CHAPTER 1

INTEGRAL AND SERIES EQUATIONS

During the last fifty years the special type of integral equations such as dual integral equations, triple integral equations etc. have been proved to be very useful tool for solving the mixed boundary value problems \[144\]. The same is also true for series equations such as dual series equations, triple series equations etc. \[144\].

In the following lines we give the historical developments of such types of integral and series equations developed in the literature.

1.1. DUAL INTEGRAL EQUATIONS

The study of dual integral equations had started as early as in 1873 by Weber \[199\] who obtained the solution by inspection. But it is only since 1945 that there have been attempts to develop the theory of dual integral equations, dual series equations and triple relations and this work was initiated by publication of Titchmarsh's book \[183\] "An introduction to the theory of Fourier integrals" which included first systematic treatment of dual integral equations.

Most of the dual integral equations with which we meet in the solution of mixed boundary value problems of Mathematical Physics are of the type
\[ \int_{0}^{\infty} w(u) \phi(u) K(x,u) du = f(x), \quad 0 < x < 1, \quad (1.1.1) \]
\[ \int_{0}^{\infty} \phi(u) K(x,u) du = g(x), \quad x > 1, \quad (1.1.2) \]

where \( w(u) \) is a function of \( u \) alone. We shall call \( w(u) \) the 'weight function' and \( K(x,u) \) the kernel of this pair of equations.

The functions \( f(x) \) and \( g(x) \) are prescribed and the unknown function \( \phi(u) \) is to be determined. In practical applications the kernels are usually either of Bessel type \( J_\nu(xu) \) or of trigonometric type, \( \sin(xu) \), \( \cos(xu) \).

At present, many methods are available for finding the solution of dual integral equations. Some of these methods are listed below.

### 1.1.1 Mellin Transform Method

The use of Mellin transform has been responsible for many developments in the theory of dual integral equations. The formal solution of the dual integral equations of the type:
\[ \int_{0}^{\infty} t^\alpha \phi(t) J_\nu(xt) dt = f(x), \quad 0 < x < 1, \quad (1.1.3) \]
\[ \int_{0}^{\infty} \phi(t) J_\nu(xt) dt = g(x), \quad x > 1, \quad (1.1.4) \]

for \( g(x) = 0 \), was given by Titchmarsh [183, p. 357]. His
solution is valid for $\alpha > 0$ and the method of solution is closely related with Wiener-Hopf technique \[100, \text{p. 171}\].

Miss Busbridge \[8\] extended the solution of Titchmarsh to $\alpha > -2$ by Mellin transforms method and analytic continuation in the complex plane. Heins \[63\] obtained the solution of (1.1.3) and (1.1.4) by Wiener-Hopf method.

Williams \[202\] obtained the solution of the equations (1.1.3) and (1.1.4) by using the theory of Mellin transforms but differently than that of Titchmarsh \[183, \text{p. 357}\]. He obtained the solution valid for $-2 < \alpha < 0$ and $0 < \alpha < 2$ separately.

By using Busbridge's form of solution Noble \[98\] found the solution of equations (1.1.3) and (1.1.4) valid for $-2 < \alpha < 0$. He also gave the method to extend the range of $\alpha$.

1.1.2. Weber Sonine-Schafheitlin Integrals

Weber Sonine-Schafheitlin integrals have been very useful in finding the solution of various type of dual integral equations. Gordon \[59\] showed that the pair of equations (1.1.3), (1.1.4) can be solved by using Sonine's integral \[198, \text{p. 373}\] and a discontinuous integral involving Bessel functions \[95, \text{p. 57}\].

Peters \[116\] solved the more general equations
\[
\int_0^\infty t^\alpha \phi(t) J_\mu(xt) \, dt = f(x), \quad 0 < x < 1, \quad (1.1.5)
\]

\[
\int_0^\infty \frac{\phi(t)}{t} J_\nu(xt) \, dt = g(x), \quad x > 1, \quad (1.1.6)
\]

and he solved these equations in a rather simple way by using Sonine's integrals. He also considered the more general equations

\[
\int_k^\infty t^\alpha \phi(t) J_\mu(xt) \, dt = f(x), \quad 0 < x < 1, \quad (1.1.7)
\]

\[
\int_0^\infty (t^2 - k^2) s \phi(t) J_\nu(xt) \, dt = g(x), \quad x > 1, \quad (1.1.8)
\]

for \( k > 0 \). Ahiæzer [4] considered various particular cases of the above equations under different conditions.

