CHAPTER 3

AN AUTOMORPHIC GROUP OF A SANDWICH NEARRING

INTRODUCTION:

It has been already established that if \( \phi_k : R \rightarrow R \) is a homeomorphism, the groups \( \text{Aut}(c(R,R),+,\phi_k) \) and \( R^* \) are isomorphic; where \( k \in C(R,R) \). Also, sandwich nearrings \( (c(R,R),+,\phi_k) \), is isomorphic to the nearring \( (c(R,R),+,0), \phi \in \text{u}(c(R,R),+,0) \). In this chapter, we have proved that \( \text{Aut}(c(R,R),+,\phi_k) \cong R^* \) on automorphism groups of laminated nearrings. A lot of work has been done by Magill [18].

**Theorem 3.1:** If \( \phi_k : R \rightarrow R \) is a homeomorphism, then the groups \( \text{Aut}(c(R,R),+,\phi_k) \) and \( R^* \) are isomorphic, where \( k \in c(R,R) \).

**Proof:** In Chapter 2, we have proved that the nearring \( (c(R,R),+,\phi_k), \phi \in \text{u}(c(R,R)) \) is isomorphic to the nearring \( (c(R,R),+,0) \).

In this chapter, we need only to show that

\[
\text{Aut}(c(R,R),+,\phi_k) \cong R^*.
\]

We have homeomorphisms:

\[
h : R \rightarrow R
\]

and

\[
t : R \rightarrow R
\]
For $F \in \text{Aut}(c(R,R),+,\phi_k)$ such that $t \in \text{Aut}(R,+)$,

$$s_{ot} = h s_f$$

for $s(x) = x$

$s$ is identity of $c(R,R)$, and such that $t \circ f = F(f) \circ h$ $\forall f \in c(R,R)$.

Since $h = t$, we have

$$F(f) = t \circ f \circ t^{-1}.$$ 

Now it remains to show that distinct $t_1, t_2 \in \text{HA}(R)$ define distinct $F_{1}(f) = t_1 \circ f \circ t_1^{-1}$.

But this follows from the uniqueness guaranteed by Theorem 1 of Chapter 2.

For our purposes here we say that a surjective $\phi_k \in c(R,R)$ is nearly increasing if there is an interval $(-r,r)$ so that $\phi_k$ is increasing on $R\vert (-r,r) = (-\infty,-r] \cup [r,\infty]$ that is if $x, x' \in R\vert (-r,r)$ and $x < x'$ then $\phi_k(x) < \phi_k(x')$.

Similarly, a surjective $\phi_k \in c(R,R)$ is nearly decreasing if there is an interval $(-r,r)$ so that $\phi_k$ is decreasing on $R\vert (-r,r)$, that is, if $x,x' \in R\vert (-r,r)$ and $x < x'$, then $\phi_k(x) > \phi_k(x')$. A surjective $\phi_k \in c(R,R)$ is nearly monotonic if it is either nearly increasing or nearly decreasing.

Theorem 3.2: Let $\phi_k \in c(R,R)$ be surjective and nearly monotonic, then the following are equivalent:
(i) The mapping $\phi_k$ is a homeomorphism,

(ii) $(c(R,R),+,0) \cong (c(R,R),+,\phi_k)$,

(iii) $(c(R,R),+,\phi_k)$ has an identity,

(iv) $\text{Aut}(c(R,R),+,\phi_k) \cong \mathbb{R}^*$,

(v) $\text{Aut}(c(R,R),+,\phi_k)$ has more than two elements.

**Proof:** The equivalence of 1), 2), 3) follow from Chapter 2 Theorem 2.3.

From the above theorem we have that 1) $\rightarrow$ 4) and 4) $\rightarrow$ 5).

We will be finished when we show that 5) $\Rightarrow$ 1), each $F_i$ has its $h_i$ and $t_i$ of (Theorem 1, Chapter 2).

Let $F_1', F_2', F_3'$ be three distinct automorphism of $(c(R,R),+,\phi_k)$.

