CHAPTER 2
DUAL INTEGRAL EQUATIONS

In this chapter we have considered the solution of certain dual integral equations and simultaneous dual integral equations. In section 2.1 we have obtained a series solution of certain dual integral equations involving Bessel functions, in section 2.2 simultaneous dual integral equations involving H-functions as kernel have been studied and finally in section 2.3 a formal solution of simultaneous dual integral equations involving H-functions of n-variables have been obtained.

2.1 SERIES SOLUTION OF A PAIR OF DUAL INTEGRAL EQUATIONS*

Of concern here is the series solution of certain dual integral equations involving Bessel functions. Srivastava's solution has been obtained as a particular case.

2.1.1 Introduction

In the analysis of mixed boundary value problems in the plane, we encounter dual integral equations of the type

\[ \int_0^\infty \xi^{-1} \psi(\xi) \cos(x\xi) \, d\xi = f(x), \quad 0 \leq x < 1, \quad (2.1.1) \]

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\[ \int_{0}^{\infty} \psi(\xi) \cos(x\xi) \, d\xi = 0, \quad x > 1, \quad (2.1.2) \]

where \( f(x) \) is known function and \( \psi(\xi) \) is to be determined.

A simple procedure for dealing with the difficulty when \( f(x) \) is a simple polynomial was given by Chong \[13\]. Chong's method consists in using an entirely different method to solve the boundary value problem which corresponds to the case in which \( f(x) \) is a constant and then to use the formal solution of the dual integral equations to construct the solution appropriate to the other terms of the polynomial. Fredricks \[57\] has given a direct method of solving the equations \((2.1.1)\) and \((2.1.2)\) for the case in which the function \( f(x) \) can be represented by half-range cosine series

\[ f(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi x), \quad 0 \leq x < 1. \]

Fredrick's solution can be written in the form

\[ \psi(\xi) = \xi J_1(\xi) \sum_{n=0}^{\infty} a_n J_0(n\pi) + 4 \sum_{n=0}^{\infty} a_n J_2(n\pi) J_2(\xi). \]

Sneddon \[141\], following the method of Fredricks, derived a simple solution of the pair \((2.1.1)\) and \((2.1.2)\). Recently Srivastava \[153\], by using the analysis of Sneddon \[141\], obtained the solution of dual-equations \((2.1.1)\)
and (2.1.2) when \( f(x) \) can be represented in a series of Jacobi polynomials. He also considered the equations when trigonometric functions were replaced by Bessel functions.

The object of this section is to find the solution of the more general equations than those considered by Srivastava [153]. The results of Srivastava are deduced as particular case.

2.1.2 The Solution

Consider for solution the dual integral equations

\[
\int_0^\infty u^\alpha \psi(u) J_\mu(xu) du = f(x), \quad 0 < x < 1, \tag{2.1.3}
\]

\[
\int_0^\infty u^\beta \psi(u) J_\nu(xu) du = g(x), \quad x > 1. \tag{2.1.4}
\]

Assuming that the functions \( f(x) \) and \( g(x) \) can be represented in series of Gauss's hypergeometric functions in the forms:

\[
f(x) = \sum_{m=1}^{\infty} a_m 2^\delta x^\mu \Gamma(\mu+\delta+m+1) \{ \Gamma(\mu+1) \Gamma(m+1) \}^{-1} x \times \, _2F_1(\mu+\delta+m+1,\mu-m;\mu+1;x^2),
\]

\[
g(x) = \sum_{m=1}^{\infty} \frac{a_m 2^{\delta+\beta-\alpha} x^{-2\delta-2m-2-\mu-\beta+\alpha}}{\Gamma(\mu+\delta+2m+2) \Gamma(-\delta-m-\beta/2+\alpha/2+v/2-\mu/2)} \times \frac{\Gamma(\delta+m+1+\mu/2+v/2+\beta/2-\alpha/2)}{\Gamma(\delta+m+1+\mu/2+v/2+\beta/2-\alpha/2)}^{-1} \, _2F_1(\delta+m+1+v+\alpha+\beta)/2, \delta+m+1+(\mu-v-\alpha-\beta)/2; \mu+\delta+2m+2; x^2}, \tag{2.1.6}
\]
and \( f(x) \) satisfies the condition

\[
\int_{0}^{1} x^{\mu+1}(1-x^2)^{\delta} 2F_1[-n, \mu+\nu+1; \mu+1; x^2] f(x) \, dx = 0; \quad (2.1.7)
\]

then the solution of the dual integral equations (2.1.1) and (2.1.2) is given by

\[
\psi(u) = \sum_{m=1}^{\infty} a_m u^{\delta-\alpha} J_{\mu+\delta+2m+1}(u), \quad (2.1.3)
\]

valid for \( n = 0, 1, 2, \ldots; m \neq n, R(\mu+1) > 0, R(\delta+1) > 0, \)

\[
R(2\mu+\delta+2m+2) > R(-\delta) > -1, R(\mu+\nu+\delta+2m+2) > R(\alpha-\beta-\delta) > -1.
\]

2.1.3 Verification

On substituting \( \psi(u) \) from (2.1.8) in (2.1.3) and (2.1.4) and interchanging the order of integration and summation, evaluating \( u \)-integrals given by Watson [198, p. 401], (2.1.3) and (2.1.4) are automatically satisfied since \( f(x) \) and \( g(x) \) are given by (2.1.5) and (2.1.6).

To verify the condition (2.1.7), we multiply (2.1.3) by \( x^{\mu+1}(1-x^2)^{\delta} 2F_1(-n, \mu+n+1; \mu+1; x^2) \) and integrate between the limits from \((0, 1)\) and interchanging the order of integration, we get

\[
\int_{0}^{1} x^{\mu+1}(1-x^2)^{\delta} 2F_1(-n, \mu+n+1; \mu+1; x^2) f(x) \, dx \]

\[
= \int_{0}^{\infty} u^\alpha \psi(u) du \int_{0}^{\pi/2} \sin^{\mu+1} \cos^{2\delta+1} j_{\mu} (u \sin \theta) \times
\]

\[
\times 2F_1(-n, \mu+n+1, \nu+1; \sin^2 \theta) \, d\theta
\]
\[
2^\delta \frac{r(u+1) r(\delta+u+1)}{r(u+n+1)} \int_0^{\infty} u^{-\delta-1} J_{\mu+\delta+2n+1}(u)\psi(u)du
\]