1.1.3. Method for Trigonometric Kernels

Fredricks [57] and Sneddon [141] gave the solution for equations with sine and cosine kernels by giving representation of \( f(x) \) in the series form. Tranter [187] obtained a very simple formal solution of equations (1.1.3) and (1.1.4) when Bessel functions are replaced by trigonometric functions for \( \alpha = -1 \) and \( g(x) = 0 \). Lowndes [78] improved upon Tranter's solution by considering general values of \( \alpha \) and more general result for Bessel function integral representation [198, pp. 48, 470]. Using Tranter's method Dwivedi [33] obtained the solution of more general dual integral equations with trigonometric kernels. Srivastava [153]
generalised Sneddon's approach \([14]\) and obtained the solution for more general set of equations.

1.1.4. Integral Representation Method

This is a convenient method for finding the solution of various dual integral equations. Tranter \([184]\) and Noble \([99]\) solved the equations (1.1.3) and (1.1.4) under different conditions by representing the unknown function in the form of an integral. Cooke \([14]\) reduced the equations (1.1.3) and (1.1.4) with \(g(x) = 0\), using Tranter's technique, to a Fredholm integral equation of second kind. Lebedev and Ufland \([77]\) touched the equations

\[
\int_0^\infty \{1 - H(t)\} \phi(t) J_0(x t) \, dt = f(x), \quad 0 < x < a, \quad (1.1.9)
\]

\[
\int_0^\infty \phi(t) J_0(x t) \, dt = 0, \quad x > a, \quad (1.1.10)
\]

and found the solution in the form of a Fredholm integral equation with symmetric kernel.

Recently Srivastav and Parihar \([177]\) found the solution of dual equations involving inverse Mellin transforms by this method. Beside this various investigators like Burlak \([9]\), Copson \([19]\), Sneddon \([14]\), Lowengrub and Sneddon \([37]\), Széfer \([180]\), Srivastav \([172]\) and Westmann \([200]\) considered different equations and found the solution by this method.
1.1.5 **Multiplying Factor Method.**

This is one of the simplest methods for finding the solution of certain dual integral equations. Noble [104] was first to apply the method to find the solution of equations (1.1.3) and (1.1.4). He also discussed its relation with Weiner-Hopf technique. Later on Dwivedi [34], Jageta [65] and other workers applied this technique to solve different equations. Recently Pathak and Prasad [115] extended the method to the more general dual integral equations involving H-functions.

1.1.6 **Operators of Fractional Integration**

Erdélyi and Köber ([50], [74]) introduced certain operators of fractional integration which are useful for finding the solution of certain dual integral equations. Erdélyi and Sneddon [51] and Sneddon [142] found the solution of various dual integral equations by means of fractional integral operators. By now these operators have been extended and many new introduced by various workers such as Erdélyi [49], Fox [56], Jageta [65], Kesavan [69], Saxena ([119],[120]), Tanno [184], Saxena and Sethi [123] and Sethi and Banerji [128] who have considered the dual integral equations involving Bessel functions, H-functions, Meijer's G-functions and inverse Mellin transforms. Saxena and Kushwaha [122] considered for solution the dual integral equations with H-functions.

1.1.7 **Solution by Generalised Functions**

Srivastav and Parihar [37] developed the technique for solution of dual equations using generalised functions.
concept and gave the solution of some dual equations. Recently Srivastav [174] has considered dual integral equation with trigonometric kernels and tempered distributions. He [175] also gave an $L_2$-theory for solving dual trigonometrical equations.

1.1.8. Reduction to Initial Value Problems

Recently Kalaba and Zugustin [66] gave the initial value method for solving certain dual integral equations and their solution is in the form of Fredholm integral equation.

