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CERTAIN TRIPLE SERIES EQUATIONS INVOLVING THE PRODUCT OF "r" JACOBI POLYNOMIALS

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The solution of certain triple series equations involving the product of "r" Jacobi polynomials is obtained in this paper by using the technique employed by Dwivedi and Trivedi [1976]. The solution of the series equation involving product of "r" Laguerre polynomials is derived as a particular case of the main result.

The solution of certain series equations involving Jacobi polynomials has recently been considered by Dwivedi and Trivedi [1976]. In this paper the solution of the following series equation involving the product of "r" Jacobi polynomials is obtained in the region $R$, given by:

$$R: \{0 < X_i < C_i \}, \quad (i=1, 2, \ldots, r).$$

$$\sum_{n_1, n_2, \ldots, n_r=0}^{\infty} \left[ A_{n_1, n_2, \ldots, n_r} \prod_{i=1}^{r} \left( \frac{\Gamma(a_i - a_i + n_i + 1)}{\Gamma(a_i + n_i + 1)} \right) \right]$$

$$\times P_{n_i}^{(\alpha_i, \beta_i + p_i)} \left( 1 - \frac{2x_i}{C_i} \right)$$

$$= f(x_1, x_2, \ldots x_r) \forall x_i \in R_1$$

$$\vdots$$ (1.1)

$$\sum_{n_1, n_2, \ldots, n_r=0}^{\infty} \left[ A_{n_1, n_2, \ldots, n_r} \prod_{i=1}^{r} \left( \frac{\Gamma(p_i + a_i + p_i + n_i + 1)}{C_i^{a_i + p_i} \Gamma(a_i + n_i + 1)} \right) \right]$$

$$\times P_{n_i}^{(\alpha_i + p_i, \beta_i)} \left( 1 - \frac{2x_i}{C_i} \right)$$

$$= 0, \forall x_i \in R_2$$

$$\vdots$$ (1.2)

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\( f(x_1, x_2, \ldots, x_r) \) is a prescribed function. The problem is that of determining the sequence \( \{ A_{n_1}, n_2, \ldots, n_r \} \) satisfying the simultaneous equations (1.1)—(1.3). The technique used is the one used earlier by Dwivedi and Trivedi [1976]. The regions \( R_1 \) and \( R_2 \) in (1.1), (1.2) are given by:

\[
R_1 : \{ 0 < X_i < b_i \} \\
R_2 : \{ b_i < x_i < C_i \}, \quad i = 1, 2, \ldots, r.
\]

The solution of the simultaneous series equations involving the product of \( "r" \) Laguerre polynomials is derived as a particular case of our findings.

Results Required

(i) The Jacobi polynomial is defined in terms of hypergeometric function as (Rainville (1971), p. 254, eqn. (1))

\[
P_n^{\alpha, \beta} \left( 1 - \frac{2x}{C} \right) = \frac{(1 + x)^n}{(n)!} \, _2F_1 \left[ -n, n+\alpha+\beta+1; 1+\alpha; \frac{X}{C} \right] \quad \ldots \ (2.1)
\]

(ii) We need the following relations

[\text{Erdelyi (1954), p. 191, eqns. (43), (44)}]

\[
\int_0^y x^\alpha (y-x)^{\mu-1} \, P_n^{(\alpha, \beta)} \left( 1 - \frac{2x}{C} \right) \, dx = \frac{\Gamma (\mu) \Gamma (n+\alpha+1)}{\Gamma (\alpha+\mu+n+1)} \, y^{\alpha+\beta} \times P_n^{(\alpha+\mu, \beta-\mu)} \left( 1 - \frac{2y}{C} \right)
\]

\[
\alpha > -1, \mu > 0 \ldots \quad \ldots \ (2.2)
\]

\[
\int_0^C \left( 1 - \frac{X}{C} \right)^{\beta} \, P_n^{\alpha, \beta} \left( 1 - \frac{2x}{C} \right) (x-y)^{\mu-1} \, dx = \frac{C^\mu \Gamma (\mu) \Gamma (n+1+\beta)}{\Gamma (\beta+\mu+n+1)} \times \left( 1 - \frac{2y}{C} \right)^{\beta+\mu} \, P_n^{(\alpha-\mu, \beta+\mu)} \left( 1 - \frac{2y}{C} \right)
\]

\[
\beta > -1, \mu > 0. \ldots \quad \ldots \ (2.3)
\]

(iii) The orthogonality relation for Jacobi polynomials given by [Rainville (1971) p. 258]

\[
\int_0^C x^\alpha \left( 1 - \frac{X}{C} \right)^{\beta} \, P_n^{\alpha, \beta} \left( 1 - \frac{2x}{C} \right) \, P_m^{\alpha, \beta} \left( 1 - \frac{2x}{C} \right) \, dx = \frac{C^{\alpha+1} \Gamma (\alpha+n+1) \Gamma (\beta+n+1)}{(n)! (2n+\alpha+\beta+1) \Gamma (n+\alpha+\beta+1)} \delta_{mn}, \quad \alpha > -1, \beta > -1
\]

\[
\ldots \quad \ldots \ (2.4)
\]

Where \( \delta_{mn} \) is Kronecker's delta.

(iv) The Abel integral equation in "r" dimension i.e.
The following formula involving Gamma function is also required:

\[ \lim_{\beta \to \infty} \left[ \frac{\Gamma(1 + \alpha + \beta)}{\Gamma(1 + \alpha)} \right] = 1 \]  


\[ L_n^{(\alpha)}(x) = \lim_{\beta \to \infty} P_n^{(\alpha, \beta)} \left( 1 - \frac{2x}{\beta} \right) \]  

\[ \beta 

The series equations

Here we consider the solution of the following triple series equations involving the product of \("r"\) Jacobi polynomials:

\[ \sum_{n_1, n_2, \ldots, n_r = 0}^{\infty} \left[ A_{n_1, n_2, \ldots, n_r} \prod_{i=1}^{r} \left( \frac{\Gamma(n_i + n_{i+1} + 1)}{\Gamma(n_i + n_{i+1} + 1)} \right) \right] \]
\( p_{n_1}^{(\alpha_i, \beta_i+p_i)} \left( 1-\frac{2x_i}{C_i} \right) = f(x_1, x_2, \ldots, x_r), \quad \forall x_i \in R_1 \quad \ldots (3.1) \)

\[
\sum_{n_1, n_2, \ldots, n_r = 0}^\infty \left[ A_{n_1, n_2, \ldots, n_r} \varpi_{i=1}^r \left( \frac{\Gamma(\beta_i+\sigma_i+p_i+n_i+1)}{C_i^{\sigma_i+p_i}} \right) \Gamma(\beta_i+n_i+1) \right.
\times \left. p_{n_i}^{(\alpha_i+p_i, \beta_i)} \left( 1-\frac{2x_i}{C_i} \right) \right] = 0, \quad \forall x_i \in R_2 \quad \ldots (3.2) \]

= 0, otherwise.

Where the parameters \( \alpha_i, p_i \) and \( \beta_i \) satisfy the following conditions:

for \( i = 1 \) to \( r \).

(i) \( \alpha_i + 1 > \sigma_i \)

(ii) \( 0 < \sigma_i < 1 \)

(iii) \( \beta_i + 1 > \max (0, -p_i) \)

(iv) \( 0 < \alpha_i + p_i < 1 \).

To solve (3.1)-(3.3), we assume that

\[
\sum_{n_1, n_2, \ldots, n_r = 0}^\infty \left[ A_{n_1, n_2, \ldots, n_r} \varpi_{i=1}^r \left( \frac{\Gamma(\beta_i+\sigma_i+p_i+n_i+1)}{C_i^{\sigma_i+p_i}} \right) \Gamma(\beta_i+n_i+1) \right.
\times \left. p_{n_i}^{(\alpha_i+p_i, \beta_i)} \left( 1-\frac{2x_i}{C_i} \right) \right] = \varpi_{i=1}^r \left( 1-\frac{X_i}{C_i} \right)^{\beta_i} \phi(x_1, x_2, \ldots, x_r), \quad \forall x_i \in R_1 \quad \ldots (3.4) \]