The $t_i$'s are distinct, for it $t_i = t_j$, then $h_i \circ \phi = \phi \circ h_i = \phi \circ t_j = h_j \circ \phi_k$. For arbitrary $x \in R$, there is $y \in R$ such that $\phi(y) = x \quad \forall \ y \in R, \ x \in R$.

Hence

$$h_i(x) = h_i(\phi(y)) = \phi \circ t_i(y) = \phi \circ h_j(y) = h_j \circ \phi(y)$$
\[ = h_j(x) \]

This makes \( h_i = h_j \),

and with \( F_i(f) = t_i \circ f \circ h_i^{-1} \)

\[ = t_j \circ f \circ h_j^{-1} \]

\[ = F_j(f) \quad \forall \ f \in C(R, R) \]

we get \( F_i = F_j \). So the \( t_i \)'s are distinct, and by this theorem

that let \( HA(R^+) \), denote the set of homeomorphisms \( f: R \rightarrow R \), such

that \( f \) is also a group automorphism of \( (R, +) \). Then \( HA(R^+) \) is a

group isomorphic to \( R^* \), the multiplicative group of the field

\( (R, +, \cdot) \).

In fact, each element \( f \in HA(R^+) \) is defined by

\[ f(x) = ax, \text{ for some } a \in R^* \]

There are \( a_i \in R^* \) such that \( t_i(x) = a_i(x) \), from

which follows that one of the \( a_i \not\in \{ \pm 1 \} \).

So, there is an automorphism \( F \) of \( (c(R, R), +, \cdot) \) which

has \( h \) and \( t \) of Theorem 1 of Chapter 2

where \( t(x) = ax \) and \( a \not\in \pm 1 \).

Our result step is to show that we may, without loss of
generality, assume \( a > 1 \). If \( a < 0 \), then \( F^2 \) has \( t^2 \) and

\[ t^2(x) = t(t(x)) \]

\[ = a^2(x) \]
and $a^2 > 0$ and $a^2 \neq 1$. So, we may assume that $0 < a$ if

$$0 < a < 1,$$

then $F^{-1}$ has $t^{-1}$, and $t^{-1}(x) = a^{-1}x$, with $1 < a^{-1}$, so without loss of generality, we take $a > 1$.

We may assume that $a$ to be very large. For with $F$ having with $t(x) = ax$ and $1 < a$, then $F^n$ has $t^n$ with $t^n(x) = a^n x$ and

$$a^n \to +\infty$$

and

$$n \to +\infty.$$

In particular, if $\phi$ is monotonic on $\mathbb{R} \cap (-r, r)$ and if $x, y \in \mathbb{R}^*$ are distinct, then there is an $n$ so that

$$a^n x, a^n y \in \mathbb{R} \cap (-r, r]$$

and

$$a^n x \neq a^n y.$$

We are ready to show that $\phi_k$ is injective, hence $\phi_k$ is a homeomorphism of $\mathbb{R}$. If

$$\phi_k(x) = \phi(y)$$

and

$$x \neq y$$

then

$$\phi_k(a^n x) = \phi_k \circ t^n(x)$$

$$= h^n \circ \phi_k(x)$$

$$= h^n \circ \phi_k(y)$$
\[
\phi_k \circ t^n(y) = \phi_k(a^n y).
\]

If \( x = 0 \), then \( \phi_k(0) = \phi_k(a^n y) \), for each positive integer \( n \) and this contradicts that \( \phi \) is nearly monotonic. If \( 0 \not\in \{x, y\} \) there is an \( n \) so that \( a^n x, a^n y \in R(-r, r) \) and \( a^n x \neq a^n y \). Since \( \phi_k \) is nearly monotonic, we also have

\[
\phi_k(a^n x) \neq \phi_k(a^n y)
\]

Since the assumption \( \phi_k(x) = \phi_k(y) \) with \( x \neq y \) gets us into this unacceptable situation, we must have \( \phi_k(x) \neq \phi_k(y) \). If \( x \neq y \).