CHAPTER IX

DISCS UNDER SIMULTANEOUS EFFECTS
OF INERTIA AND SMALL ROTATION

9.1 DISCS.

The lubrication of discs is important because they can be made kinematically equivalent to gears, if they have the same radius at their contact line and rotate at the same angular velocities as the gears. Assume that the disc is stationary at the lower surface transverse to the fluid film where pure sliding is absent and

\[ U = +U \text{ (constant)} \]  

\[ \text{(9.1.1)} \]

Let us take the pressure distribution and the film thickness as functions of the coordinate \( y \) along the line of centres of the annulus of the discs only so that we may have

\[
\begin{align*}
(\text{i}) & \quad P = P(y) \\
(\text{ii}) & \quad h = h(y) = h_0 (1 + \frac{y^2}{2Rh_0}),
\end{align*}
\]

\[ \text{(9.1.2)} \]

where \( R \) is the reduced radius given by

\[ \frac{1}{R} = \frac{1}{R_1} \pm \frac{1}{R_2}, \]

\[ \text{(9.1.3)} \]

\( R_1 \) and \( R_2 \) being radii of internal and external discs; \( h_0 \) being the minimum of oil film thickness on the line of centres.

For the solution of such discs (See figure 9.1), all the results of Section (2.4) in one dimensional form apply.
Fig. 9.1 (a and b) Film thickness of discs.
But the boundary conditions be imposed as follows for the determination of the positive region of the pressure:

(i) \( P = 0 \) at \( h = h_0 \) or \( P = 0 \) at \( y = 0 \) \( \quad (9.1.4) \)

(ii) \( P = \frac{\partial P}{\partial y} = 0 \) at \( h = h_1 \) or \( P = 0 \) at \( y = \bar{y}, \) say

Putting, \( \tan \theta = \frac{y}{\sqrt{2Rh_0}} \) \( \quad (9.1.5) \)

(ii) of the equation \( (9.1.2) \) assumes the form

\[ h = h_0 \sec^2 \theta \quad (9.1.6) \]

and boundary conditions \( (9.1.4) \) are given by the following:

(i) \( P = 0 \) at \( \theta = 0 \) \( \quad (9.1.7) \)

(ii) \( P = \frac{\partial P}{\partial \theta} = 0 \) at \( \theta = \gamma, \) say

when

\[ \tan \gamma = \frac{\bar{y}}{\sqrt{2Rh_0}} \quad (9.1.8) \]

The pressure \( P_0 \) is determined on integrating the following differential equation which is obtained with the help of the equations \( (2.4.3), (9.1.1), (9.1.2), (9.1.5) \) and \( (9.1.6) \):

\[ \frac{d}{ds} \left( \frac{h_0^3 \sec^4 \theta}{\sqrt{2Rh_0}} \frac{dP_0}{ds} \right) + \frac{d}{ds} \left( \frac{M DU}{2} h_0^3 \sec^6 \theta \right) = 0 \quad (9.1.9) \]

Integrating the equation \( (9.1.9) \) and determining the values of the arbitrary constants of integration with the help of the boundary conditions \( (9.1.7) \) supported by the equation
(9.1.8), we get

\[ P_o = \frac{M \rho U \sqrt{2Rh_o}}{64 \cos^6 \gamma} \left( \sin 4\theta + 8 \sin 2\theta + 12\theta \right. \\
\left. + 32 \cos^6 \gamma \tan \theta \right) \]  
(9.1.10)

where \( \gamma \) is determined by the following equation:

\[ \sin 6\gamma + 3 \sin 4\gamma - 3 \sin 2\gamma - 12\gamma = 0 \]  
(9.1.11)

With the help of the equations (9.1.1), (9.1.2), (9.1.5), (9.1.6), (9.1.10) and (9.1.11), the equation (2.4.2) assumes the following form:

\[
\frac{d}{d\theta} \left( \frac{\rho}{\mu} \cdot \frac{h^3 \sec^4 \theta}{\sqrt{2Rh_o}} \cdot \frac{dp}{d\theta} \right) + \frac{d}{d\theta} \left( \frac{M \rho^2 U h^3_o}{2\mu} \sec 6 \theta \right) \\
- \frac{d}{d\theta} \left( \frac{M^2 \rho^4 U^2 h^7_o \sec^4 \theta}{3360 \mu^2 \sqrt{2Rh_o}} \right) \times \\
(12 \sec^{12} \gamma \sin \theta \cos^3 \theta - 19 \sec^6 \gamma \tan \theta \sec^2 \theta \\
+ 61 \tan \theta \sec^6 \theta) = 0 \]  
(9.1.12)