= 0 \text{ when } m \neq n,

where \(\psi(u)\) is given by (2.1.8). Here use has been made of the following results given by Watson [198, p. 404] and Sonine's integral [186, p. 97] respectively:

\[
\int_0^\infty y^{-1} J_{\nu+2n+k}(y) J_{\nu+2m+k}(y) dy = 0, \ m \neq n
\]

\[
\int_0^{\pi/2} \sin^{\nu+1} \theta \cos^{2\mu+1} \theta J_\nu(z \sin \theta) \mathcal{E}_1(-n, \nu+\nu+n+1; \nu+1; \sin^2 \theta)d\theta = \frac{2^\mu R(\nu+1)}{z^{\nu+1} R(\nu+n+1)} J_{\nu+\nu+2n+1}(z),
\]

valid for \(R(\nu+1) > 0, R(\nu+1) > 0, n = 0, 1, 2, \ldots\)

Equations (2.1.5) and (2.1.6) are easily obtainable from [48, p. 48].

2.1.4 A Special Case

The solution of dual integral equations (2.1.3) and (2.1.4) for the case \(n = 0, \alpha = 2k - 3/2, \beta = 1/2, \mu = \nu, \delta = k-1, g(x) = 0\), is a known result due to Srivastava [153].

2.2. SIMULTANEOUS DUAL INTEGRAL EQUATIONS ASSOCIATED WITH \(H\)-FUNCTIONS*

In the present section we have obtained a formal solution of certain simultaneous dual integral equations, possessing \(H\)-functions as kernel, by making use of fractional integration.

operators due to Saxena, which are the extensions of Köber and Erdélyi's operators. It has been shown that Saxena's operators are more suitable in the solution of the equations considered in this section due to the fact that the application of Köber's operators eliminate only two Gamma functions from the integrand, whereas Saxena's operators are capable of eliminating four Gamma functions, two each from the numerator and the denominator.

2.2.1 Introduction

Simultaneous dual integral equations arise in the formulation of great variety of mixed boundary value problems of semi-infinite domains involving more than one unknown function such as the problems in elastostatics visco-elasticity and electrostatics. Erdogan and Bahar [53] have reduced simultaneous dual integral equations involving Bessel functions as kernels to infinite set of simultaneous algebraic equations. Westmann [200] considered special cases of the results obtained by Erdogan and Bahar.

Recently Fox [56] and Saxena [118] have used the operators of fractional integration due to Köber to obtain the solution of certain dual integral equations associated with H-functions, introduced by Fox [55] while investigating a generalised Fourier kernel. More recently Saxena [118] has further obtained the solution of dual integral equations
involving H-functions by the application of the fractional integration operators introduced by him. In this section we have obtained the formal solution of simultaneous dual integral equations of a more general nature than even considered by Dwivedi [35]. By the application of the fractional integration operators due to Saxena [118], the given integral equations have been transformed into two others with a common kernel. Consequently the problem reduces to that of solving a single integral equation. Since the common kernel comes out to be a symmetrical Fourier kernel, the formal solution is then readily obtained.

For the definition, analytic continuations and asymptotic expansions, the reader is referred to the work of Braaksma [7]. The function defined by

\[ H_{2p,2q}^{a_p,b_p}(x) \]

\[ = H_{2p,2q}^{a_p,b_p}(x) = \frac{1}{2\pi i} \int x_p,q(s)x^{-s}ds, \quad (2.2.1) \]

where

\[ x_p,q(s) = \prod_{i=1}^{q} \frac{r(b_i + s B_i)}{r(b_i + s b_i)} \prod_{j=1}^{p} \frac{r(a_j - s A_j)}{r(a_j - A_j + s A_j)}, \quad (2.2.2) \]

behaves as a symmetrical Fourier kernel as proved earlier by Fox [55].
From (2.2.2), it follows that the Mellin transform of
\[ H_{2p,2q}^{q,m}(x) \] is equal to the \( \chi_{p,q}(s) \).

(2.2.3)

The solution of the following simultaneous dual integral equations will be obtained:

\[ \int_0^\infty H_{2(p+m),2(q+m)}^{q+m}(xu) \sum_{h=1}^n a_{hk} f(u) du = \phi_k(x), \quad 0 < x < 1; \quad (2.2.4) \]

\[ \int_0^\infty H_{2(p+n),2(q+n)}^{q+n}(xu) \sum_{h=1}^n b_{hk} f(u) du = \psi_k(x), \quad x > 1; \quad (2.2.5) \]

Here \( a_{hk} \) and \( b_{hk} \) are constants, \( \phi_k(x) \) and \( \psi_k(x) \) are given functions and \( f_n(x) \) are to be determined. The \( H \)-functions used here are defined as below:

\[ H_{2(p+m),2(q+m)}^{q+m}(x) = \frac{1}{2\pi i} \int_L \chi_{p,q,m}(s)x^{-s}ds, \quad (2.2.6) \]

where

\[ \chi_{p,q,m}(s) = \prod_{i=1}^q \frac{r(b_i+sB_i)}{r(b_i+B_i-sB_i)} \prod_{i=1}^p \frac{r(a_i-sA_i)}{r(a_i-A_i+sA_i)} \]

\[ \times \prod_{i=1}^m \frac{r(a_i^k-s\tau_i)}{r(a_i^k-A_i^k+s\tau_i)} \frac{r(b_i^k+r+s\tau_i)}{r(b_i^k+r+s\tau_i)}, \quad (2.2.7) \]

and

\[ H_{2(p+n),2(q+n)}^{q+n}(x) = \frac{1}{2\pi i} \int_L \chi_{p,q,n}(s)x^{-s}ds, \quad (2.2.8) \]

where
\[ x_p, q, n(s) = \prod_{i=1}^{q} \frac{r(b_i + sB_i)}{r(b_iB_i - SB_i)} \prod_{i=1}^{p} \frac{r(a_i - sA_i)}{r(a_i - A_i + sA_i)} \]
\[ \prod_{i=1}^{n} \frac{r(\lambda_i + s\lambda_i)}{r(\delta_i + g - s\lambda_i)} \cdot (2.2.9) \]

and \( r, g = 0, 1, 2, \ldots \)

In what follows \( i, j, w \) and \( k \) will take the values from \( 1, 2, \ldots, p; 1, 2, \ldots, q; 1, 2, \ldots, m \) and \( 1, 2, \ldots, n \) respectively. We assume that the following conditions are satisfied:

(i) \( p \leq q - 1 \)

(ii) \( a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, k_i, \lambda_i, \mu_i, \rho_i \) are real or complex and \( A_i, B_i, T, \lambda_i, \mu_i, \tau_i \) are real and positive.

