CHAPTER - 6

APPLICATION TO A CRACK PROBLEM
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6.1 INTRODUCTION

In the classical work of Griffith (1921), interest in crack problems in the mathematical theory of elasticity arises from the theory of brittle fracture, which itself originated nearly eighty years ago. Since the number of materials that fail under normal conditions in a brittle fashion is relatively small, for many years this theory was regarded as of academic rather than practical interest. In service period of components it is common experience that crack can be very detrimental to strength even though cracks may be of small dimension.

In the Griffith theory it is assumed that cracks exist or are formed in a solid body when it is subjected to tensile forces and that when the tensile forces are increased to a point where the strain-energy release rate with crack extension is greater than the rate at which energy is gained by the creation of new free surface area, rapid crack extension takes place and solid fractures.

To exploit even the simplest physical model of this type, it is necessary to investigate the stress distribution in the neighbourhood of a crack and to calculate the increase of strain-energy due to the presence of the crack.

The method of curvilinear co-ordinates and integral transforms play central roles in the analysis of 2-D as well as 3-D problems in mathematical theory of elasticity relating to cracks.
Tweed [257] developed the theory of finite Mellin transforms to find formulae for the stress intensity factors and crack formation energy of a radial system of edge cracks in a circular elastic cylinder under torsion.

Tweed [251, 256] also investigated some dual and triple integral equations involving inverse Mellin type transforms and illustrated the application of these equations in the solution of certain crack problems in the theory of elasticity. Trivedi and Pandey [249] solved dual integral equations involving inverse Mellin transforms and also gave their application. Recently Dwivedi and Chandel [82] obtained the solution triple of integral equations involving the inverse of Naylor’s Mellin type transforms and the application of these equations in crack problem of an infinite elastic solid with circular hole.

6.2 APPLICATION TO A CRACK PROBLEM IN THE THEORY OF ELASTICITY

6.2.1 The Problem

As an application of triple integral equations investigated in chapter three, we shall now consider the problem of determining the stress intensity factors and crack formation energy of a crack which originates at the edge of a circular hole in an infinite elastic solid under longitudinal shear.

We shall assume that in cylindrical co-ordinates \((r, \theta, z)\), the hole is defined by the relations \(0 \leq r \leq R, \ 0 \leq \theta \leq 2\pi, -\infty < z < \infty\) and the crack is given by \(0 < R < R_b \leq r \leq R_c, \ \theta = 0, -\infty < z < \infty\). We shall also assume that the crack \((s)\) and the hole are traction free and that as \(r\) tends to infinity \(\sigma_{rz}\) tends to \(T \sin\theta\) and \(\sigma_{\theta z}\) to \(T \cos\theta\).
In the longitudinal shear problem the fields of displacement and stress in the body under consideration are such that

\begin{align*}
    u_r &= u_\theta = 0, \\ u_z &= w(r,\theta) \quad (6.2.1) \\
    \sigma_{rr} &= \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{r\theta} = 0, \quad (6.2.2) \\
    \sigma_{\theta z} &= \frac{\mu}{r} \frac{\partial w}{\partial \theta} \quad \text{and} \quad \sigma_{rz} = \mu \frac{\partial w}{\partial r} \quad (6.2.3)
\end{align*}

where \( \mu \) is the shear modulus and \( w(r,\theta) \) is a solution of the equation

\begin{equation}
    \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = 0 \quad (6.2.4)
\end{equation}

Because of the antisymmetric nature of the problem it will be sufficient to find a function \( w(r,\theta) \) which satisfies (6.2.4) in the region \( R < r < \infty, 0 < \theta < \pi \) and is such that:

(i) As \( r \) tends to infinity, \( \frac{1}{r} \frac{\partial w}{\partial \theta} \) tends to \( \frac{T}{\mu} \cos\theta \) and \( \frac{\partial w}{\partial r} \) to \( \frac{T}{\mu} \sin\theta \),

(ii) \( \frac{\partial w(R, \theta)}{\partial r} = 0, \quad 0 \leq \theta \leq \pi \),

