CHAPTER - SEVEN

SOME SIX INTEGRAL EQUATIONS INVOLVING INVERSE MELLIN TRANSFORMS*

In this chapter the solution of six integral equations involving inverse Mellin transform has been obtained by reducing them to a system of Fredholm integral equations.

7.1 INTRODUCTION

In this section we solve the six integral equations of the form:

\[ M^{-1} \left[ \frac{\Gamma (1+\eta-S/\sigma)}{\Gamma (1+\eta+\alpha-S/\sigma)} \bar{\varphi}(S);x \right] = f_1(x), \quad 0 \leq x < a_1, \quad (7.1.1) \]

\[ M^{-1} \left[ \frac{\Gamma (\xi+S/\delta)}{\Gamma (\xi+\beta+S/\delta)} \bar{\varphi}(S);x \right] = g_2(x), \quad a_1 < x < a_2, \quad (7.1.2) \]

\[ M^{-1}\left[\frac{\Gamma(1+\eta-S/\sigma)}{\Gamma(1+\eta+\alpha-S/\sigma)} \varphi(S);x\right] = f_3(x), \ a_2 < x < a_3, \quad (7.1.3) \]

\[ M^{-1}\left[\frac{\Gamma(\xi+S/\delta)}{\Gamma(\xi+\beta+S/\delta)} \varphi(S);x\right] = g_4(x), \ a_3 < x < a_4, \quad (7.1.4) \]

\[ M^{-1}\left[\frac{\Gamma(1+\eta-S/\sigma)}{\Gamma(1+\eta+\alpha-S/\sigma)} \varphi(S);x\right] = f_5(x), \ a_4 < x < a_5, \quad (7.1.5) \]

\[ M^{-1}\left[\frac{\Gamma(\xi+S/\delta)}{\Gamma(\xi+\beta+S/\delta)} \varphi(S);x\right] = g_6(x), \ a_5 < x < \infty, \quad (7.1.6) \]

where \( \alpha, \beta, \xi, \eta > 0, \ \sigma > 0 \) are real parameters; \( f_1, g_2, f_3, g_4, f_5 \) and \( g_6 \) are prescribed functions while \( \varphi(S) \) is an unknown function to be determined and

\[ M[h(x);S] = H(S) \quad (7.1.7) \]

and

\[ M^{-1}[H(S);x] = h(x) \quad (7.1.8) \]

denote the Mellin transform of \( h(x) \) and its inversion formula respectively.

The above equations are an extension of the triple integral equations solved by Lownde [73] and quadruple integral equations solved recently by Dwivedi, Kushwaha and Trivedi [38] by means of a systematic application of some slightly extended forms of Erdelyi-Kober operators of fractional integration [114].
Using the properties of extended form of Erdelyi-Kober operators given in [73] we show in purely formal manner that the solution of the six integral equations can be expressed in terms of the solution of a Fredholm integral equation of the second kind. The method of solution employed here will be seen to follow closely that used by Ahmad [1] to obtain the solution of some quadruple integral equations involving Bessel functions.

7.2 FRACTIONAL INTEGRAL OPERATORS

We recall here a few definitions and properties of operators used in solving the triple and quadruple integral equations. Lowndes [73] has defined the following operators:

\[ I_{\eta,\alpha}(a,b,\sigma)f(x) = \frac{\sigma x^{\alpha (\alpha + \eta)}}{\Gamma(\alpha)} \int_a^b (x^\sigma \sigma \tau)^{\alpha - 1} \tau (\tau - 1)^{\eta + 1} \tau f(t) dt, \quad \alpha > 0, \]

\[ = \frac{x^{1 - \alpha (\alpha + \eta + 1)}}{\Gamma(1 + \alpha)} \frac{d}{dx} \int_a^b (x^\sigma \sigma \tau)^{\alpha \tau (\eta + 1) - 1} x f(t) dt, \quad -1 < \alpha < 0 \]

\[ K_{\eta,\alpha}(c,d;\sigma)f(x) = \frac{\sigma x^{\eta}}{\Gamma(\alpha)} \int_c^d (t^\sigma \sigma \tau)^{\alpha - 1} \tau (\tau - 1 - \alpha - \eta) - 1 \]

\[ xf(t) dt, \quad \alpha > 0, \]
\[ \frac{\pi x^\sigma(\eta-1)+1}{(1+\sigma)} \frac{d}{dx} \int_c^d \frac{(t^\sigma-x^\sigma)t^\sigma(1-\alpha-\eta)-1}{(t^\sigma-x^\sigma)} \, dt \]

\[ x f(t) \, dt, \quad -1 < \alpha < 0, \]  
(7.2.4)

where \( a < x < b, \ \sigma > 0. \)

