Chapter-One

Introduction And Review of Literature
Chapter - I

Introduction and Review of Literature:

A model is an equation or a set of equations which describes a phenomenon. A statistical model incorporates a random variable whose properties are defined by a distribution function. Y represents dependent variable or response variable or variable in whose study we are interested and X is independent variable or predictor or explanatory variable.

The statistical model can be written as

\[ Y = f(X) + U \]  \hspace{1cm} (1.1)

It defines the functional form which relates Y and X. The functional form f can be either linear or non-linear. U is a random variable which has well defined statistical properties.

1.1 Linear Models:

In a linear model, all parameters appear linearly. A parameter is said to be appearing linearly in a model if the partial differential of order one of the model with respect to parameter is independent of parameter.

Suppose

\[ Y = \alpha + \beta X + U \]  \hspace{1cm} (1.2)

defines a statistical model in which \( \alpha, \beta \) are parameters of the model.

The deterministic part of the model (1.2) is
\[ Y = \alpha + \beta X \] \hspace{1cm} \text{(1.3)}

The first order partial differential of (1.3) with respect to \( \alpha \) and \( \beta \) is

\[ \frac{\partial Y}{\partial \alpha} = 1 \] \hspace{1cm} \text{(1.4)}

And

\[ \frac{\partial Y}{\partial \beta} = X \] \hspace{1cm} \text{(1.5)}

It is observed that equations (1.4) and (1.5) do not contain parameters or in other words \( \frac{\partial Y}{\partial \alpha} \) is independent of \( \alpha \) and \( \frac{\partial Y}{\partial \beta} \) is independent of \( \beta \). Thus \( \alpha \) and \( \beta \) are said to appear linearly in equation (1.3).

The use of a linear model is very common. The famous least square technique is directly applicable to estimate its parameters. The estimates of parameters are unbiased, normally distributed and have minimum variance in the class of unbiased estimators. It is therefore, Least square estimates are also known as Best Linear Unbiased Estimators (BLUE), the minimum possible variance is also known as Minimum Variance Bound (MVB). It is expected that a good estimator must possesses these properties. A linear regression model has a sum of squares surface, which are ellipse in shape for true parameter model, ellipsoid are hyper ellipsoid for three or more parameter models.

The Gauss-Newton method used for least square estimates for a linear model converges in a single step irrespective of the initial estimates,
which may be far remote from least square estimates. As a matter of fact there is no need to use Gauss-Newton method of estimation in linear model as there is an explicit formula for obtaining least square estimates for linear model using least square method.

1.2 Non-Linear Models:

A model is said to be non-linear if at least one of its parameters appear non-linearly. In other words, in a non-linear model at least one first order partial derivative with respect to its parameter should involve parameter, for example a non-linear model with its deterministic part is described as

\[ Y = e^{\theta X} \quad (1.6) \]

where \( \theta \) is parameter of the model. The partial differential of equation (1.6) with respect to \( \theta \) is

\[ \frac{\partial Y}{\partial \theta} = e^{\theta X} X \quad (1.7) \]

It is observed that partial differential of model (1.6) with respect to its parameter \( \theta \) is not independent of parameter, hence \( \theta \) appears non-linearly in model (1.6) and the model is classified as non-linear model. Non-linear regression models are different in nature as compared to linear regression models in the sense that the least square estimators of parameters of non-linear models are neither unbiased nor normally distributed nor
minimum variance estimators. The least square estimators of non-linear models have these properties only asymptotically.

Ratkowsky (1989) remarked that some non-linear regression models have estimators which are badly biased with a highly asymmetric, long tailed, non-normal distribution with a sample variance very high in magnitude as compared to minimum variance bound (MVB). He further remarked that in general, the smaller the sample size, the greater is the extent of non-linearity in non-linear models.