(iii) \( w(r, \pi) = 0, \quad R \leq r < \infty \),

(iv) \( w(r, 0) = 0, \quad (R \leq r \leq Rb) \cup (Rc \leq r < \infty) \)

(v) \( \frac{\partial w(r,0)}{\partial \theta} = 0, \quad Rb < r < Rc \)

6.2.2 The Solution

Equation (6.2.4)) can also be written as

\begin{equation}
    r^2 \frac{\partial^2 w}{\partial r^2} + r \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial \theta^2} = 0 \quad (6.2.5)
\end{equation}
Let $w_1(r, \theta)$ be a solution of equation (6.2.5) in the given region. Let it satisfy condition (ii) and (iii). Then on applying the transform $G_R$ to equation (6.2.5), we get

\[
G_R \left[ r^2 \frac{\partial^2 w}{\partial r^2} + r \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial \theta^2} \right] = 0
\]

\[
\int_0^\infty r^2 \frac{\partial^2 w}{\partial r^2} \left[ r^{s-1} - R^{2s} r^{-s-1} \right] dr + \int_0^\infty r \frac{\partial w}{\partial r} \left[ r^{s-1} - R^{2s} r^{-s-1} \right] dr \\
+ \int_0^\infty r \frac{\partial^2 w}{\partial \theta^2} \left[ r^{s-1} - R^{2s} r^{-s-1} \right] dr = 0
\]

\[
\int_0^\infty \frac{\partial^2 w}{\partial r^2} \left[ r^{s+1} - R^{2s} r^{-s+1} \right] dr + \int_0^\infty \frac{\partial w}{\partial r} \left[ r^s - R^{2s} r^{-s} \right] dr + \frac{\partial^2 w}{\partial \theta^2} = 0
\]

(6.2.6)

where $\overline{w}_1(s, \theta) = G_R \left[ w_1(r, \theta); r \to s \right]$, $|\text{Re}(s)| < 1$ (6.2.7)

Integrating equation (6.2.6) by parts, we get

\[
\left[ \frac{\partial w}{\partial r} \left( r^{s+1} - R^{2s} r^{-s+1} \right) \right]_0^\infty - \int_0^\infty \frac{\partial w}{\partial r} \left[ (s+1) r^2 + R^{2s} (s-1) r^{-s} \right] dr \\
+ \int_0^\infty \frac{\partial w}{\partial r} \left[ r^s - R^{2s} r^{-s} \right] dr + \frac{\partial^2 w}{\partial \theta^2} = 0
\]

(6.2.8)

we know $\lim_{r \to \infty} \frac{\partial w}{\partial r} = 0$

therefore equation (6.2.8) takes the form

\[
-s \int_0^\infty \frac{\partial w}{\partial r} \left[ r^s + R^{2s} r^{-s} \right] dr - \int_0^\infty \frac{\partial w}{\partial r} \left[ r^s - R^{2s} r^{-s} \right] dr \\
+ \int_0^\infty \frac{\partial w}{\partial r} \left[ r^s - R^{2s} r^{-s} \right] dr + \frac{\partial^2 w}{\partial \theta^2} = 0
\]

(6.2.9)
again integrating by parts,

\[- s \left[ w \left\{ r^s + R^{2s} \right\} \right]_R^\infty + s^2 \int_R^\infty w \left\{ r^{s-1} - R^{2s} r^{-s-1} \right\} \, dr + \frac{\partial^2 w}{\partial \theta^2} = 0 \]

\[ s^2 \bar{w}_1 + \frac{d^2 \bar{w}_1}{d\theta^2} = 0 \quad (6.2.10) \]

it follows that

\[ \bar{w}_1 (s, \theta) = s^{-1} \left[ A (s) \cos (s\theta) + B (s) \sin (s\theta) \right] \quad (6.2.11) \]

where \( A (s) \) and \( B (s) \) are arbitrary functions of \( s \).