Where \( \phi(x_1, x_2, \ldots, x_r) \) is an unspecified function. In view of the orthogonality relation (2.4), it follows from equations (3.2), (3.3) and (3.4) that

\[
A_{n_1, n_2, \ldots, n_r} = \varpi_{i=1}^r
\times \left( \frac{(n_i)! (2n_i+\alpha_i+p_i+\beta_i+1)}{C_i^{(n_i+\alpha_i+p_i+\beta_i+1)}} \right)
\times \left( \frac{\Gamma(\beta_i+\sigma_i+p_i+n_i+1)}{\Gamma(\beta_i+n_i+1)} \right)
\times \int_0^{b_1} \int_0^{b_2} \ldots \int_0^{b_r} \phi(y_1, y_2, \ldots, y_r) \varpi_{i=1}^r \left[ y_i^{\alpha_i+p_i} p_{n_i}^{(\alpha_i+p_i, \beta_i)} \right]
\times \left( 1-\frac{2y_i}{C_i} \right) dy_i \quad \ldots (3.5) \]
If we now put
\[
S_t(x_t, y_i) = \sum_{n_1, n_2, \ldots, n_r=0}^{\infty} \frac{(n_t)! \Gamma(n_t + p_t + \beta_t)}{C_t^{(n_t - \alpha_t + 1)} \Gamma(\beta_t + p_t + \alpha_t + n_t + 1)} n_t \frac{\Gamma(n_t + p_t + \beta_t + 1) \Gamma(n_t + \alpha_t + n_t + 1)}{\Gamma(p_t + n_t + 1) \Gamma(\alpha_t + n_t + 1)} \cdot \frac{1}{C_t} \left(1 - \frac{2X_t}{C_t}\right)
\]
then it can be seen that (Dwivedi and Trivedi [1976], p. 952, eqn. (2.11))
\[
S_t(x_t, y_i) = \int_0^\infty \left( \begin{array}{c}
\Gamma(\alpha_t) \\
\Gamma(\beta_t)
\end{array} \right) \frac{x_t^{a_t-1} y_t^{b_t-1}}{-1 - \frac{2y_t}{C_t}} dt, \tag{3.6}
\]
where
\[
m_t = \min(x_t, y_t), \quad g_t(t) = \frac{(t_1 + p_1)^{\alpha_t} (t_2 + p_2)^{\alpha_t}}{(t_1 - \alpha_t) (t_2 - \alpha_t)}, \quad m_t = \min(x_t, y_t) \tag{3.8}
\]
provided that \((\alpha_t - \beta_t) > 1, (p_t + \alpha_t) > 0, \sigma_t > 0, \beta_t > -1.\)

To determine the unknown function \(f(y_1, y_2, \ldots, y_r),\) we substitute for \(A_{n_1, n_2, \ldots, n_r}\) from (3.5) into (3.1) and then interchanging the orders of integrations and summations, we get, in view of (3.6), the following form of (3.1)
\[
\int_0^{b_1} \int_0^{b_2} \int_0^{b_r} \left[ \phi(y_1, y_2, \ldots, y_r) \pi_i^{\alpha_t} \left( y_i^{a_t} \right) S_t(x_t, y_i) \right] dy_t \quad \forall X_t \in R_t \tag{3.10}
\]
using (3.7), (3.10) takes the form:
\[
\left[ \int_0^{b_1} \int_0^{b_2} \int_0^{b_r} \phi(y_1, y_2, \ldots, y_r) \pi_t^{\alpha_t} S_m(x_t, y_t) dy_t \right] \pi_i^{\alpha_t} \left( y_i^{a_t} \right) S_t(x_t, y_i) \right] dy_t \quad \forall X_t \in R_t \tag{3.11}
\]
This can be written as:
\[
\int_0^{b_1} \int_0^{b_2} \int_0^{b_r} \phi(y_1, y_2, \ldots, y_r) S_{y_1}(x_t, y_1) S_{y_2}(x_t, y_2) \ldots S_{y_r}(x_t, y_r) \ dy_1 \ldots \ dy_r
\]
\[= \left[ \pi^r_{i=1} \left( \Gamma \left( \sigma_i \right) \Gamma \left( \sigma_i + p_i \right) x_i^{\sigma_i} \right) f(x_1, x_2, \ldots, x_r) \right] \]  ... (3.12)

On putting the values of \( S_{x_i}, S_{y_i} \) from (3.8) into (3.12) and then interchanging the orders of integrations in (3.12) and rearranging the terms, we get

\[
\int_0^{x_1} \int_0^{x_2} \ldots \int_0^{x_r} \left[ \pi^r_{i=1} \left( g_i \left( t_i \right) \left( x_i - t_i \right)^{\alpha_i - 1} \right) \right] \phi(t_1, t_2, \ldots, t_r) \, dt_1 \, dt_2 \ldots dt_r
\]

\[= \left[ \pi^r_{i=1} \left( \Gamma \left( \sigma_i \right) \Gamma \left( \sigma_i + p_i \right) x_i^{\alpha_i} \right) \right] f(x_1, x_2, \ldots, x_r) \]  ... (3.13)

where

\[
\phi \left( t_1, t_2, \ldots, t_r \right) = \int_{t_1}^{h_1} \int_{t_2}^{h_2} \ldots \int_{t_r}^{h_r} \left[ \pi^r_{i=1} \left( y_i - t_i \right)^{\sigma_i + p_i - 1} \right] \times \phi \left( y_1, y_2, \ldots, y_r \right) \, dy_1 \, dy_2 \ldots dy_r
\]

\[0 < t_i < b_i, \, i = 1, 2, \ldots, r \]  ... (3.14)

(3.13) is Abel integral equation in "r" dimension and inverting it with the help of (2.5), we get :

\[g_1(x_1) g_2(x_2) \ldots g_r(x_r) \phi(x_1, x_2, \ldots, x_r) = \frac{\sin \left( \sigma_1 \pi \right) \sin \left( \sigma_2 \pi \right) \ldots \sin \left( \sigma_r \pi \right)}{\pi^r} \]

\[
\times \pi^r_{i=1} \left[ \Gamma \left( \sigma_i \right) \Gamma \left( \sigma_i + p_i \right) \right] \frac{\varphi}{\partial x_1 \partial x_2 \ldots \partial x_r} \left( \int_0^{x_1} \int_0^{x_2} \ldots \int_0^{x_r} \right)
\]

\[\lambda_{1}^{r_{1}} \lambda_{2}^{r_{2}} \ldots \lambda_{r}^{r_{r}} f(t_1, t_2, \ldots, t_r) \, dt_1 \, dt_2 \ldots dt_r \]

\[\left( x_1 - t_1 \right)^{\sigma_1} \left( x_2 - t_2 \right)^{\sigma_2} \ldots \left( x_r - t_r \right)^{\sigma_r} \]  ... (3.15)

Again, inverting the Abel integral equation (3.14) with the help of (2.6), we get the desired function \( \phi \left( y_1, y_2, \ldots, y_r \right) \) in the following form :

\[\phi \left( y_1, y_2, \ldots, y_r \right) = \frac{\left( -1 \right)^r \sin \left( \sigma_1 + p_1 \right) \pi \sin \left( \sigma_2 + p_2 \right) \pi \ldots \sin \left( \sigma_r + p_r \right) \pi}{\pi^r} \]

\[
\times \frac{\varphi}{\partial y_1 \partial y_2 \ldots \partial y_r} \left( \int_{y_1}^{b_1} \int_{y_2}^{b_2} \ldots \int_{y_r}^{b_r} \right) \]

\[\frac{\phi \left( t_1, t_2, \ldots, t_r \right) \, dt_1 \, dt_2 \ldots \, dt_r}{\left( t_1 - y_1 \right)^{\sigma_1 + p_1} \left( t_2 - y_2 \right)^{\sigma_2 + p_2} \ldots \left( t_r - y_r \right)^{\sigma_r + p_r}} \]  ... (3.16)

where \( \phi \left( t_1, t_2, \ldots, t_r \right) \) is given by (3.15) or alternatively, by :

\[\phi \left( t_1, t_2, \ldots, t_r \right) = \frac{\sin \left( \sigma_1 \pi \right) \sin \left( \sigma_2 \pi \right) \ldots \sin \left( \sigma_r \pi \right)}{\pi^r} \pi^r_{i=1} \left[ \Gamma \left( \sigma_i \right) \Gamma \left( \sigma_i + p_i \right) \right] \]

\[
\times \pi^r_{i=1} \left( 1 - \frac{t_i}{C_i} \right) \left( \frac{\varphi}{\partial t_1 \partial t_2 \ldots \partial t_r} \right) \]

\[\times \left( \int_{t_1}^{l_1} \int_{t_2}^{l_2} \ldots \int_{t_r}^{l_r} \right) \left( x_1^{\sigma_1} \ldots x_r^{\sigma_r} \right) f(x_1, x_2, \ldots, x_r) \, dx_1 \, dx_2 \ldots dx_r \]
obtained by putting the values of \( g_i(t_i), (i=1 \text{ to } r) \) from (3.9) in (3.16).

The desired sequence \( \{A_{n_1, n_2, \ldots, n_r}\} \) satisfying the series equations (3.1)-(3.3) is thus given by (3.5) where \( \phi(y_1, y_2, \ldots, y_r) \) is given by (3.16).

**A Particular Case**

If for \( i=1, 2, \ldots, r \), we put \( C_i = \beta_i \) in equation (3.1) to (3.3) and let \( \beta_i \to \infty \) there, in view of (2.7), we find that these reduce to the following series equations involving the product of \( "r" \) Laguerre polynomials:

\[
\sum_{n_1, n_2, \ldots, n_r}^{\infty} \left[ A_{n_1, n_2, \ldots, n_r} \pi_{i=1}^{r} \left( \frac{\Gamma(\sigma_i - \sigma_i + n_i + 1)}{\Gamma(\alpha_i + n_i + 1)} \right) \times L_{n_i}^{(\alpha_i)}(x_i) \right] = f(x_1, x_2, \ldots, x_r) \quad \forall x_i \in R_1 \quad \ldots (4.1)
\]

\[
\sum_{n_1, n_2, \ldots, n_r=0}^{\infty} \left[ A_{n_1, n_2, \ldots, n_r} \pi_{i=1}^{r} \left( L_{n_i}^{(\alpha_i + p_i)}(x_i) \right) \right] = 0, \quad b_i < x_i < \infty, \quad \ldots (4.2)
\]

\[
= 0, \quad \text{otherwise.} \quad \ldots (4.3)
\]

where \( \alpha_i \) and \( p_i \) satisfy the conditions: for \( i=1, 2, \ldots, r \).