The Gauss-Newton method to obtain least square estimates for a non-linear model converges in several iterations. In fact, convergence of solution in non-linear least square estimation is a serious problem. Ratkowsky (1989) remarked that convergibility of solution to obtain the least square estimates in non-linear regression depends heavily on parameterization of the model and on initial guess values of estimates of the parameters. If non-linear behaviour increases, the problem of convergence becomes more and more difficult. The contours of equal residual sum of squares shifts more and more from elliptical shape. The closeness of initial guess values of the estimates of the parameters decides the fate of convergence of solution. In some cases the solution for obtaining the least square estimate in non-linear estimation may diverge also. Some times it takes very large number of iterations.
The application of non-linear models is very wide and it is frequently used in studies arising in the fields of Agriculture, Fisheries, Forestry, Biology, Animal Sciences, Economics, Business, Chemistry, Physics, Ecology, Engineering, Horticulture and Population studies etc.

1.3 Types of Non-Linear Models:

Several Authors have classified non-linear models in different categories, keeping different considerations in their mind. The first classification is due to Draper & Smith (1998).

1.3.1 Classification due to Draper and Smith:

Draper and Smith (1998) have classified non-linear models into two categories based on the fact whether the model can be transformed into linear in its parameters or not.

(a) Intrinsically Linear Non-Linear Models:

A non-linear model which can be transformed into a form which is linear in its parameters is said to be intrinsically linear non-linear model. e.g.

\[ Y = e^{\theta X} U \]  \hspace{1cm} (1.8)

on taking log transformation of (1.8)

\[ \log Y = \theta X + \log U \]  \hspace{1cm} (1.9)
In equation (1.9) parameter $\theta$ appears linearly in which dependent variable $Y$ has been transformed into $\log Y$ and the random variable $U$ has been transformed into $\log U$. The model (1.8) is said to be linear non-linear model under criterion of Draper and Smith (1998). The validation of assumptions of random variable $U$ for $\log U$ possess serious problem before a Statistician.

(b) Intrinsically Non-Linear Models:

A non-linear model which can not be transformed into a form, linear in its parameters, is said to be intrinsically non-linear model. e.g.

$$y = \alpha + \beta \rho^x$$

(1.10)

Where $\alpha, \beta$ and $\rho$ are constants. It is observed that no transformation can bring equation (1.10) in to a form in which it’s parameters appear linearly. Hence a model described by equation (1.10) is said to be intrinsically non-linear.

1.3.2 Classification due to Ratkowsky:

Another classification of non-linear models is due to Ratkowsky (1983). He has grouped non-linear models using different criteria. He grouped non-linear models as close-to-linear non-linear models and far-from-linear non-linear models. This classification utilizes
properties of least squares estimators of parameters. In linear models the least square estimators of parameters are unbiased, normally distributed and have minimum variance property. If least square estimate of a parameter of non-linear model is slightly biased with a distribution close to normal distribution with a variance slightly more than minimum variance bound, the estimator of the parameter of non-linear model is said to be close-to-linear and non-linear model is classified as close-to-linear non-linear model.

On the contrary if the least square estimator of parameter of non-linear model is badly biased with a distribution far from normal and possessing a variance very high as compared to minimum variance bound (MVB), the non-linear model is classified as far-from-linear non-linear model.

Ratkowsky (1989) remarked that a close-to-linear model has several good properties as compared to far-from-linear model. The first one is that there is almost sure convergence i.e. convergence to global optimum. The most important aspect is that far-from-linear model may possess more than one minimum in their sum of squares surface and far-from-linear model may converge at a point which is not a global optimum but a local optimum. The interpretation of results about parameter estimates in far-from-linear model become not only difficult but also dubious. It is important to note that the problem of convergence of solution is more important than number of
iterations required for convergence of solution for a particular model. It is not necessary that in a close-to-linear model, convergence will occur more rapidly or in fewer numbers of iterations as compared to far-from-linear model. What is important in close-to-linear model is that, chances of convergence of solution from a reasonable starting guess values are significantly high. The least square parameter estimates of close-to-linear non-linear model are very close to be unbiased and have distribution closely approximating a normal distribution. The confidence intervals for each parameter will be close to being exact. A far-from-linear model will have estimators of parameters distributed non-normally and their standard errors are highly biased, leading to wrong inferences. The extent and degree of non-linearity in this case is generally not known. Ratkowsky (1989) further remarked that the amount of non-normality in parameter estimates of far-from-linear model depends also on sample size; therefore, comparisons based on sample sizes will lead to differ on amount of non-normality. He strongly recommended search and use of non-linear models possessing close-to-linear behaviour.
1.3.3 Classification due to Bates and Watts:

Bates and Watts (1980) have developed new measures of assessing non-linearity of statistical models. They named it as intrinsic non-linearity and parameter-effects nonlinearity. They demonstrated that intrinsic non-linearity can not be reduced for a statistical model but parameter-effects due to reparametrisation of model can reduce the parameter-effects non-linearity. Often reparametrisation enables us to obtain estimators which exhibit close-to-linear behaviour. Recommendations have been made for search of non-linear models possessing low intrinsic non-linearity and close-to-linear behaviour.