From the theory of Abel integral equations it follows that the inverse operators are given by

\[ I_{\eta,\alpha}^{-1}(a,b;\sigma)f(x) = I_{\eta+\alpha,-\alpha}(a,b;\sigma)f(x). \]  
(7.2.5)

\[ K_{\eta,\alpha}^{-1}(c,d;\sigma)f(x) = K_{\eta+\alpha,-\alpha}(\sigma,d;\sigma)f(x). \]  
(7.2.6)

We require two lemmas given by Lowndes [73] which define the pairs of operators:

**Lemma A.** Let \( I_{\eta,\alpha}(a,b;\sigma), I_{\eta,\alpha}^{-1}(d,x;\sigma) \) be operators as defined earlier, then

\[ I_{\eta,\alpha}^{-1}(d,x;\sigma)I(a,b;\sigma)f(x) = \frac{\sigma \sin(\alpha \pi)}{\pi} \frac{x^{-\sigma \eta}}{x^\sigma-d^\sigma} \]

\[ x \int_a^b \frac{t^\sigma(\eta+1)-1(d^\sigma-t^\sigma)\alpha}{(x^\sigma-t^\sigma)} f(t) \, dt, \]  
(7.2.7)

provided \( x > d \geq b > a. \)

**Lemma B:** Let \( K_{\eta,\alpha}(a,b;\sigma), K_{\eta,\alpha}^{-1}(x,d;\sigma) \) be operators defined as earlier, then

\[ K_{\eta,\alpha}^{-1}(x,d;\sigma)K_{\eta,\alpha}(a,c;\sigma)f(x) = \frac{\sigma \sin(\alpha \pi)}{\pi} \frac{x^{\sigma(\alpha+\eta)}}{(d^\sigma-x^\sigma)} \]

\[ x \int_a^b \frac{t^\sigma(1-\alpha-\eta)-1(t^\sigma-d^\sigma)\alpha}{(t^\sigma-x^\sigma)} f(t) \, dt \]  
(7.2.8)

provided \( x < d \leq a < b. \)
Two well known results [73] which play an important part in our solution are:

$$M[I_{1,\alpha}(0,x;\alpha)f(x);S] = \frac{\Gamma(1+n-S/\alpha)}{\Gamma(1+n+\alpha-S/\alpha)} M[f(x);S] \quad (7.2.9)$$

$$M[K_{1,\alpha}(x,\infty;\alpha)f(x);S] = \frac{\Gamma(n+S/\alpha)}{\Gamma(\eta+\alpha+S/\alpha)} M[f(x);S]. \quad (7.2.10)$$

7.3 SOLUTION OF SIX INTEGRAL EQUATIONS

Let $I_1$ denote the interval $(0,a_1)$, $I_2$ the interval $(a_1,a_2)$, $I_3$ the interval $(a_2,a_3)$, $I_4$ the interval $(a_3,a_4)$, $I_5$ the interval $(a_4,a_5)$ and $I_6$ the interval $(a_5,\infty)$. For a function $f$ in $L_2(0,\infty)$ we shall write $f_1 + f_2 + f_3 + f_4 + f_5 + f_6$ where $f_i = f$ on $I_i$ and $f_j = 0$ on $I_j (i,j = 1,2,3,4,5,6; i \neq j)$ and similarly for $g$. We can write six integral equations as

$$M^{-1}\left[\frac{\Gamma(1+n-S/\alpha)}{\Gamma(1+n+\alpha-S/\alpha)} \overline{\varphi}(S);x\right] = f(x) \quad (7.3.1)$$

$$M^{-1}\left[\frac{\Gamma(\xi+S/\alpha)}{\Gamma(\xi+\beta+S/\alpha)} \overline{\varphi}(S);x\right] = g(x) \quad (7.3.2)$$

where $f_1, g_2, f_3, g_4, f_5$ and $g_6$ are prescribed functions while $g_1, f_2, g_3, f_4, g_5$ and $f_6$ are unknown functions to be determined. If we write

$$\overline{\varphi}(S) = M[\varphi(x);S], \quad (7.3.3)$$
and use the formula (7.2.9) and (7.2.10) we find that equations (7.3.1) and (7.3.2) assume the operational form

\[ I_{\eta,\alpha}(0,x;\sigma) \varphi(x) = f(x) \]  
(7.3.4)

\[ K_{\xi,\beta}(x,\sigma;\delta) \varphi(x) = g(x) \]  
(7.3.5)

Using the formula (7.2.5) and (7.2.6) and solving the above equations for \( \varphi(x) \), we obtain

\[ \varphi(x) = I_{\eta+\alpha,-\alpha}(0,x;\sigma)f(x), \]  
(7.3.6)

\[ = K_{\xi+\beta,-\beta}(x,\infty;\delta)g(x). \]  
(7.3.7)