(i) \( \alpha_i + 1 > \sigma_i \)

(ii) \( 0 < \sigma_i < 1 \)

(iii) \( 0 < \sigma_i + p_i < 1 \)

The sequence \( \{A_{n_1, n_2, \ldots, n_r}\} \) satisfying (4.1)-(4.3) is derived from (3.5) and is given by:

\[
A_{n_1, n_2, \ldots, n_r} = \pi_{i=1}^{r} \left( \frac{(n_i)!}{\Gamma(\alpha_i + n_i + 1)} \right) \times \int_{b_1}^{\beta_1} \int_{b_2}^{\beta_2} \cdots \int_{b_r}^{\beta_r} \left[ \pi_{i=1}^{r} \left( y_i^{\alpha_i + p_i} L_{n_i}^{(\alpha_i + p_i)}(y_i) \right) \right] \times \phi(y_1, y_2, \ldots, y_r) dy_1 dy_2 \cdots dy_r \quad \ldots (4.4)
\]

where \( \phi(y_1, y_2, \ldots, y_r) \) is given by (3.16) and \( \phi(t_1, t_2, \ldots, t_r) \) is given by:

\[
\phi(t_1, t_2, \ldots, t_r) = \frac{\sin(\sigma_1 \pi) \sin(\sigma_2 \pi) \cdots \sin(\sigma_r \pi)}{\pi^r} \times \pi_{i=1}^{r} \left[ \Gamma(\sigma_i) \Gamma(\alpha_i + p_i) \right] \times \frac{\sigma_i - \alpha_i}{t_i^{\sigma_i - \alpha_i}} e^{-t_i} \frac{\partial^r}{\partial t_1 \partial t_2 \cdots \partial t_r}
\]
REFERENCES


\[
\times \left( \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_r} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \cdots \alpha_r^{\alpha_r} f(x_1, x_2, \ldots, x_r)}{(t_1-x_1)^{\alpha_1} (t_2-x_2)^{\alpha_2} \cdots (t_r-x_r)^{\alpha_r}} \right) dx_1 \, dx_2 \, \ldots \, dx_r
\]

obtained by using (2.9) in (3.16).
SOME SIX INTEGRAL EQUATIONS INVOLVING INVERSE MELLIN TRANSFORMS

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ABSTRACT

In this paper the solution of six integral equations involving inverse Mellin transform has been obtained by reducing them to a system of Fredholm integral equations.

1. Introduction

In this paper we solve the six integral equations of the form:

\[ M^{-1}\left[ \frac{\Gamma(1+\eta-S/\sigma)}{\Gamma(1+\eta+a-S/\sigma)} \phi(S) ; x \right] = f_1(x), \quad 0 \leq x < a_1, \]  

\[ M^{-1}\left[ \frac{\Gamma(\xi+S/\beta)}{\Gamma(\xi+\beta+S/\beta)} \phi(S) ; x \right] = g_2(x), \quad a_1 \leq x < a_2, \]  

\[ M^{-1}\left[ \frac{\Gamma(1+\eta-S/\sigma)}{\Gamma(1+\eta+a-S/\sigma)} \phi(S) ; x \right] = f_3(x), \quad a_2 \leq x < a_3, \]  

\[ M^{-1}\left[ \frac{\Gamma(\xi+S/\beta)}{\Gamma(\xi+\beta+S/\beta)} \phi(S) ; x \right] = g_4(x), \quad a_3 \leq x < a_4, \]  

\[ M^{-1}\left[ \frac{\Gamma(1+\eta-S/\sigma)}{\Gamma(1+\eta+a-S/\sigma)} \phi(S) ; x \right] = f_5(x), \quad a_4 \leq x < a_5, \]  

\[ M^{-1}\left[ \frac{\Gamma(1+\eta-S/\sigma)}{\Gamma(1+\eta+a-S/\sigma)} \phi(S) ; x \right] = f_6(x), \quad a_5 \leq x < a_6, \]

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where \( a, \beta, \xi, \eta, \delta > 0, \sigma > 0 \) are real parameters \( f_1, \ g_2, \ g_3, \ g_4, \ g_5 \) and \( g_6 \) are prescribed functions while \( \varphi(S) \) is an unknown function to be determined and

\[
M[h(x); S] = H(S)
\]
and

\[
M^{-1}[H(S); x] = h(x)
\]
denote the Mellin transform of \( h(x) \) and its inversion formula respectively.

The above equations are an extension of the triple integral equations solved by Lowndes [4] and quadruple integral equations solved recently by Dwivedi, Kushwaha and Trivedi [8] by means of a systematic application of some slightly extended forms of Erdelyi-Kober operators of fractional integration [2].

Using the properties of extended form of Erdelyi-Kober operators given in [4] we show in purely formal manner that the solution of the six integral equations can be expressed in terms of the solution of a Fredholm integral equation of the second kind. The method of solution employed here will be seen to follow closely that used by Ahmad [7] to obtain the solution of some quadruple integral equations involving Bessel functions.

2. Fractional Integral Operators

We recall here a few definitions and properties of operators used in solving the triple and quadruple integral equations. Lowndes [4] has defined the following operators:

\[
I_{\nu+\alpha} (a, b; \sigma) f(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{I(a)} \int_a^b (x^\sigma - t^\sigma)^{\alpha-1} t^\sigma(1-\eta-1)(t)dt, \quad \alpha > 0,
\]

\[
= \frac{x^{\nu-\sigma(\alpha+\eta)}}{I(1+a)} \frac{d}{dx} \int_a^b (x^\sigma - t^\sigma)^{\alpha-1} t^\sigma(1-\eta-1)(t)dt,
\]

\( -1 < a < 0 \), \( \nu > 0 \), \( \sigma > 0 \).

\[
K_{\eta+\alpha} (c, d; \sigma) f(x) = \frac{\sigma x^\sigma}{I(a)} \int_c^d (t^\sigma - x^\sigma)^{\alpha-1} t^\sigma(1-\eta-1)(t)dt, \quad \alpha > 0,
\]

\[
= -\frac{x^{\sigma(\nu-1)+1}}{I(1+a)} \frac{d}{dx} \int_c^d (t^\sigma - x^\sigma)^{\alpha(1-\eta-1)} f(t)dt,
\]

\( -1 < a < 0 \), \( \sigma > 0 \).

where \( a < x < b, \sigma > 0 \).
From the theory of Abel integral equations it follows that the inverse operators are given by

\[ I_{11,1}^{-1}(a, b ; \sigma) f(x) = I_{11,1}(a, b ; \sigma) f(x), \]  

\[ K_{11,1}^{-1}(c, d ; \sigma) f(x) = K_{11,1}(c, d ; \sigma) f(x). \]

We require two lemmas given by Lowndes [4] which define the pairs of operators:

**Lemma A.** Let \( I_{11,1}(a, b ; \sigma) \) and \( I_{11,1}(d, x ; \sigma) \) be operators as defined earlier, then

\[
I^{-1}(d, x ; \sigma) I(a, b ; \sigma) f(x) = \frac{\sigma \sin (a \pi)}{\pi} \cdot \frac{x^{\sigma-\eta}}{(x^{\sigma}-d^{\sigma})} \times \\
\times \int_{a}^{b} \frac{t^{\sigma(\eta+1)-1}}{(t^{\sigma}-x^{\sigma})} \frac{(d^{\sigma}-t^{\sigma})^{\alpha}}{f(t) \, dt},
\]

provided \( x>d>b>a. \)

**Lemma B.** Let \( K_{11,1}(a, b; \sigma) \) and \( K_{11,1}(x, d; \sigma) \) be operators defined as earlier, then

\[
K_{11,1}^{-1}(x, d; \sigma) K_{11,1}(a, c; \sigma) f(x) = \frac{\sigma \sin (a \pi)}{\pi} \cdot \frac{x^{\sigma(\alpha+\eta)}}{(d^{\sigma}-x^{\sigma})} \times \\
\times \int_{a}^{b} \frac{t^{\sigma(1-\alpha-\eta)-1}}{(t^{\sigma}-x^{\sigma})} \frac{(t^{\sigma}-d^{\sigma})^{\alpha}}{f(t) \, dt},
\]

provided \( x<d<a<b. \)

Two well-known results [4] which play an important part in our solution are:

\[
M[I_{11,1}(0, x; \sigma) f(x); S] = \frac{\Gamma(1+\eta-S/\sigma)}{\Gamma(1+\eta+a-S/\sigma)} M[f(x); S]
\]

(17)

\[
M[X_{11,1}(x, \infty; \sigma) f(x); S] = \frac{\Gamma(\eta+S/\sigma)}{\Gamma(\eta+a+S/\sigma)} M[f(x); S]
\]

(18)

