CHAPTER - ELEVEN

ON THREE COPLANER GRIFFITH CRACKS IN INFINITE ELASTIC MEDIUM*

11.1 INTRODUCTION

Previously Tranter [155] and Willmore [165] have determined the stress distribution in the neighbourhood of two coplaner griffith cracks defined by $-b \leq x \leq -a$, $a \leq x \leq b$, $y = 0$ which are opened by constant pressure along the length of the cracks. Willmore based his work on complex variable methods developed by Green and Stevenson and Muskhelishvili [77], while Tanter, using Fourier transform methods developed by Sneddon, reduces the problem to that of solving a set of triple integral equations with cosine kernel. Tranter solves the triple integral equations by reducing them to dual trigonometric equations whose solution was well-known. After that Srivastava and Lowengrub [127] solve these equations by using finite-Hilbert transform technique developed by Srivastava and Lowengrub [122].

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Recently Parihar and Sowdamini [92] have also considered the above problem. They have reduced the problem to that of singular integral equation and obtained the solution by using the method of [91]. They have not interpreted the results graphically.

This chapter concerns the problem of determining the stress distribution in an infinite elastic medium containing three coplaner griffith cracks, defined by $b \leq |x| \leq c$, $|x| \leq a$, $y = 0$ which are opened by constant pressure along the length of the cracks. As in the paper of Srivastava and Lowengrub [127], we first reduce the problems to solving a set of quadruple integral equations and then give an exact solution based on finite-Hilbert transform technique developed by Srivastava and Lowengrub [122]. These solutions are used to calculate the stress intensity factors and crack energy. The results are shown graphically. Our method is simpler and straightforward than that of [92].

11.2 THE BOUNDARY VALUE PROBLEM AND THE RELEVANT QUADRUPLE INTEGRAL EQUATIONS

Denoting the relevant components of stress by $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$ and the displacement vector by $(u_x, u_y, 0)$, we calculate the distribution of stress in the half space $y \geq 0$, when the boundary $y = 0$ is subjected to the conditions:
\[ \sigma_{yy}(x,0) = \begin{cases} -p(x), & -a \leq x \leq a \\ -q(x), & b \leq x \leq c, \quad -c \leq x \leq -b \end{cases} \]

\[ u_y(x,0) = \begin{cases} 0, & a < x < b, \quad -b < x < -a \\ 0, & |x| > c, \end{cases} \]

\[ \sigma_{xy}(x,0) = 0, \quad -\infty < x < \infty. \]

In addition we require that all the stress and displacement components vanish at infinity and that \( p(x) = p(-x), \) \( q(x) = q(-x). \) It is a simple matter (see pp. 405-406 of [116]), to show that if we use the notation

\[ F_c[f(\xi,y); \xi \to x] = \sqrt{2/\pi} \int_0^\infty f(\xi,y) \cos \xi x \, d\xi, \]

\[ F_s[f(\xi,y); \xi \to x] = \sqrt{2/\pi} \int_0^\infty f(\xi,y) \sin \xi x \, d\xi, \]

to denote the Fourier - cosine and sine transform of \( f(\xi,y) \) respectively, a solution of the equations of elastic equilibrium appropriate to the half plane \( y \geq 0 \) is given by

\[ u_x(x,y) = -\frac{(1+n)}{E} \sqrt{2/\pi} \ F_s[M(\xi)(1 - 2\eta - \xi y)]. \]

\[ e^{-\xi y}, \xi \to x], \quad (11.2.1) \]

\[ u_y(x,y) = \frac{(1+n)}{E} \sqrt{2/\pi} \ F_s [M(\xi) (2-2\eta + \xi y)]. \]

\[ e^{-\xi y}, \xi \to x] \quad (11.2.2) \]
where \( E \) is Young's modulus and \( \eta \) is the Poisson's ratio of the material.

