified exponential Curve belonging to the family of Convex/Concave Curves (Ratkowsky, 1989). Amongst the three parameters $\alpha, \beta$ and $\rho$, the parameter $\alpha$ represents the asymptotic value of $Y$, $\beta$ the change in $Y$ when $X$ changes from 0 to $\infty$ and $\rho$ represents the factor by which the deviation of $Y$ from its asymptotic value is reduced for a unit value increase in $X$. The change of origin of $X$, only changes the value of parameter $\beta$.

Historically, the model (1.29) has provided a great deal of motivation for using nonlinear functions. Following are important reparameterizations of the form (1.29).

Let $\rho = e^{-\gamma}$, where $\gamma$ is a constant, the function (1.29) can be written as,

$$Y = \alpha + \beta e^{-\gamma x} \quad (1.30)$$

which represents learning curve in psychology.

Let,

$$Y_0 + d = \alpha, \quad d = -\beta, \quad 10^{-k} = \rho$$

where $d$, $\beta$ and $k$ are constants and $Y_0$ is initial value of dependent variable $Y$. Then the form

$$Y = Y_0 + d \left( 1 - 10^{-kx} \right) \quad (1.31)$$

and, alternatively,

let

$$A = \alpha, \quad -A(10^{-C^*b^*}) = \beta, \quad 10^{-c^*} = \rho$$

where $A, C$ and $b$ are constants, then again the form

$$Y = A \left( 1 - 10^{-c^*(x+b^*)} \right) \quad (1.32)$$

defines Mitscherlich’s law in agriculture.

In the production function study the form (1.29) defines Spillman function (Heady and Dillon, 1960). In fisheries research, It is known as Von Bertalanffy law. It also represents
Newton's law of cooling for a body over time. The function (1.29) has several applications in the field of life sciences, Industrial engineering, Electrical engineering and Chemical engineering problems. It is also used in graduation of life table data for force of mortality and is known as Makeham's second modification. It is also used in saturation curves or magnetization curves study. Ratkowsky (1983) has mentioned several reparameterizations of model (1.29) which are similar in behavioural properties. They are,

\[ Y = \alpha - \beta \exp(-\rho x) \]  
(1.33)
\[ Y = \alpha \{1 - \exp[-(x + \beta) \rho]\} \]  
(1.34)
\[ Y = \alpha - \exp[-(\beta + \rho x)] \]  
(1.35)
\[ Y = \alpha - \exp(-\beta)\rho^x \]  
(1.36)
\[ Y = \frac{1}{\alpha} - \beta \rho^x \]  
(1.37)
\[ Y = \text{Exp}(\alpha) - \beta \rho^x \]  
(1.38)

The model (1.29) and all its forms of reparameterizations (1.30) to (1.38) are intrinsically nonlinear models. The straight-forward application of least squares method is not possible to estimate parameters of the model. The values of \( \rho \) other than \( 0 < \rho < 1 \) are not of much practical utility. It is very difficult to come to any conclusion about which of the form (1.29) to (1.38) fits the data better particularly when small sample sizes are considered. A basis for choosing among them is mentioned in section 1.5.

Some ad hoc procedures like method of partial sums, and method of three selected points (Croxton & Cowden, 1964) were in practice for estimation of parameters of modified exponential
growth curve (1.29), due to computational convenience, although estimators possess poor statistical properties. Another approach was to guess the asymptotic value $\alpha$ by visual inspection of results or their graph and then to transform the equation into linear in parameters so that Least squares method may be directly applicable, here again estimators lack properties of good statistical estimators.

Hartley (1948) suggested the method of "Internal Least Squares", which consists in bringing a nonlinear relation into linear form with the help of differential equation and one gets regression equation in which dependant variable $Y$ is related to its own repeated sums and variable $X$ as independent variables. He demonstrated the application of method to function (1.29) as well as to Gompertz and Logistic curves. Hartley (1959) and Patterson and Lipton (1959) have further reviewed the technique of "Internal Least Squares".

Stevens (1951) has described a method of successive approximation for estimation of parameters of (1.29) which depends upon a reasonably accurate initial estimator of $\rho$. He showed a procedure for constructing and solving the maximum likelihood equations and provided tables for assisting this process. Gomes (1953) had developed this technique further and improved speed of convergence. He had also extended the tables to cover a wider range of values. Gomes (1953) remarked that, in Stevens Method, if initial estimate of $\rho$ is not good enough, the convergence is, some times, rather slow. Apart from it there is no certainty, from the
theory, that the iteration process will converge. (Levenberg, 1944
and Hotelling, 1947).