(iii) Let \( s = \sigma + it \) where \( \sigma \) and \( t \) are real, then the contour \( L \) along which the integrals are taken is a straight line parallel to the imaginary axis in the complex \( s \)-plane whose equation is \( \sigma = \sigma_0 \), where \( \sigma_0 \) is a constant.

(iv) All the poles of integrands \( (2.2.6) \) and \( (2.2.8) \) are simple.

The contour \( L \) is such that all the poles of \( r(b_j + sB_j) \), \( r(\sigma_i^k + s\lambda_i^k) \), and \( r(\delta_i + s\lambda_i) \) lie on the left, and those of \( r(a_i - sA_i) \), \( r(\alpha_i^k - s\lambda_i^k) \), and \( r(\delta_i^k + s\lambda_i) \) lie to the right of it.

(v) \( e = 2 ( \sum_{j=1}^{p} B_j - \sum_{i=1}^{q} A_i ) > 0 \).

(vi) \( \sigma_0 < \frac{1}{2} - \frac{1}{6} \sum_{i=1}^{m} (\eta_i^k - \alpha_i^k) \) for \( (2.2.6) \).
From the asymptotic behaviour of Gamma function \( \Gamma(z) \), it follows that the conditions (vi) and (vii) ensure the convergence of the integrals (2.2.6) and (2.2.8) respectively.

2.2.2 The Mellin Transform

As usual we denote the mellin transform of

\[ f_h(u) \] by \( \mathcal{M}[f_h(u)] \) and if

\[ \mathcal{M}[f_h(u)] = F_h(s), \]

we also write \( f_h(u) = \mathcal{M}^{-1}[F_h(s)] \).

Formally, we have

\[ \mathcal{M}[f_h(u)] = F_h(s) = \int_0^\infty u^{s-1} f_h(u) du, \quad (2.2.10) \]

and

\[ \mathcal{M}^{-1}[F_h(s)] = f_h(u) = \frac{1}{2\pi i} \int_L F_h(s) u^{-s} ds. \quad (2.2.11) \]

Conditions of validity of (2.2.10) and (2.2.11) can be found in Titchmarsh [183]. Fox [56] has restated the Parseval's theorem on Mellin transforms in the following form:

If

\[ \mathcal{M}[h(u)] = H(s), \]

and

\[ \mathcal{M}[f_h(xu)] = x^{-s} F_h(s), \]

then

\[ \int_0^\infty h(xu) f_h(u) du = \frac{1}{2\pi i} \int_L x^{-s} H(s) F_h(1-s) ds. \quad (2.2.12) \]
From (2.2.6), (2.2.8) and (2.2.11), it follows that

\[ M[H_{2}(p+m), 2(q+m)] = \chi_{p,q,m}(s), \quad (2.2.13) \]

and

\[ M[H_{2}(p+n), 2(q+n)] = \chi'_{p,q,n}(s), \quad (2.2.14) \]

\( \chi_{p,q,m}(s) \) and \( \chi'_{p,q,n}(s) \) have the values given in (2.2.7) and (2.2.9) respectively.

On using \( M[f_{h}(u)] = F_{h}(s) \) and applying (2.2.12) to (2.2.4) and (2.2.5), we find that

\[ \frac{1}{2\pi i} \int_{L} \chi_{p,q,m}(s) x^{-s} \sum_{h=1}^{n} a_{hk} F_{h}(1-s) ds = \phi_{k}(x), \quad 0 < x < 1, \quad (2.2.15) \]

and

\[ \frac{1}{2\pi i} \int_{L} \chi'_{p,q,n}(s) x^{-s} \sum_{h=1}^{n} b_{hk} F_{h}(1-s) ds = \psi_{k}(x), \quad x > 1, \quad k=1,2,3,\ldots,n, \quad (2.2.16) \]

by virtue of (2.2.13) and (2.2.14).

2.2.3. Application of Saxena's Operators

The equations (2.2.15) and (2.2.16) will be transformed into two others with the same kernel by making use of integration operators due to Saxena [118]. The following integral \([48]\) is useful

\[ \int_{0}^{2\pi i} \left[ \sum_{k=1}^{n} - c_{k} + \frac{k}{m_{k}}, \eta_{m} + \frac{k}{m_{k}} r_{m} \frac{k}{m_{k}} + \eta_{m} \right] \left( \frac{c_{m}}{x_{m}} \right)^{s-1} dt \]
\[
\left[ \frac{R_{m}^{k} - s_{m}}{R_{m}^{k} + s_{m}} \right] \left[ \frac{R_{m}^{k} - s_{m}}{R_{m}^{k} + s_{m}} \right], \quad (2.2.17)
\]

where \( c_{m} = \frac{1}{r_{m}} \), \( r = 0, 1, 2, \ldots \); \( R(c_{m} n_{m} - s_{m}) > 0 \), \( R(c_{m} n_{m} - s_{m}) > r_{m} + 1 \).

When fractional integration operators are introduced some of these conditions may no longer be necessary. Replacing \( x \) by \( t \) in (2.2.15), multiplying by

\[
c_{m} \frac{n_{m} - 1}{t_{m}} \int \frac{\beta_{m}^{k} + n_{m} + r}{\beta_{m}^{k} + n_{m} + r} \frac{t_{m}^{c_{m}}}{x_{m}},
\]

where \( c_{m} = \frac{1}{r_{m}} \) and integrating through the integral sign with respect to \( t \) from 0 to \( x \), \( 0 < x < 1 \), we find that

\[
\frac{1}{2\pi i} \int_{L_{p, q, m-1}} x^{-s} \sum_{h=1}^{n} a_{h} F_{h}(1-s)ds = (\beta_{m}^{k} + n_{m}) \frac{c_{m}^{x} - c_{m}^{x}}{R(c_{m} n_{m} - s_{m})}.
\]

\[
\int_{0}^{x} \frac{c_{m} n_{m} - 1}{t_{m}} \int \frac{\beta_{m}^{k} + n_{m} + r}{\beta_{m}^{k} + n_{m} + r} \frac{t_{m}^{c_{m}}}{x_{m}},
\]

where \( 0 < x < 1 \) and \( c_{m} = \frac{1}{r_{m}} \) and \( k = 1, 2, \ldots, n \).