The measure of skewness and biased assessment also helps in evaluation of extent of non-normality and close-to-linear behaviour of non-linear models. Reparametrisation of parameters in non-linear model is an important tool in bringing a non-linear model into a form possessing the property of close-to-linear behaviour.
Ratkowsky (1983 and 1989) has given a great emphasis on reparametrization of non-linear models. Through reparametrization the model is brought into a form in which parameters behave more closely to linear behaviour. Ratkowsky (1989) suggested guidelines in choosing parameters with better properties. One way is to perform simulation studies and look at histograms of the estimates of each parameter separately. The parameters having estimates possessing distribution close to normal distribution and are close-to-linear, do not require reparametrization. Those parameter estimates which are far from normal distribution and have long tailed distribution require reparametrization.

Ratkowsky (1989) mentioned that intrinsic non-linearity as defined by Bates and Watts (1980) is small in almost all non-linear models of practical interest. The major contribution in non-linearity is due to parameter-effects nonlinearity. It is therefore, suggested that a suitable reparametrization of non-linear model should be carried out so as to minimize its non-linear behaviour.

Rosh (1975) has defined a class of parameters exhibiting close-to-linear behaviour as expected value parameters. The extent of non-linearity in expected value parameter is also small in the cases when values of expected value parameters are within observed range of data.
1.4 Some Important Non-Linear Statistical Models:

1.4.1 Modified Exponential or Monomolecular Growth Model:

The model with deterministic component

\[ Y = \alpha + \beta \rho^x, \quad 0 < \rho < 1 \]  

is the most useful intrinsically non-linear model, whenever the phenomena exhibits asymptotic behaviour. The form (1.11) is popularly known as modified exponential curve belonging to the family of convex / concave curves (Ratkowsky 1989).

Among the three parameters \( \alpha, \beta, \) and \( \rho \), the parameter \( \alpha \) represents the asymptotic value of \( Y \), \( \beta \) the change in \( Y \) when \( X \) changes from 0 to \( \infty \) and \( \rho \) represents the factor by which the deviation of \( Y \) from its asymptotic value is reduced for a unit value increase in \( X \). The change of origin of \( X \), only changes the value of parameter \( \beta \). Historically, the model (1.11) has provided a great deal of motivation for using non-linear functions.

1.4.2 Sigmoidal or ‘S’ Shaped Growth Curves:

Family of sigmoidal curves has several important curves which have been used in various spheres of studies by researchers. The sigmoidal models have a point of inflection but no maxima or minima, they have typically a ‘S’ shape and therefore are also known as ‘S’ shape models. There are many important curves which
belong to family of S-shaped curves. A very versatile member of this family is Logistic curve.

(a) Logistic Curve:

The first and foremost important curve in the family of sigmoidal curves is Logistic curve. It is also popularly known as Perl-Reed curve. Its functional form was first derived by Verhulst in the year 1838, in which it was used to describe growth in the size of a population or organ. It exhibits growth phenomenon in which growth rate does not decline steadily on the contrary it increases to a maximum and then it decreases steadily to zero. The point where growth rate attains its maximum value and starts declining is point of inflection. The logistic curve is symmetric about its point of inflection. The functional form of logistic curve is

\[ Y = \frac{k}{1 + e^{a+bx}} \]  \hspace{1cm} (1.12)

where k, a and b are constants. Its point of inflection is at \( X = \frac{k}{2} \), the asymptotes are at \( X = 0 \) and \( X = k \), the curve is symmetric at \( X = \frac{k}{2} \).

The curve is concave upward for \( X < \frac{k}{2} \) and convex upward for \( X > \frac{k}{2} \).