3. Solution of Six Integral Equations

Let \( I_i \) denote the interval \((0, a_i)\), \( I_2 \) the interval \((a_1, a_2)\), \( I_3 \) the interval \((a_2, a_3)\), \( I_4 \) the interval \((a_3, a_4)\), \( I_5 \) the interval \((a_4, a_5)\) and \( I_6 \) the interval \((a_5, \infty)\). For a function \( f \) in \( L_4(0, \infty) \) we shall write \( f_1 + f_2 + f_3 + f_4 + f_5 + f_6 \) where \( f_i = f \) on \( I_i \) and \( f_i = 0 \) on \( I_j \) (i, j = 1, 2, 3, 4, 5, 6; i \neq j) and similarly for \( g \). We can write six integral equations as
\[ M^{-1} \left[ \frac{I(1+\eta-S/a)}{I(1+\eta+a-S/a)} \bar{f}(S) ; x \right] = f(x) \]

\[ M^{-1} \left[ \frac{I(\xi+S/\beta)}{I(\xi+\beta+S/\beta)} \bar{g}(S) ; x \right] = g(x) \]

where \( f_1, g_2, f_3, g_4, f_5 \) and \( g_6 \) are prescribed functions while \( g_1, f_2, g_3, f_4, g_5, f_6 \) are unknown functions to be determined. If we write

\[ \bar{f}(S) = M[f(x) ; S], \]

and use the formula (17) and (18) we find that equations (19) and (20) assume the operational form

\[ I_{\eta,\alpha}(0, x ; c) \phi(x) = f(x) \]

\[ K_{\xi,\beta}(x, \sigma ; \delta) \phi(x) = g(x) \]

Using the formula (13) and (14) and solving the above equations for \( \phi(x) \), we obtain

\[ \phi(x) = I_{\eta+\alpha,0}(0, x ; c) f(x), \]

\[ = K_{\xi,\beta,0}(x, \infty ; \delta) g(x). \]

We proceed to determine \( \phi \). The subscripts on all the operators \( I' \)'s will be supposed to have the subscripts \((\eta, \alpha ; c) \) understood and all \( K' \)'s to have subscript \((\xi, \beta ; \delta) \). Evaluating (23) on \( I_1, I_2, I_3, I_4 \) and \( I_5 \), we get

\[ \phi = \left( \begin{array}{c} x \\ 0 \end{array} \right) I^{-1} f_1 \]

\[ \phi_2 = \left( \begin{array}{c} a_1 \\ 0 \end{array} \right) I^{-1} f_1 + \left( \begin{array}{c} x \\ a_1 \end{array} \right) I^{-1} f_2 \]

\[ \phi_3 = \left( \begin{array}{c} a_1 \\ 0 \end{array} \right) I^{-1} f_1 + \left( \begin{array}{c} a_2 \\ a_1 \end{array} \right) I^{-1} f_2 + \left( \begin{array}{c} x \\ a_2 \end{array} \right) I^{-1} f_3 \]

\[ \phi_4 = \left( \begin{array}{c} a_1 \\ 0 \end{array} \right) I^{-1} f_1 + \left( \begin{array}{c} a_2 \\ a_1 \end{array} \right) I^{-1} f_2 + \left( \begin{array}{c} a_3 \\ a_2 \end{array} \right) I^{-1} f_3 \left( \begin{array}{c} x \\ a_3 \end{array} \right) I^{-1} f_4 \]

\[ \phi_5 = \left( \begin{array}{c} a_1 \\ 0 \end{array} \right) I^{-1} f_1 + \left( \begin{array}{c} a_2 \\ a_1 \end{array} \right) I^{-1} f_2 + \left( \begin{array}{c} a_3 \\ a_2 \end{array} \right) I^{-1} f_3 \left( \begin{array}{c} a_4 \\ a_3 \end{array} \right) I^{-1} f_4 + \left( \begin{array}{c} x \\ a_4 \end{array} \right) I^{-1} f_5 \]
Similarly evaluating (24) on $I_2, I_5, I_4, I_5$ and $I_6$ we get

\[ \phi_2 = (a_2) K^{-1}g_2 + (a_3) K^{-1}g_3 + (a_4) K^{-1}g_4 + (a_5) K^{-1}g_5 + (\infty) K^{-1}g_6 \] (30)

\[ \phi_3 = (a_2) K^{-1}g_2 + (a_4) K^{-1}g_4 + (a_5) K^{-1}g_5 + (\infty) K^{-1}g_6 \] (31)

\[ \phi_4 = (a_2) K^{-1}g_2 + (a_5) K^{-1}g_5 + (\infty) K^{-1}g_6 \] (32)

\[ \phi_5 = (a_5) K^{-1}g_5 + (\infty) K^{-1}g_6 \] (33)

\[ \phi_6 = (\infty) K^{-1}g_6 \] (34)

Since $f_1$ and $g_6$ are known, $\phi_1$ and $\phi_6$ can be determined by the equations (25) and (34) respectively. We now solve (26) for $f_2$ and substitute its value in (27) to get

\[ \phi_2 = (a_2) I^{-1}f_1 + (a_3) I^{-1} \left( \frac{x}{a_1} \right) I \left[ \phi_2 - \left( \frac{a_1}{a_2} \right) I^{-1}f_1 \right] + (x) I^{-1}f_3, \] (35)

We solve (33) for $g_5$ and substitute its value in (32) to get

\[ \phi_4 = (a_2) K^{-1}g_2 + (a_5) K^{-1} \left( \frac{x}{a_1} \right) K \left[ \phi_4 - \left( \frac{a_1}{a_5} \right) K^{-1}g_6 \right] + (\infty) K^{-1}g_6. \] (36)

Similarly from equation (30) we find out the value of $\phi_2$ in terms of $\phi_3$ and $\phi_6$ with the help of equations (31) and (33),

\[ \phi_3 = (a_2) K^{-1}g_2 + (a_3) K^{-1}g_3 + (a_4) K^{-1}g_4 + (a_5) K^{-1}g_5 - \]

\[ - \left( \frac{a_3}{a_5} \right) K^{-1}g_6 - \left( \frac{a_4}{a_3} \right) K^{-1} \left( \frac{x}{a_3} \right) K \left\{ \phi_3 - \left( \frac{a_3}{a_5} \right) K^{-1}g_5 \right\} \]

\[ + \left( \frac{a_3}{a_4} \right) K^{-1}g_4 + \left( \frac{a_5}{a_4} \right) K^{-1} \left( \frac{x}{a_4} \right) K \left\{ \phi_5 - \left( \frac{a_3}{a_5} \right) K^{-1}g_6 \right\} + (\infty) K^{-1}g_6 \] (37)

and with the help of equations (26) and (28), that of $\phi_5$ in terms of $\phi_2$ and $\phi_4$ from equation (29),

\[ \phi_5 = (a_2) I^{-1}f_1 + (a_3) I^{-1} \left( \frac{x}{a_1} \right) I \left[ \phi_5 - \left( \frac{a_1}{a_2} \right) I^{-1}f_1 \right] \]
We have thus arrived at six simultaneous equations (25), (37), (35), (36), (38) and (34) associated with six unknown functions $\phi_1$, $\phi_2$, $\phi_3$, $\phi_4$, $\phi_5$ and $\phi_6$ respectively. From these equations, we can obtain the values of these unknowns and hence the solution of the problem will be determined by equation (20a).

4. An Application

We shall consider now the six integral equations which are extension of triple integral equations solved by Lowndes [4];

\[
\int_0^x u^{-2n} \psi(u) J_{2p}(ux) \, dx = F_1(x), \quad 0 \leq x < a_1
\]

\[= F_2(x), \quad a_2 < x < a_3 \]

\[= F_3(x), \quad a_4 < x < a_5 \]

\[= 0, \quad a_1 < x < a_2, \]

\[= 0, \quad a_3 < x < a_4, \]

\[= 0, \quad a_6 < x < \infty, \]

where $J_{2p}(ux)$ is the Bessel function of the first kind of order $2p$, $F_1(x)$, $F_2(x)$ and $F_3(x)$ are prescribed functions and $\psi(u)$ is to be determined. When $p = q$ and $a_4 = a_5 = \infty$ these are equations investigated by Ahmad [7]. We now show, in a fairly straightforward manner, that the above equations can be transformed into equations of the type (1) to (6) with $g_2 = g_4 = g_6 = 0$.

Denoting the Mellin transform of $\psi(u)$ by

\[M[\psi(u) ; S] = \bar{\psi}(S),\] (41)

and using the result

\[M\left[ \xi^{-2n} J_{2p}(\xi) ; S \right] = 2^{s-1-2n} \frac{\Gamma(q-\frac{S}{2})}{\Gamma(1+n+q-S/2)},\] (42)

We have, on applying the Faltung theorem for Mellin transforms [3], that the integral equations (39) and (40) can be written in the form
\[ M^{-1} \left[ \frac{\Gamma (1+p+S/2)}{\Gamma (1+n+q-S/2)} \bar{\phi}(S) ; x \right] = 2^{1+2n} x^{-2n} F_1(x), \quad 0 < x < a_1 \]
\[ \quad = 2^{1+2n} x^{-2n} F_2(x), \quad a_2 < x < a_3 \]
\[ \quad = 2^{1+2n} x^{-2n} F_3(x), \quad a_4 < x < a_5 \]

\[ M^{-1} \left[ \frac{\Gamma (p+S/2)}{\Gamma (q-n+S/2)} \phi(S) ; x \right] = 0, \quad a_4 < x < a_3 \]
\[ \quad = 0, \quad a_3 < x < a_4 \]
\[ \quad = 0, \quad a_6 < x < \infty, \] (43)

where
\[ \bar{\phi}(S) = 2^{s} \frac{\Gamma (q-n+S/2)}{\Gamma (1+p-S/2)} \bar{\phi}(1-S). \] (45)

These equations are the same as equations (1) to (6) with \( \sigma = \delta = 2, \xi = n = p, \alpha = q - p + n, \beta = q - p - n \) and

\[ f_1(x) = 2^{1+2n} x^{-2n} F_1(x) \]
\[ f_2(x) = 2^{1+2n} x^{-2n} F_2(x) \]
\[ f_3(x) = 2^{1+2n} x^{-2n} F_3(x) \]
\[ g_2 = g_4 = g_6 = 0. \] (46)

Using the results of previous section we have therefore, that the solution of equations (43) and (44) can be found in terms of a function \( \phi(x) \) given by

\[ \bar{\phi}(S) = M[\phi(x) ; x], \] (47)

where the functions \( \phi_1, \phi_2, \phi_3, \phi_4, \phi_5 \) and \( \phi_6 \) are obtained from equations (25), (37), (35), (36), (38) and (34) with the parameters \( \xi, n \) etc. given by equation (46).