We proceed to determine \( \varphi \). The subscripts on all the operators \( I \)'s will be supposed to have the subscripts \((\eta,\alpha;\sigma)\) understood and all \( K \)'s to have subscript \((\xi,\beta;\delta)\). Evaluating (7.3.6) on \( I_1, I_2, I_3, I_4 \) and \( I_5 \), we get

\[ \varphi_1 = (x_0) I^{-1} f_1 \]  
(7.3.8)

\[ \varphi_2 = (a_1) I^{-1} f_1 + (x_0) I^{-1} f_2 \]  
(7.3.9)

\[ \varphi_3 = (a_1) I^{-1} f_1 + (a_2) I^{-1} f_2 + (x_0) I^{-1} f_3 \]  
(7.3.10)

\[ \varphi_4 = (a_1) I^{-1} f_1 + (a_2) I^{-1} f_2 + (a_3) I^{-1} f_3 \]  
(7.3.11)

\[ + (x_0) I^{-1} f_4 \]
\[ \varphi_5 = (a_1^{-1}f_1 + a_2^{-1}f_2 + a_3^{-1}f_3 + a_4^{-1}f_4 + a_5^{-1}f_5) \] (7.3.12)

Similarly evaluating (7.3.7) on \( I_2, I_3, I_4, I_5 \) and \( I_6 \), we get

\[ \varphi_2 = (x_k^{-1}g_2 + a_3^{-1}g_3 + a_4^{-1}g_4 + a_5^{-1}g_5 + a_6^{-1}g_6) \] (7.3.13)

\[ \varphi_3 = (x_k^{-1}g_3 + a_4^{-1}g_4 + a_5^{-1}g_5 + a_6^{-1}g_6) \] (7.3.14)

\[ \varphi_4 = (x_k^{-1}g_4 + a_5^{-1}g_5 + a_6^{-1}g_6) \] (7.3.15)

\[ \varphi_5 = (x_k^{-1}g_5 + a_6^{-1}g_6) \] (7.3.16)

\[ \varphi_6 = (x_k^{-1}g_6) \] (7.3.17)

Since \( f_1 \) and \( g_6 \) are known, \( \varphi_1 \) and \( \varphi_6 \) can be determined by the equations (7.3.8) and (7.3.17) respectively. We now solve (7.3.9) for \( f_2 \) and substitute its value in (7.3.10), to get

\[ \varphi_3 = (a_1^{-1}f_1 + a_2^{-1}f_2 + x_k^{-1}f_2 - (a_1^{-1}f_1 + a_2^{-1}f_3) \] (7.3.18)
We solve (7.3.16) for $g_5$ and substitute its value in (7.3.15) to get

$$\varphi_4 = (x_4)K^{-1}g_4 + (a_4)K^{-1}(a_5)K[\varphi_5 - (a_5)K^{-1}g_6] + (a_5)K^{-1}g_6.$$  \(7.3.19\)

Similarly from equation (7.3.13) we find out the value of $\varphi_2$ in terms of $\varphi_3$ and $\varphi_5$ with the help of equations (7.3.14) and (7.3.16)

$$\varphi_2 = (x_2)K^{-1}g_2 + (a_2)K^{-1}(x_3)K[\varphi_3 - (a_3)K^{-1}g_4] - \left(\frac{\varphi_5}{a_5}ight)K^{-1}g_6 - (a_4)K^{-1}(x_5)K[\varphi_5 - (a_5)K^{-1}g_6] \right) + (a_4)K^{-1}g_4 + (a_5)K^{-1}(x_5)K[\varphi_2 - (a_5)K^{-1}g_6] + (a_5)K^{-1}g_6.$$  \(7.3.20\)

and with the help of equations (7.3.9) and (7.3.11), that of $\varphi_5$ in terms of $\varphi_2$ and $\varphi_4$ from equation (7.3.12),

$$\varphi_5 = (a_1)I^{-1}f_1 + (a_2)I^{-1}(a_1)I[\varphi_2 - (a_1)I^{-1}f_1]$$

$$+(a_3)I^{-1}f_3 + (a_4)I^{-1}(a_3)I[\varphi_4 - (a_1)I^{-1}f_1]$$
We have thus arrived at six simultaneous equations (7.3.8), (7.3.20), (7.3.18), (7.3.19), (7.3.21) and (7.3.17) associated with six unknown functions \( \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \) and \( \varphi_6 \) respectively. From these equations, we can obtain the value of these unknowns and hence the solution of the problem will be determined by equation (7.3.3).

