CHAPTER III

INFINITELY LONG JOURNAL BEARINGS
UNDER SIMULTANEOUS EFFECTS OF
INERTIA AND SMALL ROTATION

3.1 INFINITELY LONG JOURNAL BEARING:

If we assume the bearing to be infinitely long in
the y-direction, it implies no variation of pressure in the
x-direction and hence \( \frac{\partial P}{\partial x} \) is negligible in comparison with \( \frac{\partial P}{\partial y} \)
and also \( h \) is a function of \( y \) alone. Further, assume that
the bearing is stationary at the lower surface transverse to
the fluid film where pure sliding is absent and

\[
U = -U \tag{3.1.1}
\]

For the solution of such a bearing (See figure 3.1), all
the results of the Section (2.4) in one-dimensional form
modified by (3.1.1) apply. As usual, we take

\[
\begin{align*}
    h &= c(1 + e \cos \theta) \\
    y &= R\theta
\end{align*}
\]

whence

\[
\frac{dh}{dy} = -\frac{c e \sin \theta}{R}
\]

and transform the boundaries \( \theta = 0 \) and \( \theta = 2\pi \) into the same
boundaries in the \( \Psi \)-coordinate by employing the following
Sommerfeld’s substitutions[7]:
Fig. 3.1 Clearance in journal bearings.
\[ \tan^{-1} \left\{ \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta}{2} \right\} = \frac{\psi}{2} \]

whence

\[ \cos \theta = \frac{\cos \psi - e}{(1-e \cos \psi)} \]
\[ \sin \theta = \frac{(1-e^2)^{1/2} \sin \psi}{(1-e \cos \psi)} \]
\[ 1+e \cos \theta = \frac{(1-e^2)}{(1-e \cos \psi)} \]
\[ \cos \psi = \frac{(e+\cos \theta)}{(1+e \cos \theta)} \]
\[ d\theta = \frac{(1-e^2)^{1/2} d\psi}{(1-e \cos \psi)} \]

The boundary conditions are:

(i) \( P = P_a \) where \( P_a \) is the pressure at \( \theta = 0 \) or \( \psi = 0 \) which can be evaluated from the condition at a point where a given inlet pressure \( P_1 \) corresponds to a given angle \( \theta_1 \); if the inlet hole is at \( \theta_1 = 0 \), then, of course, \( P_a \) is the value of inlet pressure.

(ii) \( \{P(\theta)\}_{\theta=0}^{\theta=2\pi} = \{P(\psi)\}_{\psi=0}^{\psi=2\pi} \) or, \( \{P(\psi)\}_{\psi=0}^{\psi=2\pi} = \{P(\psi)\}_{\psi=0}^{\psi=2\pi} \)
The pressure $P_0$ is determined on integrating the following differential equation which is obtained with the help of the equations (2.4.3), (3.1.1), (3.1.2) and (3.1.3) and employing the boundary conditions (3.1.4):

$$\frac{d}{d\psi} \left\{ \frac{1}{(1-e \cos \psi)^2} \frac{dP_0}{d\psi} \right\} - \frac{d}{d\psi} \left\{ \frac{M \rho U R \sqrt{1-e^2}}{2(1-e \cos \psi)^3} \right\} = 0 \quad (3.1.5)$$

Thus

$$P_0(\psi) = P_a + \frac{M \rho U R}{2} \cos^{-1} \left\{ \frac{(\cos \psi - e)}{1 - e \cos \psi} \right\} \quad (3.1.6)$$

For the determination of the pressure distribution excluding negative regions the boundary conditions imposed are [7]:

- $P = 0$ at $\psi = 0$,
- $\frac{dP}{d\psi} = 0$ at $\psi = \psi_2$,
- $\dot{P} = 0$ at $\psi = \psi_2$ \quad (3.1.7)

where $\psi_2$ corresponds to $\theta_2$ in the equation (3.1.3). $P_0$, for nonnegative regions, is determined on integrating the equation (3.1.5) with the conditions (3.1.7) by the usual procedure [7] as follows:

$$P_0(\psi) = M \rho U R \tan^{-1} \left\{ \sqrt{\frac{1+e}{1-e}} \tan \frac{\psi}{2} \right\}$$

$$- \frac{M \rho U R \sqrt{1-e^2}}{4(1-e \cos \psi_2)^3} \left[ (2+e^2) \psi - 4e \sin \psi + e^2 \sin \psi \cos \psi \right]$$