In this case, we can easily deduce that

\[
\sigma_{yy}(x,y) = -\sqrt{\frac{2}{\pi}} F_c [\xi M(\xi)(1+\xi y) e^{-\xi y}, \xi \to x]
\]

(11.2.3)

and

\[
\sigma_{xy}(x,y) = -\sqrt{\frac{\pi}{2}} F_s [\xi^2 M(\xi) e^{-\xi y}; \xi \to x]
\]

(11.2.4)

so the boundary conditions are satisfied provided that \( M(\xi) \) is determined by quadruple integral equations:

\[
\int_0^\infty \xi M(\xi) \cos(\xi x) d\xi = \frac{\pi}{2} p(x), \quad 0 < x < a,
\]

(11.2.5)

\[
\int_0^\infty M(\xi) \cos(\xi x) d\xi = 0, \quad a < x < b,
\]

(11.2.6)

\[
\int_0^\infty \xi M(\xi) \cos(\xi x) d\xi = \frac{\pi}{2} q(x), \quad b < x < c,
\]

(11.2.7)

\[
\int_0^\infty M(\xi) \cos(\xi x) d\xi = 0, \quad x > c,
\]

(11.2.8)

11.3 \textbf{SOLUTION OF QUADRUPLE INTEGRAL EQUATIONS}

Let a trial solution of the equation (11.2.5)-(11.2.8)

\[
M(\xi) = \int_0^a m(t) J_0(\xi t) dt + \frac{1}{\xi} \int_0^c h(u^2) \sin(\xi u) du
\]

(11.3.1)

where \( m(t) \) and \( h(u^2) \) are unknown functions. It is clear
this choice of $M(\xi)$ satisfies equations (11.2.6) and (11.2.8) if $h(u^2)$ satisfies

$$\int_{b}^{c} h(u^2) du = 0 \quad (11.3.2)$$

The equations (11.2.5) and (11.2.7) may be written in the form

$$\frac{d}{dx} \left[ \int_{0}^{\infty} M(\xi) \sin(\xi x) d\xi \right] = \begin{cases} \frac{1}{2} \pi p(x), & 0 < x < a \\ \frac{1}{2} \pi q(x), & b < x < c \end{cases} \quad (11.3.3)$$

Substituting from (11.3.1) for $M(\xi)$ into the first of equations (11.3.3) and using the results [54], we get

$$\frac{d}{dx} \left[ \int_{0}^{a} \frac{m(t)}{\sqrt{x^2 - t^2}} dt \right] = \frac{1}{2} p(x) - \frac{d}{dx} \left[ \frac{1}{2} \int_{b}^{c} h(u^2) du \right].$$

$$\log \left| \frac{u+x}{u-x} \right| du, \quad 0 < x < a \quad (11.3.4)$$

Now using the results [54] and Abel integral equation above reduces to

$$m(t) = t \int_{0}^{t} \frac{p(s)ds}{\sqrt{t^2 - s^2}} - t \int_{b}^{c} \frac{h(u^2)}{\sqrt{u^2 - t^2}} du$$

$$0 < t < a, \quad (11.3.5)$$

Again substituting the value of $M(\xi)$ into the second of equations (11.3.3), we obtain by adopting the above process as
Now substituting the value of \( m(t) \) from (11.3.5) into (11.3.6) after some manipulations and using (54),

\[
\int_{a}^{c} \frac{t \, dt}{(t^2-s^2)(u^2-t^2)^{3/2} = \frac{(a^2-s^2)^{1/2}}{(u^2-s^2)(u^2-a^2)^{1/2}}}
\]

we obtain

\[
\frac{d}{dx} \left[ \frac{1}{2} \int_{b}^{c} h(u^2) \log \frac{\sqrt{x^2-a^2} + \sqrt{u^2-a^2}}{\sqrt{x^2-a^2} - \sqrt{u^2-a^2}} \right] \]

\[
= \frac{1}{2} \pi q(x) + \frac{x}{\sqrt{x^2-a^2}} \int_{0}^{a} \frac{(a^2-s^2)^{1/2} \rho(s)}{(x^2-s^2)} ds,
\]

\[
b < x < c, \quad (11.3.7)
\]