7.4 AN APPLICATION

We shall consider now the six integral equations which are extension of triple integral equations solved by Lowndes [73]:

\[
\int_0^{a_2} u^{-2n} \psi(u) J_{2q}(ux) \, dx = F_1(x), \quad 0 \leq x < a_1 \\
= F_3(x), \quad a_2 < x < a_3 \\
= F_5(x), \quad a_4 < x < a_5 \\
\int_0^{a_2} \psi(u) J_{2p}(ux) \, dx = 0, \quad a_1 < x < a_2, \\
= 0, \quad a_3 < x < a_4, \\
= 0, \quad a_5 < x < \infty,
\]

where \( J_{2p}(ux) \) is the Bessel function of the first kind of order \( 2p \), \( F_1(x) \), \( F_3(x) \) and \( F_5(x) \) are prescribed.
functions and $\Psi(u)$ is to be determined. When $p=q$ and $a_4=a_5=\infty$ these are equations investigated by Ahmad [1]. We now show, in a fairly straightforward manner, that the above equations can be transformed into equations of the type (7.1.1) to (7.1.6) with $g_2=g_4=g_6=0$.

Denoting the Mellin transform of $\Psi(u)$ by

$$M[\Psi(u);s] = \Psi(s), \quad (7.4.3)$$

and using the result

$$M[\xi^{2n} J_{2q}(\xi);s] = 2^{s-1-2n} \frac{\Gamma(q-n+S/2)}{\Gamma(1+n+q-S/2)} \quad (7.4.4)$$

We have, on applying the Faltung theorem for Mellin transforms [112], that the integral equations (7.4.1) and (7.4.2) can be written in the form

$$M^{-1}\left[ \frac{\Gamma(1+p-S/2)}{\Gamma(1+n+q-S/2)} \varphi(s);x \right] = 2^{1+2n} x^{-2n} F_1(x), \quad 0<x<a_1$$

$$= 2^{1+2n} x^{-2n} F_3(x), \quad a_2<x<a_3$$

$$= 2^{1+2n} x^{-2n} F_5(x), \quad a_4<x<a_5 \quad (7.4.5)$$

$$M^{-1}\left[ \frac{\Gamma(p+S/2)}{\Gamma(q-n+S/2)} \varphi(s);x \right] = 0, \quad a_1<x<a_2,$$

$$= 0, \quad a_3<x<a_4,$$

$$= 0, \quad a_5<x<\infty \quad (7.4.6)$$
where,
\[
\overline{\varphi}(S) = 2^S \frac{\Gamma(q-n+S/2)}{\Gamma(1+p-S/2)} \overline{\psi}(1-S)
\]  \hspace{1cm} (7.4.7)

These equations are the same as equations (7.1.1) to (7.1.6) with \( \sigma = \delta = 2, \xi = \eta = p, \alpha = q-p+n, \beta = q-p-n \) and

\[
\begin{align*}
  f_1(x) &= 2^{1+2n} x^{-2n} F_1(x) \\
  f_3(x) &= 2^{1+2n} x^{-2n} F_3(x) \\
  f_5(x) &= 2^{1+2n} x^{-2n} F_5(x)
\end{align*}
\]  \hspace{1cm} (7.4.8)

\[
 g_2 = g_4 = g_6 = 0.
\]

Using the results of previous section we have therefore, that the solution of equations (7.4.5) and (7.4.6) can be found in terms of a function \( \phi(x) \) given by

\[
\overline{\varphi}(S) = M[\phi(x);S],
\]  \hspace{1cm} (7.4.9)

where the functions \( \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \) and \( \varphi_6 \) are obtained from equations (7.3.8), (7.3.20), (7.3.18), (7.3.19) (7.3.21) and (7.3.17) with the parameters \( \xi, \eta \) etc. given by equation (7.4.8).

Finally, in order to find the solution of integral equations (7.4.1) and (7.4.2) in terms of \( \phi(x) \) we proceed in the following way.
From equation (7.4.3) we have that the solution is

$V(u) = M^{-1} [\mathcal{T}(S);u]$  

which is

$$V(u) = M^{-1} \left[ 2^{S-1} \frac{\Gamma(\frac{1}{2} + p + S/2)}{\Gamma(\frac{1}{2} + q - n - S/2)} \right]_{ij(S)}(1-S);u], \quad (7.4.10)$$

On using equations (7.4.7) and (7.4.9). Inverting the order of integration in the last equation we get

$$\Psi(u) = \int_0^\infty \varphi(x) M^{-1} \left[ 2^{S-1} \frac{\Gamma(\frac{1}{2} + p - S/2)}{\Gamma(\frac{1}{2} + q - n - S/2)} \right]_{ij(S)}(ux)dx, \quad (7.4.11)$$

$$\Psi(u) = \int_0^\infty \left( \frac{ux}{2} \right)^{1+n+p+q} \varphi(x) J_{p+q-n}(ux)dx, \quad (7.4.12)$$

after applying the result (7.4.4). When $p = q$ and $a_4 = a_5 = \infty$ this solution is exactly same as that found by Ahmad [1].