\quad (3.1.8)
Finally, the pressure distribution, for non-negative regions, is determined on integrating the following differential equation which is obtained with the help of the equations (2.4.2), (3.1.1), (3.1.2), (3.1.3) and (3.1.8) and employing the boundary conditions (3.1.7):

\[
\frac{d}{d\psi} \left\{ \frac{1}{(1-e \cos \psi)^2} \frac{dP}{d\psi} \right\} - \frac{d}{d\psi} \left\{ \frac{M \rho U R}{2} \frac{\sqrt{1-e^2}}{(1-e \cos \psi)^3} \right\}
\]

\[+ \frac{d}{d\psi} \left\{ \frac{M^2 \rho^3 U^2 c^4}{6720 \mu^2 (1-e \cos \psi)} \right\} \sin \psi \times \]

\[
\frac{61}{(1-e \cos \psi)^6} + \frac{12}{(1-e \cos \psi_2)^6} - \frac{19}{(1-e \cos \psi_2)^3 (1-e \cos \psi)^3} = 0 \quad (3.1.9)
\]

Thus

\[
P(\psi) = [M \rho U R \tan^{-1} \{ \sqrt{\frac{1+e}{1-e}} \tan \frac{\psi}{2} \}]
\]

\[- \frac{M \rho U R \sqrt{(1-e^2)}}{4(1-e \cos \psi_2)^3} \left\{ (2+e^2) \Psi - 4e \sin \Psi + e^2 \sin \Psi \cos \Psi \right\}
\]

\[+ \frac{-M^2 \rho^3 U^2 c^4}{6720 \mu^2} \left[ \frac{-19e(1-e^2)^3}{(1-e \cos \psi_2)^3} \cdot \frac{(1+e)(1-\cos \psi)}{(1-e \cos \psi)^3} \right]
\]

\[+ \frac{6(1-e^2)^4}{(1-e \cos \psi_2)^6} \{(1-e \cos \psi)^2 - (1-e)^2 \}
\]

\[\quad - \frac{61}{4} (1-e)^4 \left\{ \frac{(1-e)^4}{(1-e \cos \psi)^4} - 1 \right\}
\]
\[(2+e^2)\psi_2 - 4e \sin \psi_2 + e^2 \sin \psi_2 \cos \psi_2 \}^{-1} x \]

\[\frac{19e(1+e)^4(1-e)^3(1-\cos \psi_2)}{(1-e \cos \psi_2)^4} + \frac{6(1-e)^6(1+e)^4}{(1-e \cos \psi_2)^6} \]

\[- \frac{61}{4} (1+e)^4 + \frac{37}{4} (1-e^2)^4 (1-e \cos \psi_2)^4 \] > (3.1.10)

where \(\psi_2\) determined by the condition: \(P = 0\) at \(\psi = \psi_2\), is given by the following equation:

\[\{(2+e^2)\psi_2 - 4e \sin \psi_2 + e^2 \sin \psi_2 \cos \psi_2 \} = 4(1-e^2) - \frac{1}{2} (1+e \cos \psi_2)^3 \times \tan^{-1} \left\{ \sqrt{\frac{1+e}{1-e}} \tan \frac{\psi_2}{2} \right\} \] (3.1.11)

The equation (3.1.10) along with (3.1.11) gives a pressure profile satisfying all the conditions embodied in equation (3.1.7). The equation (3.1.10) reproduces the pressure distributions by the quantity within the first bracket \[\cdot\] in the inertia-free rotatory frame of reference and yields the pressure distribution by the quantity within the second bracket \[\cdot\] in the inertia-rotatory frame of reference which is directly proportional to \(M^2 \rho^2 U^2\) and inversely proportional to \(\mu^2\).

3.2 **LOAD CAPACIT**: 

The load capacity is given by
\[ W = \sqrt{\left( W_X^2 + W_Y^2 \right)} \quad (3.2.1) \]

where \[ \theta = \theta_2 \]
\[ W_X = \int_{\theta=0}^{\theta=\theta_2} \text{LRP} \sin \theta \, d\theta \quad (3.2.2) \]

and \[ \theta = \theta_2 \]
\[ W_Y = \int_{\theta=0}^{\theta=\theta_2} \text{LRP} \cos \theta \, d\theta \quad (3.2.3) \]