Let us introduce the first operator of fractional integration denoted by \( R^{*} \):

\[
R^{*}[\alpha, \beta, \eta, \eta; \mu; f(x)] = \frac{\beta^{n-1}}{r^{1-\alpha}} \int_{0}^{x} \int \frac{t^{\mu}}{x_{m}^{\eta}} f(t)dt.
\]

(2.2.19)
When \( \mu = 1 \), (2.2.19) reduces to the one given by Saxena [118].

On the other hand if we set \( r = 0 \), we obtain Erdélyi's [46] operators, whereas \( \mu = 1, r = 0 \) give rise to Köber's [74] operators.

The operator \( \mathcal{R} \) exists provided \( p > 1, q < \infty, p^{-1} + q^{-1} = 1, \mu > 0, R(\alpha) > 0, R(n) > -q^{-1}, R(1-\alpha) > r, \beta \neq 0, -1, -2, \ldots; r = 0, 1, 2, \ldots \) and \( f(x) \in L_p(0, \infty) \).

For brevity, we write

\[
(\beta_n^{k+1} w, k)_w R(1-\alpha)(k+1)_w R(n)_w -1; r; c, f(x) = R_w[f(x)]
\]

where \( c_w = \frac{1}{w} \). It can be easily seen that the right hand side of (2.2.18) is equal to \( R_w[(\phi_k(x))] \) with \( 0 < x < 1 \) as \( k = 1, 2, \ldots, n \). On transforming the equation (2.2.15) step by step by the application of operators \( R_w \),

\[
w = m, (m-1), \ldots, 2, 1; \text{ it is observed that}
\]

\[
\frac{1}{z_n} \int_0^1 \chi_p, q(x) x^{-s} \sum_{h=1}^n a_h K_h(1-s) ds = R_1 R_2 \ldots R_m \left[ \phi_k(x) \right] \ldots \right)
\]

for \( 0 < x < 1, k = 1, 2, \ldots, n \).

(2.2.21)

We next proceed to transform the integral equation (2.2.16).

We know that

\[
\int_0^1 2^{k+1} n \frac{d^{n-k-1}}{d^{n-k-1}} \frac{d^{n-k-1}}{d^{n-k-1}} \left[ x \right]
\]

We have

\[
\int_0^1 2^{k+1} n \frac{d^{n-k-1}}{d^{n-k-1}} \frac{d^{n-k-1}}{d^{n-k-1}} \left[ x \right]
\]
where \( d_n = \frac{1}{\lambda_n} \), \( R(\sigma_n - \rho_n) > g+1 \) and \( d_n > 0 \),

and \( R(s + d_n \rho_n) > 0 \).

In (2.2.16) replace \( x \) by \( t \), multiply by

\[
\left( t^{\rho_n-1} \right) P(1 - \sigma_n^{\rho_n}, \delta_n^{\rho_n} + g; \rho_n^{\delta_n} + \delta_n; \frac{x}{d_n})
\]

where \( d_n = \frac{1}{\lambda_n} \) and integrate through integral sign with respect
to \( t \) from \( x \) to \( \infty \), \( x > 1 \), we then obtain

\[
\frac{1}{2\pi i} \int_{\gamma} x_p, q, n-1(s) x^{-s} \sum_{h=1}^n b_{hk} F_n(1-s) ds
\]

\[
= \frac{\left( \delta_n^{\rho_n} + \rho_n^{\delta_n} \right) d_n}{R(\sigma_n - \rho_n)} \int_x^\infty t^{-\rho_n-1} x^\rho_n \left\{ x^\frac{\mu}{\lambda} P(1-\sigma_n^{\rho_n}, \delta_n^{\rho_n} + g; \rho_n^{\delta_n} + \delta_n; \frac{x}{d_n}) \psi_k(t) dt \right\}
\]

\( k = 1, 2, \ldots, n \).

We now introduce the second operator of fractional
integration denoted by \( K \):

\[
K[\sigma, \delta, \rho; g; u; f(x)] = \frac{\mu^\rho}{\Gamma(1-\delta)} \int_x^\infty P(\sigma, \delta+\delta; \delta; \frac{x^\mu}{t^\nu}) t^{-\rho-1} f(t) dt.
\]

(2.2.24)
As in case of \( f(x) \in L_p(0,\infty), \ p \geq 1, \ R(p) > -p^{-1} \) then the operator \( K \) exists.

In the contracted form, we write

\[
(\delta_{p_k} + \rho_{p_k})^k K \left[ 1 - (\delta_{p_k} + \rho_{p_k})^k \right] g, d_p f(x) = K_p \left[ f(x) \right],
\]

where \( d_p = \frac{1}{\lambda_p} \), \( (\delta_{p_k} + \rho_{p_k})^k = \frac{r(\delta_{p_k} + \rho_{p_k}^k + g)}{r(\delta_{p_k} + \rho_{p_k})} \) .

From (2.2.24), it is evident that right hand side of (2.2.23) is \( K_p \left[ \psi_k(x) \right] \), where \( x > 1 \) for \( k = 1, 2, ..., n \).

The successive application of the operator \( K_p \) for \( l = n, (n-1), ..., 2, 1 \) to (2.2.16) transforms it into the desired form:

\[
\frac{1}{2\pi i} \int L \chi_{p,q}(s) x^{-s} \sum_{h=1}^{n} c_{hk} F_h(1-s) ds
\]

\[
= \Sigma_{h=1}^{n} c_{hk} K_1 \left[ K_2 \left[ K_n \left[ \psi_k(x) \right] \right] \right],
\]

\[ k = 1, 2, ..., n, \]

where \( c_{hk} \) are the elements of the matrix \( \left[ a_{hk} \right] \left[ b_{hk} \right]^{-1} \).

On setting

\[
G_k(x) = \left[ \begin{array}{c} R_1 \left[ R_2 \left[ \phi_k(x) \right] \right] \cdots \right], \ 0 < x < 1, \\
\Sigma_{h=1}^{n} c_{hk} K_1 \left[ K_2 \left[ K_n \left[ \psi_k(x) \right] \right] \right] \cdots \right], \ x > 1,
\]

(2.2.21) and (2.2.26) can put into a compact form:
Applying (2.2.12) to the L.H.S. of (2.2.29), we see that it can be expressed by an integral involving the product of 

\[ \mathcal{H}_{2n,2q}(ux) \text{ and } f(u). \]

The result is

\[ \int_{0}^{\infty} \mathcal{H}_{2n,2q}(ux) \sum_{h=1}^{n} a_{hk} f_h(u) du = G_k(x), \quad k=1,2,\ldots,n, \]  

(2.2.30)

where \( H(xu) \) is defined in (2.2.1).