1.1.9. Function Theoretic Approach


1.1.10. Approximate Methods of Solution

In discussing an acoustical problem King [72] developed an approximate method which he claimed could be used to solve dual integral equations of the type (1.1.3), (1.1.4) to any desired degree of accuracy.
Another method of obtaining approximate solutions of dual integral equations has been suggested by Noble \[100\]. The idea put forward by Noble is to reduce the pair of dual integral equations to a single integral equation and then to show that this equation is itself equivalent to a variational principle. Linz \[76\] suggested an alternative method for the numerical solution of some dual integral equations. More recently Kelman and Koper \[68\] suggested the least square approximations for dual trigonometrical series.

1.2 SIMULTANEOUS DUAL INTEGRAL EQUATIONS

The problem of solving simultaneous dual integral equations of the form

\[
\int_0^\infty \sum_{j=1}^n c_{ij}(x) \psi_j(x) J_{\mu i}(xy) \, dx = p_i(y), \quad y \in I_1, \quad (1.2.1)
\]

\[
\int_0^\infty \psi_j(x) J_{\mu i}(xy) \, dx = 0, \quad y \in I_2, \quad (1.2.2)
\]

\((i=1,2,\ldots,n)\) has been considered by Erdogan and Bahar \[53\]. The special case of \(n=2\) was independently discussed by Westmann \[200\]. Erdogan \[52\] considered further simultaneous equations with trigonometric and Bessel kernels. Khadem \[74\] and Dwivedi \[35\] have also considered certain simultaneous dual integral equations.

In chapter two of this thesis various types of simultaneous dual equations have been treated.
1.3 TRIPLE INTEGRAL EQUATIONS

The following are the methods for finding the solution of triple integral equations.

1.3.1 Reduction to Dual Series

Tranter [189] was the first investigator who took interest in triple integral equations. He obtained the solution of the equations:

\[ \int_0^\infty \psi(u) J_\nu(xu) \, du = f(x), \quad 0 < x < a, \quad (1.3.1) \]

\[ \int_0^\infty u^\alpha \psi(u) J_\nu(xu) \, du = g(x), \quad a < x < b, \quad (1.3.2) \]

\[ \int_0^\infty \psi(u) J_\nu(xu) \, du = h(x), \quad x > b, \quad (1.3.3) \]

with \( h(x) = 0 \) and reduced the above equations to dual series equations with Jacobi polynomials as kernels. Lowndes [32] reduced the above set also to Fredholm integral equations.

Cooke [15] considered the above equations first with \( f(x) = h(x) = 0 \) and later extended the method for \( f(x) \neq h(x) \neq 0 \).

1.3.2 Operators of Fractional Integration

Cooke [16] took a lot of interest in finding the solution of triple integral equations of Titchmarsh type. He used Erdélyi–Köber operators [50] and their extended form to find the solution of the most general set of triple integral equations. His solutions were in the form of Fredholm integral
equations and contained other solutions as particular cases. Recently Cooke [17] showed the analogy between the triple integral equations and triple Fourier-Bessel series equations. He found the solution of triple integral equations and wrote the solution of triple series by the analogy. Saxena and Sethi [125] using fractional integral operators, solved some triple integral equations.

1.3.3. **Inverse Mellin Transform Method**

Srivastav and Parihar [177] solved the triple integral equations of the type

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s \psi(s) \rho^{-s} ds = f_1(\rho), \quad 0 < \rho < a, \quad (1.3.4)
\]

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tan \alpha s \psi(s) \rho^{-s} ds = f_2(\rho), \quad a < \rho < 1, \quad (1.3.5)
\]

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s \psi(s) \rho^{-s} ds = f_3(\rho), \quad \rho > 1, \quad (1.3.6)
\]

by reducing them to dual cosine series equations and writing solution by Tranter's [183] method.