On substituting for \( P \) from the equation \((3.1.10)\) in the equations \((3.2.2)\) and \((3.2.3)\) respectively and making use of the equations \((3.1.3)\) it is obtained that

\[ W_X = (W_X)_0 + (W_X)_I \quad (3.2.4) \]

and

\[ W_Y = (W_Y)_0 + (W_Y)_I \quad (3.2.5) \]

where \((W_X)_0\) and \((W_Y)_0\) represent the load capacities in the \( x \) and \( y \) directions of the corresponding inertia less bearing while \((W_X)_I\) and \((W_Y)_I\) denote the contribution of the inertia forces in the respective directions in the inertia-rotatory frame of reference. These are expressed as follows:

\[
(W_X)_0 = \frac{1}{2} (M \rho U R^2 L) \times \\
< \frac{1}{2} (\sin \theta_2 - \theta_2 \cos \theta_2) - \frac{(1+e \cos \theta_2)^3}{4(1-e^2)^{3/2}} \times \\
[\frac{(2+e^2)}{e} \frac{\theta_2}{\sqrt{(1-e^2)}} - 2 \tan^{-1}\left(\frac{\sqrt{1-e^2}}{1+e} \tan \frac{\theta_2}{2}\right) \left(1+e \cos \theta_2\right)]
\]
\[
+ \frac{4}{e} \left\{ \frac{(e \sin \theta_2 - \theta_2)}{\sqrt{(1-e^2)}} \right\} + 2 \tan^{-1} \left( \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta_2}{2} \right)
\]

\[
- \frac{1}{e} \left\{ \frac{e \sin \theta_2}{\sqrt{(1-e^2)}} \cdot \frac{(2+e \cos \theta_2 - e^2)}{(1+e \cos \theta_2)} - \frac{(2-e^2) \theta_2}{\sqrt{(1-e^2)}} \right\}
\]

\[
+ 4 \tan^{-1} \left( \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta_2}{2} \right) \] > \] (3.2.6)

\[
(W_X)_I = - \frac{M^2 \rho U c^4 L R(1-e^2)}{6720 \mu^2} x
\]

\[
< -19(1+e)(1+e \cos \theta_2)^3 x
\]

\[
\left[ \frac{(1-e)}{2e} \left\{ \frac{(1+e \cos \theta_2)^2 - (1-e)^2}{(1-e^2)^2} \right\} + \frac{(1-e^2)}{(1-e^2)} \right] + 6e(1+e \cos \theta_2)^5 (1-\cos \theta_2)^2 (1-e)
\]

\[
+ \frac{(1+e \cos \theta_2)^5 (1-\cos \theta_2)^2 (1-e)}{(1+e)(1-e^2)^2}
\]

\[
- \frac{61}{4}(1+e)^4 \left[ \frac{(1-e^4)}{5e} \left\{ \frac{1}{(1-e)^5} - \frac{(1+e \cos \theta_2)^5}{(1-e^2)^5} \right\} \right.
\]

\[
\left. - \frac{(1-\cos \theta_2)^2}{(1-e^2)} \right] > \]

\[
+ < \frac{(1+e \cos \theta_2)^3}{\theta_2}
\]

\[
2(1-e^2)^{\frac{3}{2}} \theta_2
\]

\[
[19e(1+e \cos \theta_2)^3 (1-\cos \theta_2) + \frac{6}{(1+e)^2} (1+e \cos \theta_2)^6
\]

\[
- \frac{61}{4} (1+e)^4 + \frac{37}{4} (1+e \cos \theta_2)^4 \] x
\[
\begin{align*}
&\left[\frac{(2+e^2)}{e}\right] \left\{ \frac{\theta_2}{\sqrt{(1-e^2)}} - \frac{2 \tan^{-1}\left(\frac{\sqrt{1-e} \tan \frac{\theta_2}{2}}{1+e}\right)(1+e \cos \theta_2)}{(1-e^2)} \right\} \\
&\quad + \frac{4}{e} \left\{ \frac{e \sin \theta_2}{\sqrt{(1-e^2)}} - \frac{\theta_2}{\sqrt{(1-e^2)}} \right\} + 2 \tan^{-1}\left(\frac{\sqrt{1-e} \tan \frac{\theta_2}{2}}{1+e}\right) \\
&\quad - \frac{1}{e} \left\{ \frac{e \sin \theta_2}{\sqrt{(1-e^2)}} \right\} \left(\frac{2+e \cos \theta_2-e^2}{1+e \cos \theta_2} \right) - \frac{(2-e^2)\theta_2}{\sqrt{(1-e^2)}} \\
&\quad + 4 \tan^{-1}\left(\frac{\sqrt{1-e} \tan \frac{\theta_2}{2}}{1+e}\right) \right\} > \}
\end{align*}
\]