Since \( H(xu) \) is a symmetrical Fourier kernel we, therefore, obtain the formal solution as

\[ f_h(x) = \sum_{k=1}^{n} d_{hk} \int_{0}^{\infty} \mathcal{H}_{2n,2q}(ux) G_k(u) du \]

\[ = \sum_{k=1}^{n} d_{hk} \left[ \int_{0}^{1} \mathcal{H}_{2n,2q}(ux) \mathcal{R}_1 \{ \mathcal{R}_2 \ldots \mathcal{R}_m (\psi_k(u) du) \right] \]

\[ + \int_{1}^{\infty} \mathcal{H}_{2n,2q}(ux) \sum_{h=1}^{n} c_{hk} k_1 k_2 \ldots k_n (\psi_k(u) du) \]

(2.2.31)

where \( d_{hk} \) is the element of the matrix \( [a_{hk}]^{-1} \). Since our method is purely formal, it does not give the condition for the validity of the solution.
2.2.4. Particular Cases.

(i) Dwivedi's [35] result can be derived from (2.2.31) by taking \( r = 0 \) and \( g = 0 \).

(ii) For \( p = 0, g = 0, q = m = n = 1 \),

\[
B_1 = \gamma_1 = \lambda_1 = \frac{1}{2}, \quad b_1 = \rho_1 = \beta_1 = \beta \frac{1}{2},
\]

\[
\sigma_1 = \beta^{k+1},
\]

\[
n_1 = 1 + 
\]

H-function reduces to an ordinary Bessel function \( J_{\beta k}(x) \) in view of the identity [47] and it is seen that the formal solution of

\[
2 \int_{0}^{\infty} (xu)^{2\beta k-n} J_{\beta k}(2xu) \sum_{n=1}^{\infty} a_{nk} f(u) du = \phi_k(x), \quad 0 < x < 1, \quad (2.2.32)
\]

and

\[
2 \int_{0}^{\infty} (xu)^{k-2\beta k} J_{\beta k}(2xu) \sum_{n=1}^{\infty} b_{hk} f(u) du = \psi_k(x), \quad x > 1, \quad (2.2.33)
\]

\( r = 0, 1, 2, \ldots; \) is given by

\[
f_h(x) = \sum_{n=1}^{\infty} d_{hk} \left\{ \int_{0}^{1} R \left[ \frac{1}{2} + n - \beta k \right] \right. \]

\[
1 + n_k; \quad 2(n_k - \beta k) + 1; r; 2; \varphi(u) ] J_{2\beta k}(2xu) du
\]

\[
+ 2 \int_{1}^{\infty} \sum_{h=1}^{n} c_{hk} K \left[ 1 - \frac{\beta k}{2} + \beta k; \delta + \beta k, 2\beta k; 0; 2 \varphi(u) \right] J_{2\beta k}(2xu) du, \]

\( k = 1, 2, \ldots, n. \)
2.3 FORMAL SOLUTION OF SIMULTANEOUS DUAL INTEGRAL EQUATIONS INVOLVING H-FUNCTIONS OF n-VARIABLES

In this section the formal solution of simultaneous dual integral equations involving H-function of n-variables has been obtained. The method followed is that of fractional integration. Here also it has been shown that the given simultaneous dual integral equations can be transformed, by the application of fractional integration operators, into two others with a common kernel. Since in this case also the common kernel comes out to be a symmetrical Fourier kernel given earlier by Fox, the formal solution is readily obtained.

2.3.1 Introduction

Recently Saxena and Sethi [123] have obtained the formal solution of dual integral equations associated with H-functions of two variables.

In this section we have obtained the formal solution of simultaneous dual integral equation involving H-functions of n-variables by using fractional integral operators defined later.

H-function of n-variables used here is defined in the following manner:

\[
\frac{1}{(2\pi i)^n} \int \prod_{j=1}^{l} r(e_j + e_j s_{kk}) \prod_{j=1}^{r} r(e_j + a_j s_{kk}) \prod_{j=1}^{v_j} r(\gamma_j + c_j s_{kk}) \prod_{j=1}^{m_k} r(\beta_j + b_j s_{kk}) \prod_{j=1}^{q_k} r(\delta_j + e_j s_{kk}) \prod_{j=1}^{r} r(\delta_j + a_j s_{kk}) \prod_{j=1}^{y_k} r(\beta_j + b_j s_{kk}) \prod_{j=1}^{v_j} r(\gamma_j + c_j s_{kk})
\]

where the repeated suffix represents a sum from 1 to \( n \) i.e.

\[
\frac{1}{(2\pi i)^n} \int \prod_{j=1}^{l} r(e_j + e_j s_{kk}) \prod_{j=1}^{r} r(e_j + a_j s_{kk}) \prod_{j=1}^{v_j} r(\gamma_j + c_j s_{kk}) \prod_{j=1}^{m_k} r(\beta_j + b_j s_{kk}) \prod_{j=1}^{q_k} r(\delta_j + e_j s_{kk}) \prod_{j=1}^{r} r(\delta_j + a_j s_{kk}) \prod_{j=1}^{y_k} r(\beta_j + b_j s_{kk}) \prod_{j=1}^{v_j} r(\gamma_j + c_j s_{kk})
\]

where the repeated suffix represents a sum from 1 to \( n \).

\[
\phi(s_{kk}) = \prod_{j=1}^{l} r(e_j + e_j s_{kk}) \prod_{j=1}^{r} r(e_j + a_j s_{kk}) \prod_{j=1}^{v_j} r(\gamma_j + c_j s_{kk}) \prod_{j=1}^{m_k} r(\beta_j + b_j s_{kk}) \prod_{j=1}^{q_k} r(\delta_j + e_j s_{kk}) \prod_{j=1}^{r} r(\delta_j + a_j s_{kk}) \prod_{j=1}^{y_k} r(\beta_j + b_j s_{kk}) \prod_{j=1}^{v_j} r(\gamma_j + c_j s_{kk})
\]

and \( \prod_{k=1}^{n} (ds_{kk}) = ds_1 ds_2 \cdots ds_n \).