Applying condition (iii), we get

\[ B (s) = - A (s) \frac{\cos \pi}{\sin \pi} \quad (6.2.12) \]

substituting the value of \( B (s) \) in equation (6.2.11), we obtain

\[ \bar{w}_1 (s, \theta) = A (s) \frac{\sin (\theta - \pi) s}{\sin \pi s} \quad (6.2.13) \]

From equations (6.2.7) and (6.2.13), it is clear that

\[ w_1 (r, \theta) = G_{R}^{-1} \left[ A (s) \frac{\sin (\theta - \pi) s}{\sin \pi s}; r \right], \quad |\Re (s)| < 1 \quad (6.2.14) \]

Hence it is easy to show that the function

\[ w (r, \theta) = \frac{T}{\mu} (r - R^{2} r^{-1}) \sin \theta + G_{R}^{-1} \left[ A (s) \frac{\sin (\theta - \pi) s}{\sin \pi s}; r \right], \quad |\Re (s)| < 1 \quad (6.2.15) \]

is a solution of equation (6.2.4) in the region \( R < r < \infty, 0 < \theta < \pi \) and that it satisfies the conditions (i), (ii) and (iii). If we now apply the conditions (iv) and (v) we find that \( A (s) \) must satisfy the triple integral.
equations involving inverse Mellin transforms

\[ G_R^{-1} [s^{-1}A(s); r] = 0, \quad R \leq r \leq R_b \quad (6.2.16) \]

\[ G_R^{-1} [A(s) \cot \pi s; r] = -\frac{T}{\mu} (r - R^2 r^{-1}); \quad R_b \leq r \leq R_c \quad (6.2.17) \]

\[ G_R^{-1} [s^{-1}A(s); r] = 0, \quad R \leq r \leq \infty \quad (6.2.18) \]

using the results (3.3.4) of chapter three, we get

\[ A(s) = \int_{R_b}^{R_c} P(t) (R^2 s t^{-s} + t^s) \, dt \quad (6.2.19) \]

where

\[ P(t) = \frac{T}{\mu \pi D_1(t)} \left\{ \frac{R^2 - t^2}{t} \frac{I_1}{J_1} + K_1 \right\} \quad (6.2.20) \]

\[ I_1 = \int_{R_b}^{R_c} \frac{1}{D_1(t)} \int_{R_b}^{R_c} \frac{(R^2 - t^2) D_1(x) (x^2 - R^2)}{(R^2 - xt) (x - t) x^2} \, dx \, dt \quad (6.2.21) \]

\[ J_1 = \int_{R_b}^{R_c} \frac{(R^2 - t^2)}{t D_1(t)} \, dt \quad (6.2.22) \]

\[ K_1 = \int_{R_b}^{R_c} \frac{(R^2 - t^2) D_1(x) (x^2 - R^2)}{(R^2 - xt) (x - t) x^2} \, dx \quad (6.2.23) \]

and

\[ D_1(t) = \left[ (t - R_b) (R_c - t) (R_b t - R^2) (R_c t - R^2) \right]^{1/2} \quad (6.2.24) \]

substituting \( t = Ru \), equations (6.2.20) (6.2.21), (6.2.22) and (6.2.23) take the form

\[ P(Rt) = \frac{T}{\mu \pi D_2(t)} \left\{ \frac{(1 - t^2)}{t} \frac{I_2}{J_2} + K_2 \right\} \quad (6.2.25) \]
where

\[
I_2 = \int_b^c \frac{1}{D_2(t)} \int_b^c \frac{(1 - t^2) D_2(x)(x^2 - 1)}{(1 - xt)(x - t)x^2} \, dx \, dt
\]  
(6.2.26)

\[
J_2 = \int_b^c \frac{(1 - t^2)}{t \, D_2(t)} \, dt
\]  
(6.2.27)

\[
K_2 = \int_b^c \frac{(1 - t^2) D_2(x)(x^2 - 1)}{(1 - xt)(x - t)x^2} \, dx
\]  
(6.2.28)

and

\[
D_2(t) = [(t - b)(c - t)(bt - 1)(ct - 1)]^{1/2}
\]  
(6.2.29)