Thus the curve attains S-shape.

Ratkowsky (1983) has listed several reparametrization forms of logistic model which are similar in its behavioral properties as possessed by
logistic model expressed by relation (1.12). The relations (1.13) to (1.18) are all reparametrized forms of logistic model.

\[
Y = \frac{\alpha}{1 + \exp(\beta - \gamma^x)} \quad \text{------ (1.13)}
\]
\[
Y = \frac{1}{\alpha + \beta \exp(-\gamma^x)} \quad \text{------ (1.14)}
\]
\[
Y = \frac{1}{\alpha + \beta \gamma^x} \quad \text{------ (1.15)}
\]
\[
Y = \frac{\alpha}{1 + \exp(\beta)\gamma^x} \quad \text{------ (1.16)}
\]
\[
Y = \frac{1}{\alpha + \exp(\beta)\gamma^x} \quad \text{------ (1.17)}
\]
\[
Y = \frac{\alpha}{1 + \beta \exp(-\gamma^x)} \quad \text{------ (1.18)}
\]

It is very interesting to note that relation (1.15) can be rearranged to write it as

\[
\frac{1}{Y} = \alpha + \beta \gamma^x \quad \text{------ (1.19)}
\]

The right hand side of equation (1.19) defines the deterministic component of asymptotic regression model (1.11). Thus the deterministic component of logistic model (1.19) is same as that of asymptotic regression model with a change that it relates to $1/Y$ and not $Y$ as in the case of asymptotic regression model. It leads to an important relationship between
asymptotic model and logistic regression model that dependent variable Y of former is written in reciprocal form for the later.

Logistic model has been found very useful in several area of studies arising from different fields. It has been extensively used in population studies, Bio-chemical problems and Health Sciences studies. Some of the important applications and references are listed below.

Robertson (1908) has studied the Bio-Chemical nature of growth process using logistic regression model and found that his data set confirms the pattern of growth process following logistic regression law. Robert et al (1999) have used logistic, Weibull, Gompertz and other models to select a model for wheat kernel growth. Santos et al (1999) had conducted their study on variation in heights of Pantaneiro horses. They have used Richards, Logistic and Gompertz models for this purpose. Their results indicate that female horses mature earlier.

Palahi et al (2003) used logistic models to predict mortality and probability of a tree to survive using growth data obtained by measuring diameter growth of Pinus Sylvestris L., developed in north-east Spain. Perlich C. (2003) have presented a large scale experimental comparison on logistic regression and tree induction assessing classification accuracy and the quality of ranking based on class membership probabilities.
Prajneshu and Chandran (2005) have studied compound growth rates in agricultural growth experiments using logistic non-linear models etc. in place of existing exponential models.

Bondel H.D. (2005) has described a new robust class of estimation procedures for estimating parameters of logistic regression model. He has compared them with maximum likelihood estimators (MLEs).

Lambe et al (2006) have studied logistic, Gompertz, Richards and other models to describe the pattern of growth in two contrasting breeds of lambs for the period starting from birth to slaughter. Vedenov and Pesti (2008) have compared logistic model with other models using nutritional response data. Avanza et al (2008) has suggested a statistical model that best describes the pattern of fruit growth of sweet orange (Valencia late) in Argentina. They have considered logistic, Gompertz and other models and compared them for deciding suitability to describe the pattern of growth.

In the studies relating to growth behaviour in Japanese quail, Aggrey (2009) has used logistic regression model and found it to be very useful. Keskin et al (2009) has used logistic regression for studying growth in live weight of Konya Merono. Akbar et al (2009) has used logistic regression in citrus food data obtained from fruit crop of Pakistan and have compared it with other models for the purpose of statistical modeling for citrus yield in Pakistan. Riazoshams and Midi (2009) carried out analysis of
chicken growth data. They have used six models and arrived at a model which is best suited for chicken growth data.

(b) Gompertz Curve:

Gompertz curve was developed by Benjamin Gompertz in the year 1825. He has used it in discussing the law of human mortality. It was used in Actuarial studies, Biological studies and also in Economic phenomenon. Its functional form can be written as

\[ Y = k.e^{-e^{(a-bx)}} \]  

--- (1.20)

k and b are positive quantities. It is apparent that as X becomes negatively infinite, Y will approach zero and as X becomes positively infinite Y will approach k. Thus its asymptotes are zero and k. It is asymmetric about its point of inflection.