For estimation of parameters of the model (1.29)
from observations on \( Y \) and \( X \) variables by the least squares
method, \( \rho \) is the parameter which creates hurdle. If \( \rho \) is known, \( \alpha \)
and \( \beta \) can be easily estimated from a linear regression of \( Y \) on \( \rho^X \).
Patterson (1956) has proposed a simple method of estimating \( \rho \) as
the ratio of two linear functions of dependant variable \( Y \) and
showed it as highly efficient for a moderate number of values of
\( X \). Patterson (1958) had further extended the study and alternative
estimates of \( \rho \) were suggested.

The technique of estimating \( \rho \) by \( \hat{\rho} \) and then
regressing \( Y \) on \( \hat{\rho}^X \) for obtaining estimates of \( \alpha \) and \( \beta \) was further
Khatri (1965) have suggested inclusion of a linear term in the
functional form (1.29) which is expected to improve fitting in
several cases.

Finney (1958) discussed in length the efficiencies
of techniques for estimating parameters of the functional form
(1.29), suggested by Hartley(1948), Patterson (1956,1958) and
several other authors. The properties of the various estimators
are considered both for constant variance model and Increasing
variance model.

1.4 Sigmoidal or 'S' - shaped Growth Curves

There are several functions belonging to the family
of sigmoidal or 'S' shaped growth curves out of which Gompertz and Logistic Curves are widely used in practice. Following are the forms of various sigmoidal curves.

Gompertz: \[ Y = e^{\alpha + \beta e^{\rho x}} \] (1.39)

Logistic: \[ 1/Y = \alpha + \beta e^{\rho x} \] (1.40)

Richards: \[ \alpha / [1 + \exp(\beta - \rho x)]^{1/\delta} \] (1.41)

Morgan-Mercer-Flodin: \[ Y = \left(\beta \rho + \alpha x^\delta\right)/(\rho + x^\delta) \] (1.42)

and Weibull type: \[ \alpha - \beta \exp(-\rho x^\delta) \] (1.43)

\(\alpha, \beta, \rho, \) and \(\delta\) are parameters of the model. Ratkowsky (1983) mentioned several reparameterizations of models (1.39) to (1.43), which are similar in behavioural properties and remarked that, it is very difficult to decide about which of the models (1.39) to (1.43) fits the data better particularly for small sample sizes.

Sigmoidal Curves do not possess maxima or minima but have an inflection point. These curves are widely used in the field of biology, engineering, economics and population studies.

The Gompertz and Logistic forms can also obtained by allowing certain transformations only in the functional form (1.29). The transformed equation with

\[ Z = \log Y = \alpha + \beta e^{\rho x} \] (1.44)

leads to Gompertz equation, where as,

\[ Z = 1/Y = \alpha + \beta e^{\rho x} \] (1.45)

gives Logistic Curve.

The other reparameterizations of Gompertz function reported in Ratkowsky (1983); and Draper and Smith (1981), which are similar in behavioural properties are
\[ Y = \alpha \exp(-\beta e^{-\rho X}) , \quad (1.46) \]
\[ Y = \alpha \exp[-\exp(\beta - \rho X)] , \quad (1.47) \]

Richards (1959) remarked that Gompertz curve is of more utility in population studies and animal growth than in botanical field. Medawar (1940) used Gompertz function in the study of growth of a chicken's heart, however, Amer and Williams (1957) used it in the study of leaf-area growth.

Similarly there are several other reparameterizations of Logistic or auto catalytic growth function mentioned in Ratkowsky (1983) which are similar in behavioural properties. They are,

\[ Y = \alpha / 1 + \exp(\beta - \rho X) \quad (1.48) \]
\[ Y = 1 / \alpha + \beta \exp(-\rho X) \quad (1.49) \]
\[ Y = \alpha / 1 + \exp(\beta) \rho^X \quad (1.50) \]
\[ Y = 1 / \alpha + \exp(\beta) \rho^X \quad (1.51) \]
\[ Y = \alpha / 1 + \exp(-\rho X) \quad (1.52) \]

Oliver (1966) used the functional form (1.52) for the maximum likelihood estimation of the Logistic growth function. Truett, Cornfield and Kannel (1967) highly appreciated the utility of logistic curve in heart study data, in fact it was a historical paper which was a landmark in the history of application of logistic regression and since then it has been tremendously used in the study of health sciences.