Lowndes [84] extended the operators of fractional integration considered earlier by Cooke and obtained the solution of the following set of triple equations

\[
M^{-1} \left\{ \frac{\Gamma(\xi + s/6)}{\Gamma(\xi + \beta + s/6)} \phi(s); \ x \right\} = 0 \left\{ \begin{array}{ll} 0 < x < a, \\ b < x < \infty, \end{array} \right. \quad (1.3.7)
\]
\[ M^{-1}\{ \frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\xi-s/\sigma)} \} \phi(s); x = f_2(x), a < x < b, \quad (1.3.8) \]

where \( \alpha, \beta, \xi, \eta, \delta > 0, \sigma > 0 \) are real parameters. His form of solution is a Fredholm integral equation of second kind. He also considered the equations

\[ \int_0^\infty \psi(u) J_{2p}(ux)dx = 0, \quad 0 \leq x < a, \quad b < x < \infty, \quad (1.3.9) \]
\[ \int_0^\infty u^{-2n} \psi(u) J_{2q}(ux)dx = F(x), \quad a < x < b, \quad (1.3.10) \]

by changing them in the form above.


1.3.4. Finite Hilbert Transform Technique

In a recent paper Srivastava and Lowengrub [165] obtained the solution of triple integral equations with trigonometric kernels. Their equations were of the form:

\[ \mathcal{J}[A(u);y] = 0, \quad 0 < y < a, \quad b < y < \infty, \quad (1.3.11) \]
\[ \mathcal{J}[u^p A(u);y] = f(y), \quad a < y < b, \quad (1.3.12) \]

where \( \mathcal{J} \) denotes Fourier sine or cosine transform and \( p = \pm 1 \).

By representing \( A(u) \) by a suitable function, they transformed the above equations to a Hilbert problem whose solution is
well known. They also determined the solution of the above set when there is an extra factor \((1+H(\delta))\) in equation (1.3.12) and in this they first reduced the problem to Hilbert problem and then to Fredholm integral equations of second kind.

Singh [13] gave an exact solution of the equations (1.3.11) and (1.3.12) by Hilbert transform technique.

Recently Tweed and Longmuir [194] considered some triple integral equations involving inverse Mellin transforms and used equations to solve some crack problems of elasticity.

Spence [15] developed the Wiener Hopf solution for certain triple integral equations for Hankel transforms which arise in the electrified disc in a coplanar gap.

1.4 SIMULTANEOUS TRIPLE INTEGRAL EQUATIONS

Till date very less number of research papers have been published on simultaneous triple integral equations which arise in physical situations.

In chapter three of the present thesis we have given the solution of various type of simultaneous triple integral equations involving different kernels of special functions.

1.5 QUADRUPLE INTEGRAL EQUATIONS

Ahmad [3] was the first investigator to introduced to us the quadruple integral equations of the following type:
\[ \int_0^\infty t^{-2\alpha} \phi(t) J_\nu (xt) dt = F_1(x), \quad 0 < x < a, \quad (1.5.1) \]
\[ \int_0^\infty t^{-2\beta} \phi(t) J_\nu (xt) dt = G_2(x), \quad a < x < b, \quad (1.5.2) \]
\[ \int_0^\infty t^{-2\alpha} \phi(t) J_\nu (xt) dt = F_3(x), \quad b < x < c, \quad (1.5.3) \]
\[ \int_0^\infty t^{-2\beta} \phi(t) J_\nu (xt) dt = G_4(x), \quad c < x < \infty, \quad (1.5.4) \]

where \( F_1, G_2, F_3 \) and \( G_4 \) are prescribed functions of \( x \) and \( \phi(x) \) is to be determined. By using the Erdélyi-Köber operators he reduced the above equations to Fredholm integral equation of second kind.

Dange and Singh \[24\] obtained the formal solution of the four equations

\[ \int_0^\infty \phi(t) P_{-1/2+it} (\cosh x) dt = f_1(x), \quad 0 < x < a, \quad (1.5.5) \]
\[ \int_0^\infty t \tanh \pi t \phi(t) P_{-1/2+it} (\cosh x) dt = f_2(x), \quad a < x < b, \quad (1.5.6) \]
\[ \int_0^\infty \phi(t) P_{-1/2+it} (\cosh x) dt = f_3(x), \quad b < x < c, \quad (1.5.7) \]
\[ \int_0^\infty t \tanh \pi t \phi(t) P_{-1/2+it} (\cosh x) dt = f_4(x), \quad c < x < \infty, \quad (1.5.8) \]

where \( \phi(t) \) is to be determined in the form of Fredholm integral equation which can be solved numerically.