The Gompertz curve have point of inflection at \( X = \frac{a}{b} \) and the ordinate, at point of inflection will be \( Y = \frac{k}{e} \). approximately when 37% of the final growth have been reached.

Ratkowsky (1983) has given some reparameterized form of Gompertz model which are listed below.

\[ Y = \alpha \exp[-\exp(\beta - \gamma X)] \]  

--- (1.21)

\[ Y = \exp(\alpha - \beta \gamma X) \]  

--- (1.22)
The relation (1.22) can also be written as

\[ \log Y = \alpha - \beta \gamma^x \]  

The relation (1.23) is log transformation of equation (1.22). It is noted that the right hand side of equation (1.23) is identical in nature as equation (1.11) which was defined in asymptotic regression model. From equation (1.23) we learn that the Gompertz model relates log(Y) with a deterministic component \( \alpha - \beta \gamma^x \), which is identical to deterministic component of asymptotic regression model. These discussions provide an inherent relationship between Gompertz model and asymptotic regression model.

Wrist (1926) have used Gompertz curve in Biological growth studies. Davidson (1928) used it to represent the growth in body weight of cattle.

Weymouth et al (1931) have used Gompertz curve in studies of growth in shell size of the razor clam. Weymouth and Thompson (1931) applied Gompertz growth curve for studying growth of pacific cockle.

Medawar (1940) has applied Gompertz model to data-sets relating to studies on growth of a chicken’s heart. Amer and Williams (1957) has studied the pattern of growth in leaf-area using Gompertz model.

Richards (1959) remarked that Gompertz model is most powerful model in population studies and animal growth studies.
Alessandra et al (2002) have studied several growth models including Gompertz logistic and models using data on weight and age of Holstein Heifers females. The comparison between several models was made and Gompertz model was found to be the most suitable for his data sets.

Roush and Branton (2004) have compared Gompertz model with other models in the studies related to poultry growth model regarding Genetic behaviour. Sengul and Kiraz (2005) has made comparative study of Gompertz, Logistic, MMF (Morgan-Mercer-Flodin) and Richards curve using data on male and female white turkey poults.

Vuori K et al (2006) has described simulation study for estimation of Gompertz model. Cetin et al (2007) has conducted their study to compare growth curves of male and female partridges bird. They have used Gompertz, Richards's models and concluded that Gompertz model had better explanation for growth phenomena of male and female partridges as compared to other models.


Yamano T. (2009) has studied Gompertz function in the analytical entropy expression in terms of growth velocity. Mazmanoglu A. and Unlu A.R. (2009) have used Gompertz in non-linear regression for
studies relating to live eye lens weight and age measurements of the European rabbits in Australia.

(c) Richard's Model:

Richard's model represents a very flexible growth model and is represented by

\[ Y = c + a(1 + b e^{gY})^\lambda \quad \text{(1.24)} \]

where \( a, b, c, g \) and \( \lambda \) are parameters of the model.

The Richard's model translates into Logistic model when \( \lambda = -1 \), similarly it represents Gompertz model for \( \lambda = \pm \infty \). And it assumes Bertalanffy function for \( \lambda = 3 \).

The point of inflection of Richard's model is \(-\frac{1}{g} \log(-b \lambda)\), which is not in proportion to its asymptote.

Ratkowsky (1983) has given some other reparametrization forms of Richard's model.

\[ Y = \frac{\alpha}{[1 + \exp(\beta - \gamma x)]^{1/\delta}} \quad \text{(1.25)} \]

\[ Y = \frac{\alpha}{[1 + \beta \exp(-\gamma x)]^{1/\delta}} \quad \text{(1.26)} \]

\[ Y = \frac{\alpha / \beta}{[1 + \exp(-\beta(1 + \delta)(X - \gamma))]^{1/(1+\delta)}} \quad \text{(1.27)} \]
where $\alpha, \beta, \gamma, \rho$ and $\delta$ are parameters of the model.