Assuming that the left hand side of (11.3.7) is differentiable with respect to \( x \), we obtain

\[
\int_{0}^{c} \frac{\sqrt{u^2-a^2}}{(u^2-x^2)} h(u^2) \, du = \frac{1}{2} \pi f(x),
\]

\[
b < x < c \quad (11.3.8)
\]

where

\[
f(x) = \frac{\sqrt{x^2-a^2}}{x} q(x) + \frac{2}{\pi} \int_{0}^{a} \frac{(a^2-s^2)^{1/2} \rho(s)}{(x^2-s^2)} \quad (11.3.9)
\]
The solution of (11.3.8), using the finite Hilbert-transform technique is given by

\[ h(u^2) = -\frac{2}{\pi} \frac{u^2(u^2 - b^2)}{(u^2 - a^2)(c^2 - u^2)} \left[ \frac{c}{b} \frac{(c^2 - y^2)^{1/2}}{y^2 - b^2} \right] \int_\frac{c}{b}^{u} y f(y) \, dy + \frac{u c_1}{\{(u^2 - a^2)(u^2 - b^2)(c^2 - u^2)\}^{1/2}}, \]

\[ b < u < c \quad (11.3.10) \]

where \( c_1 \) is determined from (11.3.2) and has the value

\[ c_1 = \frac{2}{\pi F} \oint_b^{c} \frac{\left( \frac{c^2 - b^2}{y^2 - b^2} \right)^{1/2}}{y^2 - u^2} \cdot y f(y) \, dy. \]

\[ \cdot \frac{c}{b} \frac{u^2(u^2 - b^2)}{(u^2 - a^2)(c^2 - u^2)} \left[ \frac{du}{y^2 - u^2} \right] \quad (11.3.11) \]

with \( F = F(\pi/2, K) \), \( K = \left[ \frac{(c^2 - b^2)/(b^2 - a^2)}{1/2} \right] \) is an elliptical integral of the first kind. Now \( m(t) \) determined from (11.3.5) has the value

\[ m(t) = t \oint_0^{\frac{t}{\sqrt{t^2 - s^2}}} \frac{p(s) \, ds}{\sqrt{t^2 - s^2}} + \frac{2t}{\pi} \oint_b^{c} \frac{(c^2 - y^2)^{1/2}}{y^2 - b^2} \cdot y f(y), \]

\[ \cdot \frac{c}{b} \frac{u^2(u^2 - b^2)}{(u^2 - a^2)(c^2 - u^2)} \left[ \frac{du \, dy}{y^2 - u^2} \right] \]

\[ - \frac{t}{\sqrt{c_1}} \left[ \int_{\frac{1}{2}}^{\frac{\pi}{2}} F\left[ \frac{1}{2}, \frac{c}{b} \frac{(c^2 - b^2)(a^2 - t^2)}{(c^2 - a^2)(b^2 - t^2)} \right] \right], \quad (11.3.12) \]
In the above analysis, we have used the following result [54].

\[
\int_{b}^{c} \frac{u \, du}{\sqrt{(u^2-a^2)(u^2-b^2)(u^2-t^2)(c^2-u^2)}}
\]

\[
= \frac{1}{\sqrt{(c^2-a^2)(b^2-t^2)}} \, F\left[\frac{1}{2}, \frac{c^2-b^2}{c^2-a^2} \, \sqrt{\frac{(c^2-a^2)(b^2-t^2)}{(c^2-a^2)(c^2-t^2)}}\right]
\]

\[
c > b > a > t.
\]

Since we have the identity

\[
\left[\frac{(u^2-b^2)(c^2-y^2)}{(c^2-u^2)(y^2-b^2)}\right]^{1/2} \left[1 + \frac{y^2-u^2}{u^2-b^2}\right]
\]

\[
= \left[\frac{(c^2-u^2)(u^2-b^2)}{(u^2-b^2)(c^2-y^2)}\right]^{1/2} \left[1 - \frac{y^2-u^2}{c^2-u^2}\right],
\]

we see that (11.3.10) may be written in the following form

\[
h(u^2) = -\frac{2}{\pi} \left[\frac{u^2(c^2-u^2)}{(u^2-a^2)(u^2-b^2)}\right]^{1/2} \int_{b}^{c} \left(\frac{y^2-b^2}{c^2-y^2}\right)^{1/2} \frac{y f(y)}{y^2-u^2} \, dy + \frac{u c_2}{\sqrt{(u^2-a^2)(u^2-b^2)(c^2-u^2)}}
\]

\[
(b < u < c), \quad (11.3.13)
\]

where from (11.3.11), we have
11.4 PARTICULAR CASE AND QUANTITIES OF PHYSICAL INTEREST