Finally, in order to find the solution of integral equations (39) and (40) in terms of \( \phi(x) \) we proceed in the following way.

From equation (41) we have that the solution is \( \psi(u) = M^{-1} [\bar{\phi}(S) ; u] \)

\[ = M^{-1} \left[ 2^{1+1-S} \frac{\Gamma(1/2+p+S/2)}{\Gamma(1/2+q-n-S/2)} M[\phi(x) ; (1-S)]; u \right]. \] (48)

On using equations (45) and (47). Inverting the order of integration in the last equation we get

\[ \psi(u) = \int_{0}^{\infty} \phi(x) M^{-1} \left[ 2^{1+1-S} \frac{\Gamma(1/2+p+S/2)}{\Gamma(1/2+q-n-S/2)} ; ux \right] dx, \] (49)
\[
\left( \frac{ux}{2} \right)^{1+n+p+q} \phi(x) J_{p+q-n}(ux) \, dx,
\]

after applying the result (42). When \( p = q \) and \( a_4 = a_6 = \infty \) this solution is exactly same as that found by Ahmad [7].

REFERENCES

On simultaneous dual series equations involving Konhauser biorthogonal polynomials

(Konhauser biorthogonal polynomials/dual series equation/Abel integral equations)

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ABSTRACT By using Abel integral equations, we solve simultaneous dual series equations involving Konhauser biorthogonal polynomials.

Dual series equations

\[ \sum_{n=0}^{\infty} \sum_{j=1}^{s} \frac{A_{n_j}}{\Gamma(\delta + 2\beta + kn_j)} Z^\delta_{n_j} (x ; k) = f_i (x) \]

and

\[ \sum_{n=0}^{\infty} \sum_{j=1}^{s} \frac{A_{n_j}}{\Gamma(\delta + \beta + 1 + kn_j)} Z^\delta_{n_j} (x ; k) = g_i (x) \]

where \([Z^\delta_{n_j} (x ; k)]_{n=0}^{\infty}\) is the Konhauser biorthogonal polynomial set, \(\delta = 0, 0 > \beta > -1, f_i (x)\) and \(g_i (x)\) are known functions and \(A_{n_j}\) is unknown constant which is to be determined, have been solved.

We require the biorthogonal properties of the Konhauser biorthogonal polynomials:\n
\[ \int_{0}^{\infty} \exp (-x) x^\delta Z_n^\delta (x ; k) dx = 0, \text{ if } m \neq n \]

\[ = \frac{\Gamma(1 + \delta + kn)}{n!} \text{ if } m = n \]

where \(\delta > -1\).

The second formula required is the Weyl integral given by Karande and Thakare:\n
\[ \int_{0}^{\infty} \frac{\exp (-x) (x-\xi)^{\beta-1} Z_n^\delta + \beta (x ; k) dx}{\Gamma(\delta + \beta + 1 + kn)} = \frac{\Gamma(\beta)}{\Gamma(\delta + 2 \beta + kn)} Z_n^\delta + 2 \beta (\xi ; k) \]

where \(\delta + 2 \beta > 0, \beta > 0, \delta > -1\).

The third result that we require is

\[ \frac{d}{d\xi} \int_{0}^{\infty} (x-x-\xi)^{\beta-1} x^{\delta + \beta} Z_n^{\delta + \beta} (x ; k) dx = \frac{\Gamma(\beta) \Gamma(\delta + \beta + 1 + kn)}{\Gamma(\delta + 2 \beta + kn)} Z_n^\delta + 2 \beta - 1 (\xi ; k) \]

where \(\delta + 2 \beta > 0, \beta > 0, \delta > -1\).

We have the Riemann–Liouville fractional integral\(^3\) given by Prabhakar:\n
\[ \int_{0}^{\infty} x^{\delta + \beta} (\xi-x)^{\beta-1} Z_n^{\delta + \beta} (x ; k) dx = \frac{\Gamma(\delta + \beta + 1 + kn) \Gamma(\beta) \xi^{\delta + 2 \beta} Z_n^{\delta + 2 \beta} (\xi ; k)}{\Gamma(1 + \delta + 2 \beta + kn)} \]

where \(\beta > 0, \delta + 1 > 0\).

If \(f(\xi)\) and \(f'(\xi)\) are continuous in \(0 \leq x < \infty\) and if \(0 < \beta < 1\), then the solutions of Abel integral equations

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\[ f_1(\xi) = \int_{0}^{\xi} \frac{F_1(x)}{(x-\xi)\beta} \, dx \]

and

\[ f_2(\xi) = \int_{\xi}^{\infty} \frac{F_2(x)}{(x-\xi)\beta} \, dx \]

are respectively given by

\[ F_1(x) = \frac{\sin \beta \pi}{\pi} \frac{d}{dx} \int_{0}^{x} \frac{f_1(\xi)}{(x-\xi)^{1-\beta}} \, d\xi \]

and

\[ F_2(x) = -\frac{\sin \beta \pi}{\pi} \frac{d}{dx} \int_{x}^{\infty} \frac{f_1(\xi)}{(x-\xi)^{1-\beta}} \, d\xi \]

**Solution of the Equations:**

From (5) and (1), we get

\[ \frac{d}{d\xi} \int_{0}^{\xi} (\xi-x)^{\beta-1} x^{\delta+\beta} Z_{n_j}^\beta(x) \, dx = \Gamma(\beta) \exp(-\xi) g_i(\xi) \]

Let

\[ f_{1i}(\xi) = x^{\delta+\beta} p_i(x) \]

where

\[ p_i(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{a} \frac{A_{nj}}{\Gamma(\delta+\beta+1+kn_j)} \]

Multiplying both sides of (13) by \((\xi-x)^{\beta-1}\) and integrating with respect to \(x\) over \((0,\xi)\) and then differentiating with respect to \(\xi\), we get

\[ \frac{d}{d\xi} \int_{0}^{\xi} (\xi-x)^{\beta-1} f_{1i}(x) \, dx \]

Now using (9) and (11), we get

\[ F_{1i}(\xi) = \frac{1}{\pi} \left( \sin \beta \pi \Gamma(\beta) \right) \xi^{\delta+2\beta-1} f_i(\xi) \]

Again dividing both sides of (15) by \((x-\xi)^{\beta}\), integrating with respect to \(\xi\) over \((0, x)\) and then using (7), we get

\[ f_{1i}(x) = x^{\delta+\beta} p_i(x) \]

Let

\[ f_{2i}(x) = \exp(-x) p_i(x) \]

where \(p_i(x)\) is given by (14).

Similarly, multiplying both sides of (17) by \((x-\xi)^{\beta-1}\) and integrating with respect to \(x\) over \((\xi, \infty)\) and differentiating with respect to \(\xi\), we get by using eqns. (10) and (12).

\[ F_{2i}(\xi) = -\frac{\sin \beta \pi \Gamma(\beta)}{\pi} \frac{d}{d\xi} \left( \exp(-\xi) g_i(\xi) \right) \]
Dividing both sides of (18) by \((\xi - x)^\beta\), integrating with respect to \(\xi\) over \((x, \infty)\) and then using (8), we get
\[
f_{2,1}(x) = \exp(-x) p_1(x)
\]
\[
= -\frac{\sin \beta \pi \Gamma(\beta)}{\pi} \int_x^\infty \frac{(d/d\xi) \exp(-\xi) g_1(\xi)}{(\xi - x)^\beta} d\xi
\]
(19)
from (16) and (19), we write respectively
\[
p_i(x) = \frac{x^{\delta+\beta} \sin(\beta \pi \Gamma(\beta))}{\pi} \int_0^x \frac{\xi^{\delta+2\beta-1} f_i(\xi)}{(x - \xi)^\beta} d\xi
\]
(20)
and
\[
p_i(x) = -\frac{\exp(x) \sin(\beta \pi \Gamma(\beta))}{\pi} \int_x^\infty \frac{(d/d\xi) \exp(-\xi) g_1(\xi)}{(\xi - x)^\beta} d\xi
\]
(21)
The L.H.S. of (20) and (21) are identical, hence multiplying both by \(x^{\delta+\beta} \exp(-x) Y^\delta_{n_j}(x; k)\), integrating (20) with respect to \(x\) over \((0, y)\), integrating (21) with respect to \(x\) over \((y, \infty)\); adding and using the orthogonality relation (3), we get, with the help of (14), the solution of the dual series eqn. (1) and (2) in the form
\[
\sum_{j=1}^s \frac{A_{n_j}}{x} = \frac{1}{\pi} (\sin(\beta \pi \Gamma(\beta)) \int_0^y \exp(-x) \times \{ \int_x^\infty \frac{(d/d\xi) \exp(-\xi) g_1(\xi)}{(\xi - x)^\beta} d\xi \} dx
\]
(22)
or
\[
\sum_{j=1}^s A_{n_j} = \frac{1}{\pi} \frac{\sin(\beta \pi \Gamma(\beta))}{(n_j !) \Gamma(\beta)} \int_0^y \exp(-x) \times \{ \int_x^\infty \frac{(d/d\xi) \exp(-\xi) g_1(\xi)}{(\xi - x)^\beta} d\xi \} \}
x Y_{n_j}^\delta(\xi; k) f_j^*(\xi) d\xi
\]
(23)
with \(\delta > 0, \beta > 0\), where
\[
f_j^*(x) = \int_0^x \frac{\xi^{\delta+2\beta-1} f_i(\xi)}{(x - \xi)^\beta} d\xi
\]
Particular case: If we set \(s = 1\) in eqn. (1) and (2) then these reduce to dual series equations involving Konhauser biorthogonal polynomials and our solution (23) is in complete agreement with that of Patil and Thakare\(^5\).