Also \( (\delta_s, a_s) \) represents a sequence of \( s \) terms \( (\delta_1, a_1) \),
\( (\delta_2, a_2), \ldots, (\delta_s, a_s) \) \((\gamma_{t_1}^{n}, c_{t_1}^{n}) \) represents the sequence
\( (\gamma_1^{1}, c_1^{1}), (\gamma_2^{1}, c_2^{1}), \ldots, (\gamma_{t_1}^{n}, c_{t_1}^{n}) \); \((\gamma_{t_1}^{2}, c_1^{2})\),
\( (\gamma_2^{2}, c_2^{2}) \) \ldots; \((\gamma_{t_1}^{n}, c_{t_1}^{n}) \) \dots; \((\gamma_{t_2}^{n}, c_{t_2}^{n}) \) \ldots; \((\gamma_{t_2}^{n}, c_{t_2}^{n}) \) \ldots; \((\gamma_{t_n}^{n}, c_{t_n}^{n}) \).
are suitable contours and the positive integers $l, l_1, l_2, \ldots, l_n; p, p_1, p_2, \ldots, p_n; r, r_1, r_2, \ldots, r_n$ satisfy the following inequalities:

$$0 \leq l \leq p, \quad 0 \leq v_n \leq t_n, \quad 0 \leq m_n \leq q_n.$$ 

The integral (1.1) converges if

$$n(l+v_n+m_n) > p+r+t_k+q_k, \quad \arg\{x_k\} < \left[ \frac{1}{2} + v_n + m_k - \frac{1}{n}(p+s+t_k+q_k) \right] \pi$$

or

$$p+t_k < r+q_k,$$

or else $p+t_k = r+q_k$, with $|x_k| < 1$ where $k=1,2,\ldots,n$.

### 2.3.2 Notations and Results

Let

$$\theta[\bar{s}_{kk}] = \frac{\prod_{j=1}^{l} r(\xi_j - e^j s_{kk})}{\prod_{j=1}^{p} r(1 - \xi_j + e^j l + e^j l s_{kk})}$$

$$\psi_n(s_k)$$ will be denoted by $\tilde{\psi}_n(s_k)$, when $(y, c)$ and $(\beta, b)$ are replaced by $(\xi, c)$ and $(n, b)$ respectively. When $s_k$ is replaced by $-s_k$ in $\psi_n(s_k)$ and $\tilde{\psi}_n(s_k)$, they will be denoted by $\psi_n(-s_k)$ and $\tilde{\psi}_n(-s_k)$ respectively. $\mathcal{I} \mathcal{I} \mathcal{I} \ldots \ n$ integrals.

Writing Mellin transform of $n$-variables symbolically and employing the condition given by Reed [1117]

$$M\{g_\lambda(x_n)\} = F_\lambda(s_n) = n \int_0^\infty g_\lambda(x_n) \prod_{k=1}^{n} x_k^{-s_k-1} ds_k,$$
and

\[ M^{-1}\{F_h(s_n)\} = e_h(x_n) = \frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} F_h(s_n) \prod_{k=1}^{n} x_k^{-s_k} ds_k. \]  \hspace{1cm} (2.3.4)

Rewriting Fox [56] parseval theorem for mellin transform in \(n\)-variables as follows:

If \( M(h(u_n)) = H(s_n) \) and \( M(f_h(u_n x_n)) = F_h(s_n) \prod x_k^{-s_k} \), when \( M \{f_h(u_n)\} = F_h(s_n) \) then

\[ n \int_0^\infty h(x_n u_n) f_h(u_n) \prod_{k=1}^{n} (du_k) = \frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} H(s_n) F_h(1-s_n) \prod_{k=1}^{n} x_k^{-s_k} ds_k. \]  \hspace{1cm} (2.3.5)

From (2.3.1) and (2.3.4) it is easily seen that

\[
\sqrt{M} \begin{bmatrix}
\alpha_n, (v_n), (m_n) \\
p+\alpha_n, (t_n+v_n), (q_n+m_n)
\end{bmatrix} \begin{bmatrix}
((c e, e_p, e\alpha_p)) \\
((e n, c n, c t_n, t_n+v_n))
\end{bmatrix} = \theta \left[ x_k^{-s_k} \right] \psi_n(-s_k). \]  \hspace{1cm} (2.3.6)

Erdélyi [46] type fractional integral operators in \(n\)-variables as introduced by Aggrawala and Goyal [1] are:

\[
\int \left\{ (\alpha_n), (\delta_n); (s'_n): w(x_n) \right\} = \prod_{k=1}^{n} x_k^{-s'_k} \left[ x_k^{-s_k} \right] \left[ x_k^{-s_k} \right] \left[ x_k^{-s_k} \right] \left[ x_k^{-s_k} \right] w(v_n),
\]  \hspace{1cm} (2.3.7)
\[
R \left\{ (\alpha_n), (\delta_n) \right\} : (s'_n) : w(x_n) \right\} \equiv \sum_{k=1}^{n} \frac{s'_k}{r(\alpha_k)} \frac{\delta_k}{x_k} \int_{x_k}^{\infty} \frac{s'_k}{x_k - x_k} \alpha_k^{-1} x_k^{-\delta_k - s'_k - \alpha_k} (dv_k) \cdot w(v_n). \quad (2.3.8)
\]

where \( \{(\alpha_n), (\delta_n)\} = (\alpha_1, \delta_1), \ldots (\alpha_n, \delta_n) \).

Equations (2.3.7) and (2.3.8) will reduce to Köber [74] operator for \( (x_n) = x, (s'_n) = s_1, (\alpha_n) = \alpha \) and \( (\delta_n) = \delta \).