Park and Lim (1985) have used Richard’s model with other models for estimation of asymptotes of disease progress curve. They have used Marquardt’s compromise method of estimation. Zhang S.Y. and Lei Y.C. (2004) have compared Richard’s model with other growth models from forestry prospective.

Khamis and Ismail (2004) have made comparative study of growth behaviour in tobacco leaf to the data obtained on tobacco leaf growth. They fitted Richard’s and other models and compared their results. Ersoy et al (2006) have studied the growth phenomena in American Bronz Turkeys. They have recorded weekly body weight for both male and female turkeys. The Richard’s growth model was fitted to turkey weight and age data. They have calculated estimates of mature body weight for male as well as female turkeys and found that male turkeys mature more slowly and needed more time to reach mature body weight as compared to female turkeys.

Damgaard and Weiner (2008) have used Richard’s model for modeling the growth of individual in crowded plant populations. Ahmadi and Golian (2008) have used Richard’s Gompertz and other models for
studies relating to use of non-linear growth models for describing the growth pattern in classical strain of broiler chicken.

(d) **Morgan-Mercer-Flodin (MMF) Model:**

The model with the deterministic part as

\[ Y = \frac{\beta \gamma + \alpha X^\delta}{\gamma + x^\delta} \]  \hspace{1cm} (1.28)

is known as Morgan-Mercer-Flodin or MMF model after the name of their inventor (Morgan et. al., 1975). The model is an extension of model of Hill (1913) and Michaelis-Menten (1913) model.

When \( \beta = 0 \), model (1.28) reduces to Hill (1913) model and when \( \beta = 0 \) and \( \delta = 1 \), it reduces to Michaelis-Menten (1913) model.

Ratkowsky (1983) has given another reparameterized form of MMF model as

\[ Y = \frac{\beta \exp(\gamma) + \alpha X^\delta}{\exp(\gamma) + x^\delta} \]  \hspace{1cm} (1.29)

Ratkowsky (1983) has shown with the help of his data sets that the parameter effect non-linearity in model (1.29) is considerably less as compared to parameter effect non-linearity in model (1.28), and has therefore recommended use of reparameterized model (1.28) in preference to other models.

Streek (2003) has used MMF model as a vernalization model which is expected to describe response of development to vernalization.
temperature to duration of vernalization period in onion. He has demonstrated through his data set that MMF model is a vernalization model; he had successfully described the phenomena in onion plantation.

Santos and Nova (2004) have used MMF non-linear regression model for their studies related to Statistical fitting and validation of non-linear metamodels. Khamis et al (2005) have used MMF model along-with other sigmoidal non-linear models for their study on growth phenomena in oil palm yield over time.

Vrahnikis et al (2006) have used MMF model in studying growth changes of plants as a response variable and environment constraints as explanatory variable. They have found that MMF model is appropriate best for both patterns of T. angustifolium and M. disciformis. Clode et al (2008) have used MMF model for estimation of regional species richness in marine benthic communities. They have also used Jackknifes techniques of estimation.
(e) **Weibull Type Model:**

The Weibull type model is defined as

\[ Y = \alpha - \beta \exp(-\gamma x^\rho) \] \hspace{1cm} (1.30)

where \( \alpha, \beta, \gamma, \rho \) and \( \delta \) are parameters of the model. The equation (1.30) defines the deterministic part of the model. Ratkowsky (1983) have given reparameterized form of weibull type model as

\[ Y = \alpha - \beta \exp\left(\frac{1}{\gamma \exp\left(-\gamma x^\rho\right)}\right) \] \hspace{1cm} (1.31)

With the help of data sets he has shown that parameter effect non-linearity is considerably reduced in model (1.31) as result of reparametrization.

Ratkowsky (1983) has given another reparameterized form of model (1.30) as

\[ Y = \exp(\alpha) - \exp(\beta - \exp(-\gamma x^\rho)) \] \hspace{1cm} (1.32)

Rawlings and Cure (1985) have studied field experiments for describing ozone effect on crop yields. They have used Weibull model as a Dose-Response model. Cimeza et al (1999) have derived an estimator in Weibull model with experimental error. Lai et al (2007) has studied Weibull model in reliability studies. They have used Weibull model allowing nearly instantaneous failure.