REFERENCES

1. INTRODUCTION

Dual series equations involving Jacobi polynomials have been the subject of study in the recent past ([1], [3], [4], [5]). In the present paper, we obtain the exact solution of some dual series equations involving the product of 'r' Jacobi polynomials, which have not been considered as yet in the literature. The solution of dual series equations involving product of 'r' Laguerre polynomials has been deduced as particular case. The results obtained here are generalizations of those obtained earlier by Dwivedi and Trivedi [1].

2. DUAL SERIES EQUATIONS

In this section we give an exact solution of the dual series equations involving the product of 'r' Jacobi polynomials, we consider the equations

\[ \sum_{n_1, n_2, \ldots, n_r = 0}^{\infty} a_{n_1, n_2, \ldots, n_r} \prod_{i=1}^{r} \left( \frac{\Gamma(\alpha_i - \sigma_i + n_i + 1)}{\Gamma(\alpha_i + n_i + 1)} \right) \frac{(\alpha_i, \delta_i + p_i)}{p_n^i} (1 - \frac{2x_i}{C_i}) \]

\[ = f(x_1, x_2, \ldots, x_r), \ 0 < x_i < a_i; \ i = 1, 2, \ldots, r; \quad (2.1) \]

\[ \sum_{n_1, n_2, \ldots, n_r = 0}^{\infty} a_{n_1, n_2, \ldots, n_r} \prod_{i=1}^{r} \left( \frac{\Gamma(\alpha_i + p_i + n_i + 1)}{\Gamma(\delta_i + n_i + 1)} \right) \frac{(\alpha_i + p_i, \delta_i)}{C_i^{\delta_i + p_i} \Gamma(\delta_i + n_i + 1)} (1 - \frac{2x_i}{C_i}) \]
\begin{align*}
\mathcal{G}(x_1, x_2, \ldots, x_r), \quad a_i < x_i < C_i \\
i = 1, 2, \ldots, r.
\end{align*}

Where the parameters \(a_i, p_i, \sigma_i\) and \(\delta_i (i = 1 \text{ to } r)\) satisfy for some non-negative integers \(m_i\) and \(k_i\), the inequalities

(i) \(a_i + 1 > \max (0, \sigma_i, -p_i)\),

(ii) \(m_i - \sigma_i > 0\),

(iii) \(p_i + \delta_i + \sigma_i + 1 > m_i\),

(iv) \(p_i + \sigma_i + k_i > 0\) and

(v) \(\delta_i + 1 > k_i\).

The dual series equations (2.1) and (2.2) have been considered by Noble \([3]\) for \(p_i = 0, \delta_i > -1, a_i + 1 > \sigma_i\), \(0 < \sigma_i < 1, C_i = 1\) that is, when \(p_i = 0, m_i = 1, k_i = 0, C_i = 1\). In equations (2.1) and (2.2) if we put \(C_i = \delta_i\) and let \(\delta_i \to \infty\) then, on using \([1, \text{eqns.} (2.4) \text{ and } (2.5)]\), we find that they become

\begin{align*}
\sum_{n_1, n_2, \ldots, n_r} \mathcal{A}_{n_1, n_2, \ldots, n_r} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r} \\
\mathcal{F}(\alpha_1) \mathcal{L}_{n_1}(x_1) \mathcal{F}(\alpha_2) \mathcal{L}_{n_2}(x_2) \cdots \mathcal{F}(\alpha_r) \mathcal{L}_{n_r}(x_r) \\
= f(x_1, x_2, \ldots, x_r), \quad 0 < x_i < a_i.
\end{align*}

\begin{align*}
\sum_{n_1, n_2, \ldots, n_r} \mathcal{A}_{n_1, n_2, \ldots, n_r} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r} \\
\mathcal{L}_{n_1}^{(\alpha_1 + p_1)}(x_1) \mathcal{L}_{n_2}^{(\alpha_2 + p_2)}(x_2) \cdots \mathcal{L}_{n_r}^{(\alpha_r + p_r)}(x_r) \\
= g(x_1, x_2, \ldots, x_r), \quad a_i < x_i < \infty, \quad i = 1, 2, \ldots, r
\end{align*}

and the conditions (2.3) on the parameters reduce to the only genuine condition \(a_i + 1 > \max (0, \sigma_i, -p_i)\).

\(a_i + 1 > \max (0, \sigma_i, -p_i)\).
Equations (2.4) and (2.5) have been solved by Lowndes [2] for \( p_i = 0, \, 0 < \sigma_i < 1 \). More recently Srivastava [5] has given the solution of (2.4) and (2.5) valid under the condition (2.6) and an additional condition \( u_i + p_i > 0 \).

We proceed now to give the solution of the dual equation (2.1) - (2.2) under the conditions (2.3). Multiplying (2.1) by \( \pi^r \sum_{i=1}^{\alpha_i} (y_i - x_i)^{m_i - \sigma_i} \) and integrating over \((0, y_i)\) \( r \) times, (with \( y_i < a_i \)) we find, using [1, eqn. (2.8)], that

\[
\sum_{n_1, n_2, \ldots, n_r} = 0 \quad A_{n_1, n_2, \ldots, n_r} \pi^r
\]

\[
\left[ \frac{\Gamma(\alpha_i - \sigma_i + n_i + 1)}{\Gamma(\alpha_i - \sigma_i + m_i + n_i + 1)} \right]^{\left( \frac{\alpha_i + m_i - \sigma_i}{\alpha_i + m_i + n_i + 1} \right)}^{\left( \frac{\alpha_i + m_i - \sigma_i}{\alpha_i + m_i + n_i + 1} + \frac{\sigma_i - m_i}{\alpha_i + m_i + n_i + 1} \right)}
\]

\[
F \left[ \frac{1}{\Gamma(m_i - \sigma_i)} \right] \]

\[
(1 - \frac{2y_i}{C_i}) = \pi^r \left[ \frac{1}{\Gamma(m_i - \sigma_i)} \right]^{\left( \frac{\alpha_i + m_i - \sigma_i}{\alpha_i + m_i + n_i + 1} \right)}^{\left( \frac{\alpha_i + m_i - \sigma_i}{\alpha_i + m_i + n_i + 1} + \frac{\sigma_i - m_i}{\alpha_i + m_i + n_i + 1} \right)}
\]

\[
x \int_0^{y_i} \frac{f(x_1, x_2, \ldots, x_r)}{(y_i - x_i)} \left[ \frac{1}{\Gamma(m_i - \sigma_i)} \right] \left[ \frac{1}{\Gamma(m_i - \sigma_i)} \right] \]

\[
0 < y_i < a_i, \quad (i = 1 \text{ to } r)
\]

Differentiating (2.7) \( m_i \) times \( (i = 1 \text{ to } r) \) with respect to \( y_i \), and using [1, eqn. (2.6)], we get

\[
\sum_{n_1, n_2, \ldots, n_r} = 0 \quad A_{n_1, n_2, \ldots, n_r} \pi^r
\]

\[
\left[ \frac{\Gamma(\alpha_i - \sigma_i + n_i + 1)}{\Gamma(\alpha_i - \sigma_i + m_i + n_i + 1)} \right]^{\left( \frac{\alpha_i + m_i - \sigma_i}{\alpha_i + m_i + n_i + 1} \right)}^{\left( \frac{\alpha_i + m_i - \sigma_i}{\alpha_i + m_i + n_i + 1} + \frac{\sigma_i - m_i}{\alpha_i + m_i + n_i + 1} \right)}
\]

\[
(1 - \frac{2y_i}{C_i}) = \pi^r \left[ \frac{1}{\Gamma(m_i - \sigma_i)} \right]^{\left( \frac{\alpha_i + m_i - \sigma_i}{\alpha_i + m_i + n_i + 1} \right)}^{\left( \frac{\alpha_i + m_i - \sigma_i}{\alpha_i + m_i + n_i + 1} + \frac{\sigma_i - m_i}{\alpha_i + m_i + n_i + 1} \right)}
\]

\[
\left[ \frac{1}{\Gamma(m_i - \sigma_i)} \right] \left[ \frac{1}{\Gamma(m_i - \sigma_i)} \right] \left[ \frac{1}{\Gamma(m_i - \sigma_i)} \right]
\]

\[
x \left[ \frac{y_i}{\Gamma(m_i - \sigma_i)} \right] f_i(y_i) \]

\[
0 < y_i < a_i, \quad (i = 1 \text{ to } r)
\]
where
\[
\pi^r f_i(y_i) = \pi^r \left[ \frac{d}{dy_i} \frac{m_i}{m_i} \right] \cdot \int_{y_i}^{x_i} f(x_1, x_2, \ldots, x_r) \, dx_1 \left( \frac{1 - \sigma_{i-1}}{y_i - x_i} \right)
\] (2.9)