Generalized Fox [54] operators in \( n \)-variables can be easily written down as

\[
J_{j} \left\{ (v_{j+k} - \xi_j^k); (\xi_j^k(c_j^k)^{-1}); (c_j^k)^{-1}; \phi'(x_n) \right\} = J_{j} \{ \phi'(x_n) \}, \quad (2.3.9)
\]

\[
R_{j} \left\{ (n_j^k - \beta_j^k); (\beta_j^k(b_j^k)^{-1}); (b_j^k)^{-1}; \psi'(x_n) \right\} = R_{j} \{ \psi'(x_n) \}, \quad (2.3.10)
\]

\[
R_{j} \left\{ (v_{j+\nu_k} - \xi_j^k(v_k^k) \xi_j^k(c_j^k)^{-1}); (c_j^k)^{-1}; \psi'(x_n) \right\} = R_{j} \{ \psi'(x_n) \}. \quad (2.3.12)
\]

2.3.3 Simultaneous Dual Equations

We shall establish the formal solution of simultaneous dual integral equations

\[
\int_{0}^{\infty} \int_{0}^{1} \int_{0}^{p+1} \int_{0}^{p+1} \left( \begin{array}{c}
\left( c_{\nu+p}^n, e_{\nu+p}^n \right) \\
(\nu_n u_n) \\
((\nu_n t_n + v_n, c_{t_n+v_n}^n)) \\
(q_{n+m}^n)
\end{array} \right) \left| \begin{array}{c}
(q_{n+m}^n) \\
(q_{n+m}^n)
\end{array} \right| \left( \begin{array}{c}
\left( e_{\nu+p}^n, e_{\nu+p}^n \right) \\
(\nu_n u_n) \\
((\nu_n t_n + v_n, c_{t_n+v_n}^n)) \\
(q_{n+m}^n)
\end{array} \right) \left| \begin{array}{c}
(q_{n+m}^n) \\
(q_{n+m}^n)
\end{array} \right|
\]

\[
x \sum_{h=1}^{n} a_{h}(u_k) \int_{0}^{\infty} \int_{0}^{p+1} \int_{0}^{p+1} \left( \begin{array}{c}
\left( e_{\nu+p}^n, e_{\nu+p}^n \right) \\
(\nu_n u_n) \\
((\nu_n t_n + v_n, c_{t_n+v_n}^n)) \\
(q_{n+m}^n)
\end{array} \right) \left| \begin{array}{c}
(q_{n+m}^n) \\
(q_{n+m}^n)
\end{array} \right|
\]

\[
0 < x_n < 1, \quad (2.3.13)
\]
Here $a_{h\nu}$ and $b_{h\nu}$ are constants, $\phi'_\mu(x_n)$ and $\psi'_\mu(x_n)$ are given functions and $f_h(x_n)$ is to be determined.

We shall assume that $H[(x_n)]$ of (2.3.14) satisfies all the conditions given earlier for $H[(x_n)]$ with $(\gamma^n_{j\nu}, c^n_j)$ replaced by $(\xi^n_j, c^n_j)$ for $j=1,2,\ldots, t_n+n$ and $(\beta^n_j, b^n_j)$ replaced by $(\eta^n_j, b^n_{j\nu})$ for $j=1,2,\ldots, q_n+m_n$.

On using $M\{f_h(u_n)\} = F_h(s_n)$ and applying (2.3.5) to (2.3.13) and (2.3.14) and from (2.3.6), we obtain

\[
\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{i s_n \cdot x_n} \left[ \sum_{k=1}^{n} a_{h\nu} F_h(1-s_n) \right] ds_n = \phi'_\mu(x_n), \quad 0 < x_n < 1, \quad (2.3.15)
\]

\[
\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{i s_n \cdot x_n} \left[ \sum_{k=1}^{n} b_{h\nu} F_h(1-s_n) \right] ds_n = \psi'_\mu(x_n), \quad x_n > 1, \quad \mu = 1,2,\ldots,n. \quad (2.3.16)
\]

Now to replace the equations (2.3.15) and (2.3.16) into two others with a common kernel we shall transform
Now in order to make the first transformation in (2.3.15) we replace x's by v's and multiply both sides by

\[ \prod_{k=1}^{n} \left\{ \prod_{j=1}^{n} r(\nu_j - c_j k s_k) \right\} \]

into

\[ \prod_{k=1}^{n} \left\{ \prod_{j=1}^{n} r(\nu_j^* - c_j^* k s_k) \right\}, \]

where \( c_j^* = \frac{1}{c_j} \) and apply the well known Beta function formula to obtain,

\[ \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} e^{-\sum_{h=1}^{n} a_{h}} F_n(1-s_{n}) \]

\[ = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} e^{-\sum_{h=1}^{n} a_{h}} F_n(1-s_{n}) \]

\[ = \int_{0}^{\infty} \phi_n (x_{n}) \]

\[ 0 < x_{n} < 1, u = 1, 2, \ldots, n. \]
where \( c_{\nu}^k = \frac{1}{c_{\nu}^k} \), by virtue of (2.3.7) and (2.3.9).

Similarly applying \( J_j \) successively for \( j=\nu_k, \ldots, 2, 1 \), we get

\[
\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \theta \left[ s_{k \nu_k} \right] \prod_{k=1}^{n} \left( \sum_{j=1}^{v_k} \prod_{k=1}^{m_k} r(\xi_j^k - c_j^k s_k) \prod_{j=1}^{q_k} r(\beta_j^k + b_j^k s_k) \prod_{j=1}^{n_k} r(\eta_j^k + m_j^k s_k) \right) x_k^s \{ \phi_{\mu}^i (x_n) \},
\]

\( \mu = 1, 2, \ldots, n \). \hspace{1cm} (2.3.18)

Also applying \( J_j^* \) operator successively for \( j=q_k, \ldots, 2, 1 \) we obtain

\[
\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \theta \left[ s_{k \nu_k} \right] \prod_{k=1}^{n} \left( \sum_{j=1}^{v_k} \prod_{k=1}^{m_k} r(\xi_j^k - c_j^k s_k) \prod_{j=1}^{q_k} r(\beta_j^k + b_j^k s_k) \prod_{j=1}^{n_k} r(\eta_j^k + m_j^k s_k) \right) x_k^s \{ \phi_{\mu}^i (x_n) \} \right].
\]

\( \mu = 1, 2, \ldots, n \). \hspace{1cm} (2.3.19)

In the same way by the application of the operator \( R_j \) and \( R_j^* \) given by (2.3.11) and (2.3.12) for \( j=m_k, \ldots, 2, 1 \), and \( j=t_k, \ldots, 2, 1 \) respectively to (2.3.16) it can be easily seen that it transforms into desired form.
\[
\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \Theta \left[ s_{k} k \right] \prod_{k=1}^{n} \frac{1}{\pi} \frac{r(\xi_j - c_j s_k)}{\pi} \frac{r(\beta_j + b_j s_k)}{\pi} \frac{1}{\pi} \frac{r(\eta_j + m_j - b_j s_k)}{\pi} \frac{1}{\pi} \frac{r(\eta_j + m_j + b_j + m_j s_k)}{\pi} \]

\[
\sum_{h=1}^{n} a_{h \mu} F_h(1-s_n) = \sum_{h=1}^{n} c_{h \mu} R_1 \cdots R_n \left\{ \psi_{\mu} (x_n) \right\}, \quad 0 < x_n < 1,
\]

where \(c_{h \mu}\) are the elements of the matrix \([a_{h \mu}] [b_{h \mu}]^{-1}\).