In particular, if we choose \( p(x) = q(x) = p_0 \), where \( p_0 \) is a constant, then

\[
c_1 = (c^2 - a^2)p_0 \left[ \frac{b^2 - a^2}{c^2 - a^2} - \frac{E}{F} \right]
\]

(11.4.1)

\[
c_2 = (c^2 - a^2)p_0 \left[ 1 - \frac{E}{F} \right]
\]

(11.4.2)

with \( E = E(\pi/2, K) \), an elliptic interest of the second kind, we then find that

\[
h(u^2) = p_0 u \left[ \frac{(u^2 - a^2) - (c^2 - a^2)E/F}{\gamma(u^2 - a^2)(u^2 - b^2)(c^2 - u^2)} \right], \quad b < u < c.
\]

(11.4.3)

\[
m(t) = \frac{1}{2} \pi p_0 t - \frac{(b^2 - a^2)t p_0}{\gamma(c^2 - a^2)(b^2 - t^2)} \left[ \pi \left( \frac{1}{2} \pi, \frac{c^2 - b^2}{c^2 - a^2} \right) \right.
\]

\[
\left. \gamma \left( \frac{c^2 - b^2}{c^2 - a^2} \right) \right] - \left( \frac{c^2 - a^2}{b^2 - a^2} \right) \frac{E}{F}.
\]

\[
\cdot F \left( \frac{1}{2} \pi, \frac{(c^2 - b^2)(a^2 - t^2)}{(c^2 - a^2)(b^2 - t^2)} \right),
\]

(11.4.4)
where we have used the result [54]

\[
\int_{b}^{c} \left[ \frac{u^2(u^2-a^2)}{(u^2-t^2)(u^2-b^2)(c^2-u^2)} \right]^{1/2} du = \frac{(b^2-a^2)}{\sqrt{(c^2-a^2)(b^2-t^2)}}
\]

\[
\Pi(\frac{1}{2} \pi, (\frac{c^2-b^2}{c^2-a^2}), \frac{(c^2-b^2)(a^2-t^2)}{(c^2-a^2)(b^2-t^2)})
\]

\[c > b > a > t, \quad (11.4.5)\]

and

\[
\Pi(\frac{1}{2} \pi, r, s) = \int_{0}^{\pi/2} \frac{d\theta}{(1+r \sin^2 \theta)(1-s^2 \sin^2 \theta)}
\]

The **Stress Intensity Factor**

Expressions for the stress intensity factor are of great importance for workers in fracture mechanics, these expressions are given by

\[
N_a = \lim_{x \to a^+} (x-a)^{1/2} \sigma_{yy}(x,0),
\]

\[
N_b = \lim_{x \to b^-} (b-x)^{1/2} \sigma_{yy}(x,0), \quad (11.4.6)
\]

\[
N_c = \lim_{x \to c^+} (x-c)^{1/2} \sigma_{yy}(x,0),
\]

From the results of sections 11.2 and 11.3, we find that
\[ [\sigma_{yy}(x,0)]_{a \lt x \lt b} = \frac{x}{\gamma(x^2-a^2)} \int_{0}^{a} \frac{p(s)(a^2-s^2)^{1/2}}{(x^2-s^2)} \, ds \\
- \frac{2}{\pi} \frac{x}{\gamma(x^2-a^2)} \int_{b}^{c} \frac{\sqrt{(u^2-a^2)} h(u^2) du}{(u^2-x^2)} \]  
\( (11.4.7) \)

\[ [\sigma_{yy}(x,0)]_{x \gt c} = \frac{x}{\gamma(x^2-a^2)} \int_{0}^{a} \frac{p(s)(a^2-s^2)^{1/2}}{(x^2-s^2)} \, ds \\
+ \frac{2}{\pi} \frac{x}{\gamma(x^2-a^2)} \int_{b}^{c} \frac{\sqrt{(u^2-a^2)} h(u^2) du}{(x^2-u^2)} \]  
\( (11.4.8) \)