(3.2.7)

\[ (\omega_y)_o = \frac{1}{4} (M \rho U R^2 L) \times \]

\[ < \frac{1}{2} \left\{ \theta_2 \sin \theta_2 - (1 - \cos \theta_2) \right\} - \frac{(1+e \cos \theta_2)^3}{4(1-e^2)^2} \times \]

\[ \left[ \frac{2(2+e^2)}{\sqrt{(1-e^2)}} \sin \theta_2 \tan^{-1}\left\{ \frac{\sqrt{(1-e)} \tan \frac{\theta_2}{2}}{1+e} \right\} + 3(\cos \theta_2 - 1) \right] \]

\[ + \frac{(1-e)(1-\cos \theta_2)}{(1+e \cos \theta_2)} \right\} > \}
\]

(3.2.8)

and

\[ (\omega_y)_I = - \frac{M^2 \rho^3 U^2 c^4 L R(1-e^2)^2}{6720 \mu^2} \times \]

\[ \left\{ \frac{19e(1+e \cos \theta_2)^3}{2(1-e^2)^{1/2}} \left\{ \theta_2 + (\cos \theta_2 - 2) \sin \theta_2 \right\} \right\} \]

\[ + \frac{6}{(1-e^2)^{3/2}} (1+e \cos \theta_2)^6 \right\} \]

(3.2.9)
\[
\frac{\sin \theta_2}{(1+e \cos \theta_2)} - \frac{2e}{\sqrt{(1-e^2)}} \tan^{-1}\left[\sqrt{\frac{(1-e^2)}{1+e}} \tan \frac{\theta_2}{2}\right] - \frac{(1-e)}{(1+e)} \sin \theta_2
\]

\[\frac{61}{4} \left[ \frac{\sin \theta_2}{\left(\frac{1}{2}\right)} \right] \left\{ 6(1+e \cos \theta_2)^4 + 6(1+e \cos \theta_2)^3 + 2(3+4e^2)(1+e \cos \theta_2)^2 + (6+29e^2)(1+e \cos \theta_2) \right\}
\]

\[\frac{e(4+3e^2) \theta_2}{2\sqrt{(1-e^2)}} + \frac{(1+e \cos \theta_2)^3}{2(1-e^2)^{1/2}} \theta_2
\]

\[\left\{ 19e(1+e \cos \theta_2)^3(1-\cos \theta_2) + \frac{6}{(1+e)^2} (1+e \cos \theta_2)^6 \right\}
\]

\[\frac{61}{4} (1+e)^4 + \frac{37}{4} (1+e \cos \theta_2)^4 \right\} x
\]

\[\frac{2(2+e^2)}{(1-e^2)^{1/2}} \sin \theta_2 \tan^{-1}\left[\sqrt{\frac{(1-e^2)}{1+e}} \tan \frac{\theta_2}{2}\right] - (2+e+3e \cos \theta_2) \cdot \frac{(1-\cos \theta_2)}{(1+e \cos \theta_2)} \geq \frac{1}{3} \quad (3.2.9)
\]

3.3 SHEAR STRESS, FRICTIONAL FORCE AND FRICTIONAL COEFFICIENT:

The aggregate of the equations (2.4.4), (2.4.5), (3.1.1), (3.1.3), (3.1.8), (3.1.10) and (3.1.11) yields the components of the shear stress as follows:
\[ \tau_x = \frac{\mu u}{c(1 + e \cos \theta)} \]
\[ + \left\{ -\frac{M \rho^2 U^2 c^3 e \sin \theta}{120 \mu R} \right\} \left( 1 + e \cos \theta_2 \right)^3 \]
\[ \left( \frac{M^2 \rho^2 U c^3}{48 \mu R} \right) \left\{ (1 + e \cos \theta_2)^3 - (1 + e \cos \theta)^3 \right\} \]
\[ \left( \frac{M^3 \rho^4 U^2 c^7}{645120 \mu^3 R} \right) \times \]
\[ \left[ \frac{(1 + e \cos \theta_2)^3}{\theta_2} \right] \left\{ 76 \left( 1 + e \cos \theta_2 \right)^3 \left( 1 - \cos \theta_2 \right) - 61 (1 + e)^4 \right\} \]
\[ + 24 (1 + e \cos \theta_2)^6 (1 + e)^{-2} + 37 (1 + e \cos \theta_2)^4 \}
\[ + 4 e \sin \theta \left\{ -130 (1 + e \cos \theta)^3 (1 + e \cos \theta_2)^3 \right\} \]
\[ + 169 (1 + e \cos \theta)^6 + 15 (1 + e \cos \theta_2)^6 \} \]
\[ > (3.3.1) \]