Hung and Ho (2008) have applied Weibull model to study of unobserved heterogeneity on estimated mortality among the middle aged and
elderly population in Taiwan. Moh. Satt N.Z. et al (2008) have used sleep apnea data for comparing Weibull and Gamma distributions. Their method gives us some basis for selection of a model even when the sample size is small.

Nygard et al (2008) has studied a relationship between reading rate with print size in a group of 132 old readers with normal age related vision. They have used Weibull model for their data and compared it with logistic and Gompertz models.

Patel et al (2008) have conducted their studies on comparison of in vitro dissolution profiles of oxcarbazepine–Hp β-CD tablet formulations with those of marketed excarbazepine tablets using Weibull model. They have observed that Weibull model was more useful for comparing the release profiles.


Parks et al (2009) have used Weibull model for describing failure density of the products. In their work they have compared wear out
characteristics of underground cables of an electric distribution network using Weibull model.

1.5 Estimation Procedures in Non-Linear Statistical Models:

One of the serious handicaps in estimation of parameters in non-linear statistical models is that the famous least square method of estimation of parameters is not directly applicable in estimation of parameters of non-linear models. Often for estimating parameters of intrinsically linear non-linear models, the researchers firstly transform the original model into the model in which parameters appear linearly and then apply least square method of parameter estimation to transformed linear model. From parameter estimates of this transformed linear model, the parameter estimates of the original model are obtained. The parameter estimates, thus obtained from the transformed model do not possess all those properties as possessed by parameter estimates of transformed model. The application of least squares method in the transformed model gives the Best Linear Unbiased Estimator (BLUE) of its parameters. However parameter estimates of original model obtained from parameter estimates of transformed model do not possess the properties of being Best Linear Unbiased Estimator. In spite of this shortcoming, the application of least
squares method to transformed model for obtaining parameter estimates of original model is still preferred due to its computational simplicity.

For estimation of parameters in intrinsically non-linear models, there are some computationally convenient methods of estimation viz. method of partial sums, method of three selected points etc. These methods provide estimators which have poor statistical properties. The direct application of least squares method for estimation of parameters in non-linear models is iterative in nature and convergence of parameter solution is their serious shortcoming. Several iterative methods of parameter estimation have appeared in literature, out of which following three are main methods of estimation.

1.5.1 Linear Approximation with the Help of Taylor's Series:

Suppose we have n observations on X and Y, for a known non-linear statistical model defined as

\[ Y = f(X_1, X_2, ..., X_k, \theta_1, \theta_2, ..., \theta_p) + U \]  \hspace{1cm} (1.33)

where u’s are independently and identically distributed random variables with mean zero and common variance \( \sigma^2 \) and \( \theta \)'s are parameters of the model.

Let

\[ X = (X_1, X_2, ..., X_k) \] \hspace{1cm} and \hspace{1cm} \[ \theta = (\theta_1, \theta_2, ..., \theta_p) \]
The model (1.33) can be written as

\[ Y = f(X, \theta) + U \]  

--------- (1.34)

Or

\[ E(Y) = f(X, \theta) \]  

--------- (1.35)

The deterministic component of non-linear statistical model (1.33) is expanded using Taylor's series expansion with the help of known guess values of \( \theta \) as \( \theta_0 \) and considering the expansion at the first derivatives only we have

\[ f(X, \theta) = f(X, \theta_0) + \sum_{i=1}^{p} \left[ \frac{\partial f(X, \theta)}{\partial \theta_i} \right]_{\theta=\theta_0} (\theta_i - \theta_0) \]  

--------- (1.36)

Let us write

\[ f^0 = f(X, \theta_0) \]

\[ \beta^0 = \theta - \theta_0 \]

\[ Z^0_i = \left[ \frac{\partial f(X, \theta)}{\partial \theta_i} \right]_{\theta=\theta_0} \]  

--------- (1.37)

To the selected order of approximation, we have equation (1.34) of the form:

\[ Y - f^0 = \sum_{i=1}^{p} \beta^0_i Z^0_i + U \]  

--------- (1.38)

The equation (1.38) is an approximate linear form of non-linear model (1.34). This method is popularly known as linearization method. The parameters \( \beta^0 \) can be estimated from equation (1.38) by the direct application of least squares method. From the estimates of \( \beta^0 \), an estimate of
\( \theta_i \) is considered as next guess value of \( \theta \) and the whole above described procedure is repeated till the solution converges i.e. until in successive iterations \( j \) and \( (j+1) \)
\[
\left| \left( \theta_{i,j+1} - \theta_{i,j} \right) \right| < \delta \quad i = 1, 2, ..., p. \quad \text{(1.39)}
\]
where \( \delta \) is some prescribed very low amount (e.g. 0.000001).