From equations (6.2.1) and (6.2.19) it follows that

\[
u_z(r, 0, z) = \int_r^{R_c} P(t) \, dt, \quad R_b < r < R_c
\]  
(6.2.30)

We shall now calculate two quantities which of interest to workers in fracture mechanics. The first is the stress intensity factors \(K_b\) and \(K_c\) and the second is the crack formation energy \(W\) which are defined by the equations

\[
K_b = \mu \lim_{r \to R_b^+} \sqrt{2 (r - R_b)} \left[ \frac{\partial u_z(r, 0, z)}{\partial r} \right]
\]  
(6.2.31)

\[
K_c = -\mu \lim_{r \to R_c^-} \sqrt{2 (R_c - r)} \left[ \frac{\partial u_z(r, 0, z)}{\partial r} \right]
\]  
(6.2.32)

and

\[
\dot{W} = T \int_{R_b}^{R_c} \left(1 - \frac{R^2}{r^2}\right) u_z(r, 0, z) \, dr
\]  
(6.2.33)
respectively. Now taking $K_0 = T \sqrt{[R(c-b)/2]}$ and $W_0 = R^2 T^2 (c-b)^2 \pi / 8 \mu$
where $k_0$ and $W_0$ are the stress intensity factor and crack energy respectively of a crack of length $R(c-b)$ in an infinite elastic solid which
is subjected to a uniform longitudinal shear $T$ parallel to the crack faces.

Now by virtue of equation (6.2.30)

$$
\frac{K_b}{K_0} = \frac{-2 \mu}{T \sqrt{[R(c-b)]}} \lim_{r \to Rb^+} \sqrt[\mu \pi D_2 \left( \frac{r}{R} \right)} P(r)
$$

$$
= \frac{-2 \mu}{T \sqrt{[R(c-b)]}} \lim_{r \to Rb^+} \sqrt{r-Rb} \frac{T}{\mu \pi D_2 \left( \frac{r}{R} \right)} \left[ \frac{1-r^2}{r/R} \frac{I_2}{J_2} + K_2 \right]
$$

$$
= \frac{-2}{\pi \sqrt{[R(c-b)]}} \lim_{r \to Rb^+} \sqrt{(r-Rb)} \left[ \frac{1-r^2}{r/R} \frac{I_2}{J_2} + \int_b^c \frac{1-r^2}{r/R} \frac{I_2}{J_2} \left( 1- \frac{r}{R} \right) \left( x^2 - 1 \right) dx \right]
$$

$$
= \frac{-2}{\pi \sqrt{[R(c-b)]}} \left[ \frac{1-R^2b^2}{Rb/R} \frac{I_2}{J_2} + \int_b^c \frac{1-R^2b^2}{Rb/R} \frac{I_2}{J_2} \left( 1- \frac{r}{R} \right) \left( x-Rb/R \right) \left( x^2 - 1 \right) dx \right]
$$

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\[
\begin{align*}
&= \frac{-2}{\pi \sqrt{[R (c-b)]}} \frac{\sqrt{R}}{\sqrt{[(c-b) (b^2-1) (cb-1)]}} \\
&\quad \cdot \left[ \frac{1-b^2}{b} \frac{I_2}{J_2} + \int_b^c \frac{(1-b^2) D_2(x) (x^2-1) \, dx}{(1-bx) (x-b) x^2} \right] \\
&= \frac{-2}{\pi (c-b)} \frac{1}{\sqrt{[(b^2-1) (bc-1)]}} \left[ \frac{1-b^2}{b} \frac{I_2}{J_2} \\
&\quad + \int_b^c \frac{(1-b^2) \sqrt{[(x-b) (c-x) (bx-1) (cx-1) (x^2-1)]}}{(1-bx) (x-b) x^2} \, dx \right] \\
\end{align*}
\]

\[
\frac{K_b}{K_0} = \frac{-2}{\pi (c-b)} \frac{1}{\sqrt{[(b^2-1) (bc-1)]}} \left[ \frac{1-b^2}{b} \frac{I_2}{J_2} \\
&\quad + \int_b^c \frac{(1-b^2) \left( \frac{(c-x) (cx-1)}{(bx-1) (x-b)} \right)^{1/2}}{x^2} \frac{1-x^2}{\sqrt{x^2}} \, dx \right] 
\]  (6.2.34)