and

\[ \tau_y = \left\{ \frac{M \rho U c}{12 (1 + e \cos \theta)} \right\} \left\{ (1 + e \cos \theta)^3 + 3 (1 + e \cos \theta_2)^3 \right\} \]
\[ + \left\{ \frac{M^2 \rho^3 U^2 c^5}{\mu^2 R} \right\} \times \]
\[ \left[ \frac{(1 + e \cos \theta_2)^3}{53760 \theta_2 (1 + e \cos \theta)} \right] \left\{ 76 \left( 1 + e \cos \theta_2 \right)^3 \left( 1 - \cos \theta_2 \right) - 61 (1 + e)^4 \right\} \]
\[ + 24 (1 + e)^{-2} (1 + e \cos \theta_2)^6 \]
\[ + 37 (1 + e \cos \theta_2)^4 \} \]
On substituting the values of $\tau_x$ and $\tau_y$ from the equations (3.3.1) and (3.3.2) in (2.3.3), the shear stress is completely determined.

The frictional force on the infinitely long journal bearing defined by

$$F_j = \int_{\theta=0}^{\theta=\theta_2} (\tau_y) \left( \frac{dv}{d\theta} \right) d\theta$$

is expressed as follows:

$$F_j = \frac{M \rho U R c}{12} \times \left[ (\theta_2 + e \sin \theta_2) + \frac{3(1+e \cos \theta_2)^3}{(1-e^2)^{3/2}} \right] \left[ 2 \tan^{-1} \left( \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta_2}{2} \right) - \frac{e \sqrt{(1-e^2)} \sin \theta_2}{(1+e \cos \theta_2)} \right]$$

$$+ \frac{M^2 \rho^3 U^2 c^5}{\mu^2} \times \left[ \frac{(1+e \cos \theta_2)^3}{53760 \theta_2} \left\{ 76e(1+e \cos \theta_2)^3(1-\cos \theta_2) - 61(1+e)^4 \right\} - 24(1+e)^{-2}(1+e \cos \theta_2)^6 + 37(1+e \cos \theta_2)^4 \right] x$$
\[
\frac{1}{(1-e^2)^{3/2}} \{ 2 \tan^{-1} \left( \frac{\sqrt{1-e^2}}{1+e} \tan \frac{\theta}{2} \right) - \frac{e\sqrt{(1-e^2)}}{(1+e \cos \theta)} \sin \frac{\theta}{2} \} \\
+ \frac{e}{40320} \left\{ \frac{11}{2e} (1+e \cos \theta)^3 \left( 1 + \frac{e^2}{1+e} \right) \right\} \\
- \frac{47}{5e} \left( 1+e \cos \theta - 1+e \right) \\
- 6(1+e \cos \theta)^5 \frac{(1-\cos \theta)}{(1+e)^{-3}} \} > (3.3.4)
\]

The frictional coefficient for this infinitely long journal bearing defined by

\[ f = \frac{F}{W} \]  

(3.3.5)

is determined with the help of the equations (3.2.1), (3.2.4) - (3.2.9) and (3.3.4).

3.4 CONCLUSIONS:

The following conclusions are derived from the results of the foregoing sections of this chapter:

1. On putting the rotation number \( M = 0 \) in the equations (3.1.10), (3.2.4) - (3.2.9), (3.3.2) and (3.3.4) respectively, we see that the expressions for the pressure distribution, load capacity, frictional shearing stress along the bearing length and the frictional force on the bearing reduce to zero which revials that the existence of certain fundamental solutions within this
extended frame work is not allowed in any of the three theories of hydrodynamic lubrication (i) the classical, (ii) the inertia-free rotatory and (iii) the rotation-free inertial.

2. The solutions of the inertia-free rotatory system of this bearing are contained in the presented solutions and are expressed by the terms directly proportional to \( M \) and independent of \( \mu \).

3. An important qualitative result is that while the load capacity increases with increasing values of the coefficient of fluid viscosity for this bearing in the classical theory, it is independent of the fluid viscosity in the inertia-free rotatory frame of reference for the first order of effect of rotation and is increased by a quantity which is directly proportional to the product of the cube of fluid density, square of the fluid velocity along the span of the bearing system and inverse square of the coefficient of the fluid viscosity in the inertio-rotatory frame of reference of the first order.

   This indicates that low viscous fluids could also be successfully utilized as lubricants.

4. The inertio-rotatory effects of the first order contribute second order effects of rotation and inertia in the respective frames of references of rotation and inertia separately besides including first order effects of rotation for this sort of infinitely long journal bearing.