Now substituting the value of \( h(u^2) \) from (11.3.13) and (11.3.10) in (11.4.7) and (11.4.8) respectively, and using the result [54].

\[ \int_{b}^{c} \frac{u \, du}{(u^2-y^2)[(u^2-b^2)(c^2-u^2)]^{1/2}} = \begin{cases} 
\frac{1}{2} \pi \left[ (b^2-y^2)(c^2-y^2) \right]^{-1/2}, & 0 \lt y \lt b \\
0, & b \lt y \lt c \\
\frac{1}{2} \pi \left[ (y^2-b^2)(y^2-c^2) \right]^{-1/2}, & y \gt c. 
\end{cases} \]  
\( (11.4.9) \)

We obtain the expressions

\[ [\sigma_{yy}(x,0)]_{a \lt x \lt b} = \frac{x}{\gamma(x^2-a^2)} \int_{0}^{a} \frac{\sqrt{a^2-s^2}}{x^2-s^2} p(s) ds \\
+ \frac{2}{\pi} \left[ \frac{x^2(c^2-x^2)}{(b^2-x^2)(x^2-a^2)} \right]^{1/2} \]
and
\[
\left[ \sigma_{yy}(x,0) \right]_{x>c} = \frac{x}{y(x^2-a^2)} \int_{b}^{a} \frac{\sqrt{a^2-s^2}}{x^2-s^2} p(s)ds \\
+ \frac{2}{\pi} \left[ \frac{\sqrt{x^2-b^2}}{(x^2-a^2)(x^2-c^2)} \right]^{1/2} \int_{b}^{c} \frac{\sqrt{c^2-y^2}}{y^2-b^2} \frac{yf(y)}{y^2-x^2} dy + \frac{c_1x}{\sqrt{(x^2-a^2)(x^2-c^2)(x^2-b^2)}} \right]^{1/2}
\]

Now from (11.4.6), we find that the stress intensity factors are

\[
N_a = \sqrt{\frac{3}{a}} \int_{0}^{a} \frac{p(s)ds}{\sqrt{a^2-s^2}} + \frac{2}{\pi} \left( \frac{a}{2(c^2-a^2)(b^2-a^2)} \right)^{1/2}.
\]

\[
[ (c^2-a^2) \int_{b}^{c} \frac{\sqrt{c^2-b^2}}{c^2-y^2} \frac{yf(y)}{y^2-a^2} dy - \frac{1}{2} \pi c_2 ],
\]

\[
N_b = \frac{2}{\pi} \left( \frac{b}{2(b^2-a^2)(c^2-b^2)} \right)^{1/2} \left[ (c^2-b^2) \int_{b}^{c} \frac{yf(y)dy}{(y^2-b^2)(c^2-y^2)} \right]^{1/2}
\]

\[- \frac{1}{2} \pi c_2 \]
\[ N_c = \frac{2}{\pi} \left( \frac{b}{2(c^2-a^2)(c^2-b^2)} \right)^{1/2} \left[ (c^2-b^2) \int_{b}^{c} \frac{y f(y) \, dy}{((c^2-y^2)(y^2-b^2))^{1/2}} \right] + \frac{1}{2} \pi c_1 \],

where \( c_1 \) and \( c_2 \) are given by (11.3.11) and (11.3.4) respectively. In case the cracks are opened under constant pressure, \( p_0 \), we obtain

\[ N_a = p_0 \sqrt{\frac{a}{2(b^2-a^2)(c^2-a^2)}} \frac{E}{F} \]

\[ N_b = p_0 \sqrt{\frac{b}{2(b^2-a^2)(c^2-b^2)}} [(c^2-a^2)\frac{E}{F} -(b^2-a^2)] \quad (11.4.13) \]

\[ N_c = p_0 \sqrt{\frac{c}{2(c^2-a^2)(c^2-b^2)}} [1-E/F] \]