Now

\[
\frac{K_c}{K_0} = \frac{2\mu}{T \sqrt{[R(c-b)]}} \lim_{r \to R \pm} \sqrt{(Rc-r)} P(r) \\
= \frac{2\mu}{T \sqrt{[R(c-b)]}} \lim_{r \to R \pm} \sqrt{(Rc-r)} \frac{T}{\mu \pi D_2 \left( \frac{r}{R} \right)} \\
\cdot \left[ \frac{1-r^2/R^2}{r/R} \frac{I_2}{J_2} + K_2 \right]
\]
\[
\frac{K_c}{K_0} = \frac{2}{\pi (c-b)} \frac{1}{\sqrt{[(bc-1)(c^2-1)]}} \left[ \frac{1-c^2}{c} \frac{I_2}{J_2} + \int_b^c \frac{(1-c^2)x}{(x-c)(c-x)(bx-1)(cx-1)(x^2-1)} \, dx \right]\]

(6.2.35)

and

\[
\frac{W}{W_0} = \frac{8\mu}{R^2 T (c-b)^2} \int_{Rb}^{Re} \left( 1 - \frac{R^2}{r^2} \right) \int_r^{Re} P(t) \, dt \, dr
\]

(6.2.36)

Inverting the order of integration, we get

\[
\frac{W}{W_0} = \frac{8\mu}{R^2 T (c-b)^2} \int_{Rb}^{Re} P(t) \, dt \int_t^{Re} \left( 1 - \frac{R^2}{r^2} \right) \, dr
\]

Substituting \( t = Ru \) and \( r = Ry \), we get

\[
\frac{W}{W_0} = \frac{8\mu}{R^2 T (c-b)^2} \int_b^c P(Ru) R \, du \int_u^R \left( 1 - \frac{1}{y^2} \right) \, dy
\]
\[
\frac{W}{W_0} = \frac{8\mu}{T(c-b)^2\pi} \left[ \int_b^c \frac{(t^2+1)}{t} P(Rt) \, dt - \frac{(b^2+1)}{b} \int_b^c P(Rt) \, dt \right]
\]

From equation (3.3.5) of chapter three, it is clear that

\[
\int_{R_b}^{R_c} P(t) \, dt = 0
\]

or

\[
\int_b^c P(Rt) \, dt = 0
\]

therefore equation (6.2.37) takes the form

\[
\frac{W}{W_0} = \frac{8\mu}{T(c-b)^2\pi} \left[ \int_b^c \frac{(t^2+1)}{t} P(Rt) \, dt - 0 \right]
\]

\[
= \frac{8\mu}{T(c-b)^2\pi} \int_b^c \frac{(t^2+1)}{t} P(Rt) \, dt
\]

\[
= \frac{8\mu}{T(c-b)^2\pi} \int_b^c \frac{(t^2+1)}{t} \left[ \frac{T}{\mu \pi D_2(t)} \left( \frac{(1-t^2)}{t} \frac{I_2}{J_2} + K_2 \right) \right] \, dt
\]

\[
\frac{W}{W_0} = \frac{8}{\pi^2(c-b)^2} \int_b^c \frac{(1+t^2)}{t D_2(t)} \left( \frac{(1-t^2)}{t} \frac{I_2}{J_2} + K_2 \right) \, dt
\]
In this way, giving proper numerical values to $b$ and $c$, the values of \( \frac{K_b}{K_0} \), \( \frac{K_c}{K_0} \) and \( \frac{W}{W_0} \) can be obtained from equations (6.2.34), (6.2.35) and (6.2.40) respectively.