Integrating the equation (9.1.12) with the help of the boundary conditions (9.1.7) supported by the equation (9.1.8) we get,

\[ P = \left\{ \frac{M \rho U \sqrt{2Rh_o}}{64 \cos^6 \gamma} \left( \sin 4\theta + 8 \sin 2\theta + 12\theta \right. \\
\left. + 32 \cos^6 \gamma \tan \theta \right) \right\} \\
+ \left. \frac{M^2 \rho^3 U^2 h^4_o}{26880 \mu^2} \times \right. \]
which reproduces the pressure distribution by the quantity within the first brace \{.\} in the \(\text{inertia-free rotatory frame}\) of reference and yields the pressure distribution by the quantity within the second bracket <.> in the \(\text{inertio-rotatory frame}\) of reference which is directly proportional to \(M^2 \rho^3 u^2\) and inversely proportional to \(\mu^2\).

\[ (9.2) \text{ LOAD CAPACITY.} \]

The normal load capacity \(W\) of the discs per unit length is given by

\[
W = \int_{0}^{\pi} p(\frac{dy}{d\theta}) \, d\theta
\]

\[ (9.2.1) \]

On substituting for \(P\) from the equation \((9.1.10)\) in the equation \((9.2.1)\) and using \((9.1.11)\) we get

\[
W = \left\{ \frac{M \rho U R h_0^8}{8} \left( -\sec^6 \gamma \frac{1+3\gamma}{\tan \gamma} \sec^4 \gamma + 4 \tan^2 \gamma \right) \right\}
\]

\[
+ \left[ \frac{M^2 \rho^3 u^2 \sqrt{2Rh_0} h_0^4}{203212800 \mu^2} \right] \times
\]
\[
\{ \tan \gamma (22680 \sec^{15} \gamma + 144585 \sec^{12} \gamma - 18900 \sec^{10} \gamma \\
+ 140280 \sec^{8} \gamma - 250080 \sec^{6} \gamma - 127917 \sec^{4} \gamma \\
- 93696 \sec^{2} \gamma + 273763) \\
+ \gamma \sec^{6} \gamma (68040 \sec^{12} \gamma - 110565 \sec^{8} \gamma \\
- 204120 \sec^{6} \gamma - 172935) \}\]

which reproduces the load capacity by the quantity within the first brace \{\} in the inertia-free rotatory frame of reference and yields the load capacity by the quantity within the second bracket \[\] in the inertio-rotatory frame of reference.

Inertio-rotatory correction in load capacity defined by the equation (6.2.3) is given by

\[
\frac{M \rho^2 U h^3 \sqrt{2R \rho}}{25401600 \mu^2} x
\]

\[
\{ \tan \gamma (22680 \sec^{15} \gamma + 144585 \sec^{12} \gamma - 18900 \sec^{10} \gamma \\
+ 140280 \sec^{8} \gamma - 250080 \sec^{6} \gamma - 127917 \sec^{4} \gamma \\
- 93696 \sec^{2} \gamma + 273763) \\
+ \gamma \sec^{6} \gamma (68040 \sec^{12} \gamma - 110565 \sec^{8} \gamma \\
- 204120 \sec^{6} \gamma - 172935) \}\]

\[
W_{\text{correc.}} = \frac{1}{4R(\sec^{6} \gamma + \tan^2 \gamma + \sec^4 \gamma + 4 \tan^2 \gamma)}
\]

(9.2.3)
9.3 **SHEAR STRESS, FRICTIONAL FORCE AND FRICTIONAL COEFFICIENT**

The aggregate of the equations (2.4.4), (2.4.5), (9.1.1), (9.1.10), (9.1.11), and (9.1.13) yields the components of the shear stress as follows:

\[ \tau_x = - \mu \frac{U}{h_o} \cos^2 \theta \]
\[ + \left[ \frac{M \rho^2 U^2 h_o^3 \sec^6 \gamma}{60U \sqrt{2Rh_o}} \sin \theta \cos \theta - \frac{M^2 \rho^2 h_o^3 U}{48\mu} (\sec^6 \gamma - \sec^6 \theta) \right] \]
\[ + \frac{M^3 \rho^4 U^2 h_o^7}{645120 \mu^3 \sqrt{2Rh_o}} \]
\[ \{ \cot \gamma \sec^6 \gamma (24 \sec^{12} \gamma - 39 \sec^6 \gamma + 76 \sec^6 \theta - 61) \]
\[ + 8 \tan \theta (-169 \sec^{12} \theta + 130 \sec^6 \gamma \sec^6 \theta - 15 \sec^{12} \theta) \} \]

\[ (9.3.1) \]

and

\[ \tau_y = - \frac{M \rho U h_o}{12} (3 \sec^6 \gamma \cos^4 \theta + \sec^2 \theta) + \frac{M^2 \rho^3 U^2 h_o^5}{\mu^2 \sqrt{2Rh_o}} \]
\[ \left[ \cot \gamma \sec^6 \gamma \cos^4 \theta (24 \sec^{12} \gamma \sin^2\gamma (1+\cos^2 \gamma) \right] \]
\[ + 61 (\sec^8 \gamma - 1) - 76 \tan^2 \gamma \sec^6 \gamma \]
\[ + \frac{\tan \theta}{20160} (-47 \sec^8 \theta + 11 \sec^6 \gamma \sec^2 \theta + 6 \sec^{12} \gamma \cos^4 \theta) \]

\[ (9.3.2) \]

On substituting the values of \( \tau_x \) and \( \tau_y \) from the equations (9.3.1) and (9.3.2) in (2.3.31) the shear stress is
completely determined.

The frictional force on the discs determined by

\[
F_j = \int_{\theta=0}^{\theta=\gamma} (\tau \, \frac{dv}{d\theta}) \, d\theta
\]

is expressed as follows:

\[
F_j = \frac{M \rho U h_o \sqrt{2Rh_o}}{72} \times \frac{[9 \sec^6 \gamma (\gamma + \sin \gamma \cos \gamma) + 2 \tan \gamma (3 + \tan^2 \gamma)]}{M^2 \rho^3 U^2 h_o^5 - \frac{1612800 \mu^2}{1612800 \mu^2} \times \{15 \cot \gamma \sec^6 \gamma (24 \sec^{12} \gamma - 39 \sec^8 \gamma + 76 \sec^6 \gamma - 61) + 360 \sec^{16} \gamma - 345 \sec^{12} \gamma + 744 \sec^{10} \gamma - 915 \sec^4 \gamma + 156\}}
\]

The frictional coefficient for the discs is determined on substituting the values of \(W\) and \(F_j\) respectively from the equations (9.2.2) and (9.3.4) in the equation (3.3.5) as follows:

\[
f = \left[\frac{\sqrt{2Rh_o}}{72} \frac{[9 \sec^6 \gamma (\gamma + \sin \gamma \cos \gamma) + 2 \tan \gamma (3 + \tan^2 \gamma)]}{M \rho^2 U h_o^4 - \frac{1612800 \mu^2}{1612800 \mu^2} \times \{15 \cot \gamma \sec^6 \gamma (24 \sec^{12} \gamma - 39 \sec^8 \gamma + 76 \sec^6 \gamma - 61) + 360 \sec^{16} \gamma - 345 \sec^{12} \gamma + 744 \sec^{10} \gamma - 915 \sec^4 \gamma + 156\}}\right] x
\]
\[
\frac{R}{\beta} \left( -\sec^6 \gamma + 3 \gamma \tan \gamma + \sec^4 \gamma + 4 \tan^2 \gamma \right)
\]

\[
\frac{M \rho^2 U \sqrt{2R_0 h}}{203212900 \mu^2} \times
\]

\[
\{ \tan \gamma (22680 \sec^{16} \gamma + 144585 \sec^{12} \gamma - 18900 \sec^{10} \gamma
\]

\[
+ 140280 \sec^8 \gamma - 250080 \sec^6 \gamma - 127917 \sec^4 \gamma
\]

\[
- 93696 \sec^2 \gamma + 273768) \}
\]

\[
+ \gamma \sec^6 (68040 \sec^{12} \gamma - 110565 \sec^9 \gamma
\]

\[
- 204120 \sec^6 \gamma - 172935) \}^{-1} \quad (9.3.5)
\]

The conclusions similar to those of the section (3.4) are easily derived for the discs also.