In deriving (2.8) from (2.1), we have used the conditions (i), (ii) and (iii) of (2.3). If we multiply the equation (2.2) by \( \pi^r (1 - \frac{1}{C_i}) \) and differentiating \( k_i \) times \((i=1 \text{ to } r)\) with respect to \( x_i \), we find using [1, eqn. (2.7)] that it becomes

\[
\sum_{n_1, n_2, \ldots, n_r} = 0 \quad A_{n_1, n_2, \ldots, n_r} \quad \pi^r
\]

\[
x \left[ \frac{\Gamma(\delta_i + \sigma_i + p_i + n_i + 1)}{\Gamma(\delta_i - k_i + n_i + 1) C_i^p} \right] \left( \frac{a_i + p_i + k_i - \delta_i}{k_i} \right)
\]

\[
x \left[ (1 - \frac{2x_i}{C_i}) \right] = \pi^r \left[ (-1)^i \frac{k_i}{(1 - \frac{x_i}{C_i})} \right] \left( \frac{a_i + p_i + k_i - \delta_i}{k_i} \right)
\]

\[
x \frac{a_i}{k_i} \left( \frac{x_i}{C_i} \right) \left( \frac{a_i + p_i + k_i - \delta_i}{k_i} \right) \left( \frac{a_i + p_i + k_i - \delta_i}{k_i} \right)
\]

\[
a_i < x_i < C_i, \quad (i = 1 \text{ to } r)\] (2.10)

Multiplying (2.10) by \( \pi^r \left[ (1 - \frac{x_i}{C_i}) \delta_i - k_i (x_i - y_i) \right] \left( \frac{a_i + p_i + k_i - \delta_i}{k_i} \right) \) and integrating with respect to \( x_i \) over \((y_i, C_i), (i=1 \text{ to } r)\) \('r'\) times. We get, using [1, eqn. (2.9)].

\[
\sum_{n_1, n_2, \ldots, n_r} = 0 \quad A_{n_1, n_2, \ldots, n_r} \quad \pi^r
\]
\[
\left[ p_{n_1} \right]_{\alpha_1 - \sigma_1, \delta_1 + \rho_1 + \sigma_1}^{2y_1} \left( 1 - \frac{y_1}{C_1} \right) = \pi^r \sum_{i=1}^{r} \frac{k_i}{\Gamma(\sigma_1 + \rho_1 + k_i)} q_i(y_i),
\]

\[
\begin{align*}
- & (1 - y_1) \left( \frac{\delta_1 + \rho_1 + \sigma_1}{C_1} \right) \\
\times & \left[ \frac{k_i}{\Gamma(\sigma_1 + \rho_1 + k_i)} q_i(y_i) \right],
\end{align*}
\]

\[
a_i < x_i < C_i, \quad (i = 1 \text{ to } r)
\]

(2.11)

where

\[
\begin{align*}
\pi^r q_i(y_i) &= \pi^r \sum_{i=1}^{r} \int_{y_1}^{x_i-y_1} \frac{k_i}{\Gamma(\sigma_1 + \rho_1 + k_i)} \left( 1 - \frac{x_i}{C_1} \right) \delta_1 \\
&\times \frac{g_i(x_1, x_2, \ldots, x_r)}{1 - \sigma_1 - \rho_1 - k_i} dx_i.
\end{align*}
\]

The conditions (i), (iv) and (v) of (2.3) have been used in obtaining (2.11) from (2.2). The left hand sides of equations (2.8) and (2.11) are now identical and using the orthogonality relation \[1, \text{ eqn. (2.10)}\], we obtain,

\[
\begin{align*}
&\pi^r \sum_{i=1}^{r} \frac{1}{\Gamma(m_1 - \sigma_1)} \\
&\int_{y_1}^{x_i-y_1} \frac{\delta_1 + \rho_1 + \sigma_1}{C_1} f_i(y_1) a_{n_1}(y_1) dy_1 + \\
&\int_{y_1}^{x_i-y_1} \frac{(-1)^k_i}{\Gamma(\sigma_1 + \rho_1 + k_i)} \frac{\delta_1 + \rho_1 + \sigma_1}{C_1} f_i(y_1) a_{n_1}(y_1) dy_1
\end{align*}
\]

(2.13)

where

\[
\begin{align*}
\pi^r a_{n_1}(y_1) &= \pi^r \sum_{i=1}^{r} \frac{(n_i)_1(2n_i + \alpha_1 + \delta_1 + \rho_1 + 1)}{\alpha_1 - \sigma_1 + 1} \\
&\times \frac{\Gamma(n_i + \alpha_1 + \delta_1 + \rho_1 + 1)}{\Gamma(\alpha_1 - \sigma_1 + n_i + 1)}.
\end{align*}
\]
(\alpha_{1,\sigma_{1},\delta_{1}+\sigma_{1}}-p_{1}+\sum_{1}) \times \pi_{n_{1}}^{(1)} \left(1 - \frac{2y_{1}}{C_{1}}\right) \tag{2.14}

The coefficients $A_{n_{1},n_{2},\ldots,n_{r}}^{1}$ satisfying the dual series equations (2.1) and (2.2) under the conditions (2.3) are thus given by (2.13), (2.14), (2.9) and (2.12). For $p_{1}=0$, $m_{1}=1$, $k_{1}=0$, $C_{1}=1$ the solution is in complete agreement with the one obtained by NOBLE \cite{3}.

THE LIMIT CASE

If we put $C_{1}=\delta_{1}$ in equations (2.13), (2.14), (2.9) and (2.12) and take the limit as $\delta_{1} \to \infty$, then on using \cite[eqns.(2.3) to (2.5)]{1}, we find that they become

\begin{align*}
\Lambda_{n_{1},n_{2},\ldots,n_{r}} &= \pi_{r} \frac{\Gamma_{1}(\sigma_{1}-\sigma_{1})}{\Gamma_{1}(\sigma_{1}+p_{1}+k_{1})} \int_{0}^{\infty} e^{-y_{1}} \frac{a_{1}}{y_{1}} dy_{1} \\
\Gamma_{1}(y_{1}) a_{n_{1}}(y_{1}) dy_{1} &= \frac{m_{1}}{\delta_{1}+\sigma_{1}} \\
x \int_{0}^{\infty} y_{1} a_{1-\sigma_{1}}^{1} g(y_{1}) a_{n_{1}}(y_{1}) dy_{1} \\
\int_{0}^{\infty} a_{1}^{1} f(y_{1}) dy_{1} &= \pi_{r} \frac{(n_{1})!}{\Gamma_{1}(\alpha_{1}-\sigma_{1}+n_{1}+1)} L_{n_{1}}^{1-\sigma_{1}}(y_{1}) \\
\pi_{r} f_{1}(y_{1}) &= \pi_{r} \frac{d_{i}}{m_{i}} dy_{i} \\
x \int_{0}^{\infty} y_{1} x_{i}^{\sigma_{1}} f(x_{1},x_{2},\ldots,x_{r}) dx_{1} \\
\pi_{r} g_{1}(y_{1}) &= \pi_{r} \int_{y_{1}}^{\infty} \frac{k_{1}}{y_{1}} dy_{1}
\end{align*} \tag{2.15}

where,

$\pi_{r} a_{n_{1}}(y_{1}) = \pi_{r} \frac{(n_{1})!}{\Gamma_{1}(\alpha_{1}-\sigma_{1}+n_{1}+1)} L_{n_{1}}^{1-\sigma_{1}}(y_{1})$ \tag{2.16}

$\pi_{r} f_{1}(y_{1}) = \pi_{r} \frac{d_{i}}{m_{i}} dy_{i}$ \tag{2.17}
The equations (2.15) to (2.18) provide us with the solution of the dual series equations (2.4) and (2.5) under the condition (2.6), where \( m_i \) and \( k_i \) are non-negative integers satisfying \( m_i - \sigma_i > 0, \ p_i + \sigma_i + k_i > 0 \). For \( \sigma_i + p_i > 0 \), i.e. \( k_i = 0 \) the solution is in complete agreement with the one obtained by Srivastava \([5]\).