The estimation of parameters of Gompertz or logistic function is oftenly done through ad hoc procedures like method of partial sums, method of three selected points etc. The
other standard techniques like linearization method, Steepest descent and Marquardt’s compromise etc. are less in practice due to computational intricacies. For further details reference can be made of Ratkowsky (1983), Hosmer and Lemeshow (1989).

1.5 Choice for Close-to-Linear Models among Nonlinear Models

In contrast to least squares estimators of linear models, nonlinear least squares estimators are not unbiased, normally distributed, minimum variance estimators, However, these estimators possess these properties only asymptotically. A nonlinear model is named as to be close-to-linear, nonlinear model if estimators of its parameters are close to being unbiased, normally distributed and minimum variance estimators. Ratkowsky (1983) strongly advocated search for close-to-linear models among nonlinear models because such models should have both a low intrinsic nonlinearity and a low parameter-effects nonlinearity. The concepts of intrinsic nonlinearity and parameter-effects nonlinearity were developed by Bates and Watts (1980) for evaluating nonlinear behaviour of a nonlinear model. For close-to-linear nonlinear models, the speed of convergence of the unmodified Gauss-Newton method will usually be very rapid. Ratkowsky (1989) discussed various basic principles of reparameterization in search for close-to-linear nonlinear models and suggested apart from other techniques the use of "Expected-Value Parameters" of type suggested by Ross (1975) which exhibit close-to-linear behaviour. The reparameterizations
containing expected-value parameters are more cumbersome in appearance than original expressions and expected-value parameters may appear more than once. However, reparameterizations of model with expected-value parameters is still preferred because for such models, there is rapid convergence to least-squares estimates, initial parameter estimates are easy to obtain and expected value parameters provide better inference than original parameters. Ratkowsky (1989) has remarked that for almost all nonlinear models of interest, the intrinsic nonlinearity is typically low. Which implies that if a nonlinear model is found to be far-from-linear, nonlinearity is most likely to be due to parameterization and a suitable reparameterization be sought. It thus appears that main thrust is on choosing nonlinear models close-to-linear behaviour.

The emphasis on models with linearly appearing parameters and nonlinear models possessing close-to-linear behaviour, is based on the fact that estimators of parameters possess good statistical properties. In case of linear models least squares estimators are unbiased, normally distributed and minimum variance estimators where as in the case of close-to-linear nonlinear models, least squares estimators, are close to being unbiased, normally distributed and minimum variance estimators. These facts motivate us to use models linear in parameters as an alternative to nonlinear models successfully approximating the later.

In the present investigations we have approximated a nonlinearly appearing parameter to its equivalent linear form,
thus bringing a nonlinear model into its approximate "linear in parameters" form. Such an approximation of a nonlinear model to a model linear in parameters allows direct use of least squares method to the approximate linear model, yielding unbiased, normally distributed and minimum variance estimators of the parameters. The usual tests of significance procedures and confidence intervals then become directly applicable.

In fact the basic problem in statistical investigations about a model is to estimate its parameters and then to apply testing of hypothesis etc. Sizeable econometric study is devoted to the problem of estimation of parameters and it is after this stage that further studies, on a good fit to the data for the purposes of representation, prediction of Y for given X, model selection, stability of relationship etc. could be done.

1.6 Frame of the Thesis

The whole thesis has been divided into six chapters including the first one on introduction.

Chapter 2 describes a three parameter model approximating modified exponential curve and its various reparameterizations. The model has been studied both mathematically and empirically on different data sets.

Chapter 3 deals with other applications of the three parameter model with an inverse term for the determination of optimum cluster size, relationship between plot size and coefficient of variation and to many other concave/convex
functions reported in Draper and Smith (1981); and in chapter 4 of Ratkowsky (1989).

Chapter 4 studies derivation and application of a five parameter model to family of Sigmoidal or "S-shaped" curves. The utility of the five parameter model has been studied empirically on data sets arising from different fields of study. The application of five parameter model has also been studied in asymptotic regression problems. It has brought a sizeable improvement over three parameter model in the most of the cases.

Chapter 5 describes an improved method of utilizing a priori information on parameters in Linearization or Gauss-Newton method. The method of mixed estimation of Theil and Goldberger (1961) has been extended to nonlinear models.

Chapter 6 summarizes results of the present study.