On setting

\[
G_{\mu} (x_n) = \sum_{h=1}^{n} c_{h \mu} R_1 \cdots R_n \left\{ \psi_{\mu} (x_n) \right\}, \quad x_n > 1, \quad \mu = 1, 2, \ldots, n.
\]

The equations (2.3.19) and (2.3.20) transformed into one with a common kernel can be put into a compact form

\[
\frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \Theta \left[ s_{k} k \right] \prod_{k=1}^{n} \frac{1}{\pi} \frac{r(\xi_j - c_j s_k)}{\pi} \frac{r(\beta_j + b_j s_k)}{\pi} \frac{1}{\pi} \frac{r(\eta_j + m_j - b_j s_k)}{\pi} \frac{1}{\pi} \frac{r(\eta_j + m_j + b_j + m_j s_k)}{\pi} \]

\[
\sum_{h=1}^{n} a_{h \mu} F_h(1-s_n) = G_{\mu} (x_n).
\]

(2.3.22) is the reduction of (2.3.15) and (2.3.16) to two equations with a common kernel. On treating the kernel of (2.3.22) as an unsymmetric Fourier kernel and following a
procedure similar to the one adopted by Fox [56] for one
variable (2.3.22) becomes

\[
fh(x_n) = \frac{1}{(2\pi)^n} \sum_{\mu=1}^{n} d_{\mu n} \int_{-\infty}^{\infty} \prod_{j=1}^{p} r(1+ne_{j+l}-e_{j+l} s_{kk}) \prod_{j=1}^{q} r(-ne_{j+\epsilon} e_{j+\epsilon} s_{kk}) \\
\prod_{j=1}^{t_k} r(\alpha_{j+k}^{\epsilon k} + \gamma_{j+k}^{\epsilon k} - \sigma_{j+k}^{\epsilon k} s_{kk}) \prod_{j=1}^{q_k} r(-b_{j+k}^{\epsilon k} + \gamma_{j+k}^{\epsilon k} s_{kk}) \\
\prod_{j=1}^{m_k} r(-\alpha_{j+k}^{\epsilon k} + \sigma_{j+k}^{\epsilon k} s_{kk}) \prod_{j=1}^{b_{j+k}^{\epsilon k}} r(\beta_{j+k}^{\epsilon k} - \beta_{j+k}^{\epsilon k} s_{kk}) \\
x_{k\delta s_k} G^\mu_{\delta} \left(1-s_{\delta n}\right), \ h = 1,2,\ldots,n,
\]

(2.3.23)

where \( M \{G'(x_n)\} = G'(s_n) \) and \( d_{\mu n} \) are the elements of the
matrix \([a_{\mu n}]^{-1}\).

This is the formal solution of (2.3.15) and (2.3.16)
and many important properties of the function \( fh(x_n) \) can be
obtained from it. Applying Parseval theorem (2.3.5) to
(2.3.23), we see that

\[
f_h(x_n) = \sum_{\mu=1}^{n} d_{\mu n} \int_{0}^{\infty} \prod_{p+\epsilon,0;\left(t_{n+p}\right),\left(q_p\right)\left(q_{n+m}\right)} \left[(x_n u_n)\right] \\
(1+ne_{1+p},^{-}\epsilon_{1+p}, e_{1+p}, p, e_{1+p}, p) \\
(\alpha_{1+p}^{\epsilon_n} t_{n+\gamma_{1+p}^{\epsilon_n}}, t_{n+\epsilon_{1+p}^{\epsilon_n}}^{\epsilon_n} t_{n+\epsilon_{1+p}^{\epsilon_n}}^{\epsilon_n}, t_{n+\epsilon_{1+p}^{\epsilon_n}}^{\epsilon_n}, t_{n+\epsilon_{1+p}^{\epsilon_n}}^{\epsilon_n}),
\]
\[(1+n_{\text{e}_2}-a_{\text{e}_2})\]
\[\left(-c_n^\nu + n_{\text{e}_2}, c_n^\nu \right) \left(-b_n^{1+m_n}, q_n + n_{\text{e}_2} + m_n, q_n, b_n, b_n^{1+m_n} q_n, (b_n^{1+m_n}, b_n^{1+m_n}) \right)\]
\[G_n^\mu (u_n) \prod_{k=1}^n (d\chi_k)\]
\[n=1,2,\ldots,n,\]
\[\sum_{\mu=1}^n d_{\mu n} \left( n \int_0^1 \frac{p, 0; (t_n)(q_n)}{H_{p+2, 0; (t_n+\nu_n), (q_n+m_n)} \left[ (x_n u_n) \right] \prod_{k=1}^n (d\chi_k) \right)\]
\[+ n \int_1^\infty \frac{p, 0; (t_n), (q_n)}{H_{p+2, 0; (t_n+\nu_n), (q_n+m_n)} \left[ (x_n u_n) \right] \prod_{k=1}^n (d\chi_k)\}
\[\times \left( R_{1} R_{2}\ldots R_{m_k} \left( \psi_n (u_n) \right) \right) \prod_{k=1}^n (d\chi_k)\),\]
\[h = 1,2,\ldots,n,\]
\[(2.3.24)\]

where \(d_{\mu n}\) is the element of the matrix \(a_{\mu n}\)^{-1}.

2.3.4 **Particular Cases**

(i) Taking \(e^{l+p} = c_n^{n_{l+m_n}} = b_n^{n_{l+m_n}} = 1\) and noting that

\(H\)-function of \(n\) variables reduces to \(G\)-function of \(n\)-variables, we get a solution of simultaneous dual integral equation involving \(G\)-function of \(n\)-variables from which the solution obtained by Aggarwala and Goyal can be deduced as particular case.
(ii) Since \( \lim H(x_n) = H(x,y) \) when \( n = 2 \), \( x_1 = x \),
\( (x_n, n) \to 0 \)
\( x_2 = y \) we obtain the formal solution of the simultaneous dual integral equations involving \( H \)-function of two variables.