3. The quantities of interest in physical applications are the values of the series in (2.1) and (2.2) on the intervals where their values are not specified. We define

\[
F(x_1, x_2, \ldots, x_r) = \sum_{n_1, n_2, \ldots, n_r = 0}^{\infty} A_{n_1, n_2, \ldots, n_r}
\]

\[
\Gamma(\alpha_i - \sigma_i + n_i + 1) \prod_{i=1}^{r} \Gamma(\alpha_i + n_i + 1) \times p_{n_i} \frac{(a_i, \delta_i + p_i)}{(1 - \frac{2x_i}{c_i})}
\]

\[
a_i < x_i < c_i, \ (i = 1 \text{ to } r)
\] (3.1)

\[
G(x_1, x_2, \ldots, x_r) = \sum_{n_1, n_2, \ldots, n_r = 0}^{\infty} A_{n_1, n_2, \ldots, n_r}
\]

\[
\Gamma(\sigma_i + \delta_i + p_i + n_i + 1) \prod_{i=1}^{r} \Gamma(\delta_i + n_i + 1)
\]

\[
(1 - \frac{2x_i}{c_i}) \prod_{i=1}^{r} \Gamma(\sigma_i + \delta_i + p_i + n_i + 1)
\]

\[
0 < x_i < a_i, \ (i = 1 \text{ to } r)
\] (3.2)

Where the parameters \( \alpha_i, \ p_i, \sigma_i \) and \( \delta_i \) satisfy the conditions (2.3) and for some non-negative integers \( t_i \) and \( s_i \) (\( i = 1 \text{ to } r \))

\[
(vi) \sigma_i + t_i > 0,
\]

\[
(vii) \delta_i + p_i + 1 > r_i,
\]

\[
(viii) s_i - p_i - \sigma_i > 0,
\]

\[
(ix) \delta_i + p_i + \sigma_i + 1 > s_i
\] (3.3)
In view of \([1, \text{ eqn. (2.6)}]\) we can write (3.1) as

\[
F(x_1, x_2, \ldots, x_r) = \pi^r \left[ \sum_{i=1}^{a_i} \frac{t_i}{\Gamma(a_i + t_i)} \right] x
\]

Substituting the value of \(A_{n_1, n_2, \ldots, n_r}\) from (2.13) in (3.4), interchanging the orders of integrations and summations and evaluating the series with the help of \([1, \text{ eqn. (2.11)}]\), we obtain

\[
F(x_1, x_2, \ldots, x_r) = \pi^r \left[ \sum_{i=1}^{a_i} \frac{t_i}{\Gamma(a_i + t_i)} \right] x
\]

Where \(f_1(y_1)\) and \(g_1(y_1)\) are given by (2.11) and (2.12) and \(m_1, t_1, k_1\) are non-negative integers satisfying (2.3) and (3.3).

To evaluate \(G(x_1, x_2, \ldots, x_r)\), we first multiply equations (2.8) and (2.11) by \(\pi^r (1 - \frac{y_i}{C_1}) \delta_i + p_i + \sigma_1\) and differentiate.
them \(s_i\) times \((i = 1 \text{ to } r)\), then simplify the expressions using \([1, \text{ eqn. (2.7)}]\) and secure with the help of the orthogonality relation \([1, \text{ eqn. (2.10)}]\) an alternative expression for the coefficients \(A_{n_1, n_2, \ldots, n_r}\) given by

\[
A_{n_1, n_2, \ldots, n_r} = \pi^r \sum_{i=1}^{r} \frac{s_i}{\Gamma(m_i - \sigma_i)} \int_{0}^{y_1} \frac{1}{1 + p_i + \sigma_i} b_{n_i}(y_1)
\]

\[
\frac{d}{dy_1} (y_1^{\sigma_i - \alpha_i} (1 - \frac{y_1}{C_i}))
\]

\[
x f_i(y_1) \frac{dy_1}{y_1} + \frac{(-1)^{s_i+k_i}}{\Gamma(\sigma_i + \delta_i + p_i)} \int_{0}^{y_1} a_i^{\sigma_i - \alpha_i + s_i}
\]

\[
x \cdot b_{n_i}(y_1) \frac{d}{dy_1} \left[ \frac{s_i q_i(y_1)}{s_i} dy_1 \right]
\]

where,

\[
\pi^r \sum_{i=1}^{r} \frac{(n_i)!}{\Gamma(n_i + \sigma_i + \alpha_i + \delta_i + p_i)}
\]

\[
x \cdot \frac{\Gamma(n_i + \sigma_i + \alpha_i + \delta_i + p_i + 1)}{\Gamma(n_i + \sigma_i + \alpha_i + \delta_i + p_i + 1 + n_i + 1)}
\]

\[
x \cdot \frac{p_i}{\Gamma(n_i + \sigma_i + \alpha_i + \delta_i + p_i + n_i + 1)}
\]

\[
x \cdot \frac{1}{\Gamma(\sigma_i + \delta_i + p_i + 1)}
\]

\[
x \cdot \frac{1}{\Gamma(\sigma_i + \delta_i + p_i + 1 + n_i + 1)}
\]

Substituting \(A_{n_1, n_2, \ldots, n_r}\) from (3.6) in (3.2), interchanging the order of integration and summation and evaluating the series with the help of \([1, \text{ eqn. (2.11)}]\), we get

\[
G(x_1, x_2, \ldots, x_r) = \pi^r \sum_{i=1}^{r} \frac{x_i^{-\delta_i}}{\Gamma(s_i - \sigma_i - p_i)}
\]

\[
x i \frac{(-1)^{s_i}}{\Gamma(m_i - \sigma_i)} \int_{x_i}^{y_i} \frac{dy_1}{y_1^{1+p_i+\sigma_i-s_i}}
\]
\begin{align}
&\frac{(1-\frac{y_i}{C_i})^{\delta_i+p_1+\sigma_i}}{C_i^1} f_1(y_i) \, dy_1 + \frac{(-1)^{s_1+k_1}}{\Gamma(\sigma_1+p_1+k_1)} \\
&\times \int_{a_i}^{x_i} \left( \frac{d^{s_1}/dy_1^{s_1} g_1(y_i)}{1+p_1+\sigma_1-s_1} \right) dy_1 \\
&0 < x_i < a_i, \quad (i = 1 \text{ to } r) \tag{3.8}
\end{align}

Where \( \pi^r f_1(y_i) \) and \( \pi^r g_1(y_i) \) are given by (2.9) and (2.12) and \( m_i, s_i, k_i \) are non-negative integers satisfying (2.3) and (3.3). For \( p_1 = 0, m_1 = s_1 = 1, r_1 = k_1 = 0 \), \( C_1 = 1 \) the results (3.5) and (3.8) are in complete agreement with those obtained by NOBLE [3]. Of course, in comparing (3.8) one has to make a trivial simplification in eqn. (2.12) of [3, p.368]. To obtain the values of the series in (2.4) and (2.5) on the intervals where their values are not specified we put \( C_i = \delta_i \) in (3.5), (3.8), (2.9) and (2.12) and let \( \delta_i \to \infty \). Using [1, eqn. (2.3)] to [1, eqn. (2.5)] we find

\begin{align}
F(x_1, x_2, \ldots, x_r) &= \prod_{i=1}^{r} \left[ \frac{x_i^{\alpha_i}}{\Gamma(\sigma_i+t_i)} \frac{t_i}{t_i} \right] \\
&\times \int_0^{x_i} \left( x_i-y_i \right)^{\sigma_i+t_i-1} f_1(y_i) \, dy_1 + \frac{1}{\Gamma(\sigma_1+p_1+k_1)} \\
&\times \int_{a_i}^{x_i} \left( y_i^{-1} \right)^{\sigma_i+t_i-1} e^y y_i^{\sigma_i+t_i-1} g_1(y_i) \, dy_1 \\
&\quad a_i > x_i < \infty, \quad (i = 1 \text{ to } r) \tag{3.9}
\end{align}
\[ G(x_1, x_2, \ldots, x_r) = \sum_{i=1}^{r} \frac{x_i}{\Gamma(s_i - \alpha_i - p_i)} \]

\[ x \left\{ \begin{array}{c}
\frac{(-1)^{s_i-1}}{\Gamma(m_i - \sigma_i)} \frac{a_i}{x_i} \frac{d}{dy_i} \frac{y_i - y_i^{-\frac{a_i}{p_i + \sigma_i - s_i}} - y_i^{-\frac{a_i}{1 + p_i + \sigma_i - s_i}} f_i(y_i)}{(y_i - x_i)} \\
+ \frac{\delta_i + k_i}{\Gamma(p_i + \sigma_i + k_i)} \frac{a_i}{x_i} \frac{d}{dy_i} \frac{g_i(y_i)}{(y_i - x_i)} \end{array} \right\} \]

\[ 0 < x_i < s_i \quad (i = 1 \text{ to } r) \quad (3.10) \]

Where \( \pi^r f_i(y_i) \) and \( \pi^r g_i(y_i) \) are given by (2.17) and (2.19) and \( m_i, k_i, t_i \) and \( s_i \) are non-negative integers satisfying \( m_i, \sigma_i > 0, p_i + \sigma_i + k_i > 0, \sigma_i + t_i > 0, \]
\( s_i - p_i - \sigma_i > 0 \). For \( p_i = 0, m_i = s_i = 1, r_i = k_i = 0 \), the results (3.9) and (3.10) are incomplete agreement with those obtained by Lowndes in [2, p.126].

**PARTICULAR CASES**

If we put \( r = 1 \) then the dual series equation (2.1) and (2.2) reduce to those considered earlier by Dwivedi and Trivedi [1] and our solution (2.13) is in complete agreement with those in [1].

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