For detailed description of the procedure of linear approximation by Taylor’s series, reference can be made of Draper and Smith (1978, Ch. 24), Saber and Wild (1988, Ch. 2) and Gallant (1977).

### 1.5.2 Steepest Descent Method:

Steepest Descent method mainly aims to focus error sum of squares \( S(\theta) \), which can be defined as
\[
S(\theta) = \sum_{i=1}^{n} \left[ y_i - f(x_i, \theta) \right]^2 \quad \text{(1.40)}
\]
and to apply an iterative procedure to obtain minimum value error sum of squares.

The crux of the method is to move from an initial known guess value of \( \theta \) say \( \theta_0 \), along the vector with components
\[
\frac{\partial S(\theta)}{\partial \theta_i}, \quad \frac{\partial S(\theta)}{\partial \theta_2}, \ldots, \quad \frac{\partial S(\theta)}{\partial \theta_p}.
\]
Whose values change continuously as the path is followed. One of the procedures to adopt it is to estimate vector slope
components at various places on surface S(0) by fitting planner approximating functions.

Box and Draper (1987) have given a detailed description of using Steepest Descent method in estimation of parameters. The Steepest Descent method was developed by A.L. Cauchy in the year 1847.

Tamprop (1939), Levenburg (1944) have used these methods of estimation for their data sets. Curry (1944) has shown that method of Steepest Descent converges. Spang (1962) had observed that convergence in Steepest Descent method is very slow involving huge number of iterations.

Draper and Smith (1998) have remarked that the Steepest Descent method is not scale variant which results in change of direction of movement if scale is changed. They have further remarked that Steepest Descent method is on the whole slightly less favoured than linearization method with the help of Taylor’s series.

1.5.3 Marquardt’s Compromise:

In the year 1963 D.W. Marquardt has given an iterative procedure of estimation of parameters of a non-linear model, utilizing best features of linearization method with the help of Taylor’s series and Steepest Descent method. He has shown that method almost always converges and does not slow down as in the case of Steepest Descent
method. However method works well only under limitations laid down in the method itself. The full description of the method is given in paper of Marquardt (1963). For further reference on the utility of this method, the reference can also be made of Bates and Watts (1988) and Saber and Wild (1989).

Valko and Vajda (1987) have used Marquardt’s compromise method of estimation of parameters of non-linear models with error in variables and also shown that the method is superior to Gauss-Markov’s method.

Mahir et al (2009) has used Marquardt’s compromise method to estimate parameters of exponential, Logistic, Gompertz and Weibull models using data related to Gauss Heavy Metal uptake in Spinach.

1.6 Layout of Thesis:

The present research work has six chapters which includes the chapter one which is on ‘Introduction and Review of Literature’.

In the second chapter a linear model has been proposed which adequately approximates Logistic model. The proposed model has been fitted to various data sets obtained from different spheres of study. The appropriateness of the proposed model has also been verified by carrying out analysis of residuals.
In chapter three a linear model has been proposed which adequately approximates Gompertz model. As in Chapter two, the appropriateness of the proposed model has also been verified using different data sets obtained from different sources.

In chapter four a model linear in its parameters has been proposed for determination of optimum cluster size using cost function. The linear model used for determination of optimum cluster size considering cost function leads to a simple solvable expression whereas the use of empirical non-linear model for determination of optimum cluster size leads to complicated expression which can only be solved by iterative procedures.

In chapter five a model having linearly appearing parameters, has been proposed as an alternative to Makeham’s law of mortality. The usefulness of proposed model as compared to Makeham’s model has also been verified with the help of various data sets obtained from different sources.

The summary of the present thesis work has been given in the chapter six.