In case \( a \to 0 \), that is when three coplaner cracks merge into two coplaner cracks, we get the results which are in complete agreement with those of Srivastava and Lowengrub [127]. It is noted that when \( a = 0 = b \) and \( c = 1 \) the cracks merge into a single cracks of width two units. In this case \( F \to \infty, c_1 = 0 \) and the third result of (11.4.12) reduce to the simple expression

\[ N = \frac{\sqrt{2}}{\pi} \int_{0}^{1} \frac{f(y)}{\sqrt{1-y^2}} \, dy \]

which is in complete agreement with the result obtained by Lowengrub [65].
The crack energy

From (11.2.2), we now find that the displacement $u_y(x,0)$ is given by

$$u_y(x,0) = \frac{4(1-\eta^2)}{\pi E} \int_0^a \frac{m(t)}{\sqrt{t^2-x^2}} \, dt, \quad 0 < x < a \quad (11.4.14)$$

and

$$u_y(x,0) = \frac{2(1-\eta^2)}{E} \int_x^c h(u^2) \, du, \quad b < x < c, \quad (11.4.15)$$

where the expressions for $m(t)$ and $h(u^2)$ are obtained in section (11.3).

The total work $W_1$ and $W_2$ done in opening the cracks $-a < x < a$ and $b < x < c$ are respectively given by

$$W_1 = -2 \int_0^a \left[ \sigma_{yy}(x,0) \cdot u_y(x,0) \right] \, dx, \quad 0 < x < a$$

$$W_1 = \frac{4p_0(1-\eta^2)}{E} \int_0^a m(t) \, dt, \quad (11.4.16)$$

$$W_2 = \frac{2(1-\eta^2)p_0}{E} \int_b^c \int_x^c h(u^2) \, du, \quad (11.4.17)$$

Substituting the value of $m(t)$ from (11.3.5) into (11.4.6) and interchanging the order of integration, we obtain

$$W_1 = \frac{4p_0(1-\eta^2)}{E} \left[ p_o \int_0^a (a^2-s^2)^{1/2} \, ds - \int_b^c h(u^2) \right].$$

$$(\sqrt{u^2-a^2} - u) \, du,$$
which on using (11.3.13) gives

\[ W_1 = \frac{2p^2(1-\eta^2)\pi}{E} \left[ (1+\pi)a - \frac{b}{2} (b^2 - c^2) + (c^2 - a^2) \pi \frac{E}{F} \right] \]

\[ + 2L(x) - \frac{2c^2}{b} \left( c^2 - a^2 \right)^{1/2} \frac{E}{F} \]

\[ \pi \left( \frac{1}{2} \pi, \frac{b^2 - c^2}{b^2}, \frac{a}{b} \left( \frac{c^2 - b^2}{c^2 - a^2} \right)^{1/2} \right), \]  

(11.4.18)

where

\[ L(x) = \int_{b}^{c} \frac{u^2 \sqrt{u - a^2}}{\sqrt{(u^2 - b^2)(c^2 - u^2)}} \, du \]

Similarly from (11.4.17), we find that

\[ W_2 = \frac{2(1-\eta^2)p^2}{E} \left[ L(x) - \frac{c^2}{b} \left( c^2 - a^2 \right)^{1/2} \frac{E}{F} \right] \]

\[ \pi \left( \frac{1}{2} \pi, \frac{b^2 - c^2}{b^2}, \frac{a}{b} \left( \frac{c^2 - b^2}{c^2 - a^2} \right)^{1/2} \right) \]  

(11.4.19)

Here it is noted that when \( a = 0 \), the cracks merge into two coplaner cracks. In this case (11.4.19) reduces to the expression

\[ W_2 = \frac{\pi(1-\eta^2)p^2}{E} \left( c^2 + b^2 - 2c^2 \right) \frac{E}{F}, \]

(11.4.20)

which is in complete agreement with the result obtained by Srivastava and Lowengrub [127].

Now when \( a = 0 = b \) and \( c = 1 \), the cracks merge into a single crack and the energy required to open it is

\[ W = \frac{\pi(1-\eta^2)p^2}{E} \]

which is incomplete agreement with the results of Lowengrub [65].
Fig. 1

\[ \frac{N_a}{P_0} \]

- \( a = 1, b = 2 \)
- \( a = 1, b = 3 \)
- \( a = 1, b = 4 \)
Fig. 2
Fig. 3