Cooke \[27\] employed Erdélyi, Köber, and Sneddon and Srivastav operators to find the solution of Bessel functions.
quadruple integral equations with arbitrary weight functions. He reduced the equations to Fredholm integral equations. Dwivedi and Trivedi ([44], [44]) considered the more general equations than Cooke and obtained the solution in the form of Fredholm integral equations.

Beside this, other authors e.g. Gupta and Chaturvedi [61], Saxena and Sethi [24], Jain and Singh [64] and Singh [134], [135] considered different sets of quadruple equations and found their solution.

1.6. DUAL SERIES EQUATIONS

We know that the integral representations of harmonic functions reduce the solution of mixed boundary value problems of certain type to that of a pair of dual integral equations. Similarly if we make use of a series solution of Laplace's equation we are lead to a pair of dual relations but this time series and not integrals.

Now we discuss various methods for solving dual series equations developed in the recent past.

1.6.1. Reduction to System of Algebraic Equations

Cooke and Tranter [18] were first to attack for the solution of dual series equations of the type

\[ \sum_{n=1}^{\infty} \lambda_n^{-2p} a_n J_\nu(\rho \lambda_n) = f(\rho), \quad 0 \leq \rho < 1, \quad (1.6.1) \]

\[ \sum_{n=1}^{\infty} a_n J_\nu(\rho \lambda_n) = g(\rho), \quad 1 < \rho \leq a, \quad (1.6.2) \]
with \( g(\rho) = 0 \) and \(-1/2 \leq \rho \leq 1/2, \nu > 0\) where \( \{\lambda_n\} \) are the positive zeros of the Bessel function \( J_\nu(\alpha \lambda) = 0\). They called the above set as a pair of dual Fourier-Bessel series equations to determine the unknown constant \( \alpha_n \). They completed the solution for special values of parameters \( \rho \) and \( \nu \) by using a method similar to Tranter's method of solution for dual integral equations \([185]\). The final form of solution is in the form of simultaneous algebraic equations.

### 1.6.2 Reduction to the Solution of an Integral Equation

Recently Sneddon and Srivastav \([148]\) obtained the solution of the equations (1.6.1) and (1.6.2) in the form of Fredholm integral equation. They divided the problem into two: (i) with \( g(\rho) = 0 \), (ii) with \( f(\rho) = 0 \) and then obtained the solution for different values of the parameters.

The special cases of the equations to dual trigonometrical series are of more importance. Tranter in series of papers discussed such equations and finally obtained suitable closed form solutions \([187]\). Other investigators for solution of such equations are Srivastav \([173]\), Das \([22]\), Noble and Whiteman \([104], [105]\).

Collins \([20]\) considered the dual equations of the type

\[
\sum_{n=0}^{\infty} (1+H_n) a_n T_{m+n}^{-m}(\cos \theta) = F(\theta), \ 0 < \theta < \phi \ , \ (1.6.3)
\]

\[
\sum_{n=0}^{\infty} (2n+2m+1)a_n T_{m+n}^{-m}(\cos \theta) = G(\theta), \ \phi < \theta \leq \pi \ , \ (1.6.4)
\]
where \( T_{m+n} \) is an associated Legendre function, \( F(\theta) \) and \( G(\theta) \) are known and \( a_n \) is to be determined. He obtained the solution in the form of Fredholm integral equation of second kind. However, the solution for \( H_n = 0 \) is in the closed form.

1.6.3. **Multiplying Factor Method**

The solution of a pair of dual series relations of different kind has been derived by Noble [102] by multiplying factor method. He considered equations of the form

\[
\sum_{n=0}^{\infty} p_n(\lambda, \beta) a_n J_n(\alpha, \beta; x) = f(x), \quad 0 \leq x < a, \tag{1.6.5}
\]

\[
\sum_{n=0}^{\infty} a_n J_n(\alpha, \beta; x) = g(x), \quad a < x \leq 1, \tag{1.6.6}
\]

where \( J_n(\alpha, \beta; x) \) denotes the Jacobi polynomial and \( p_n(\lambda, \beta) \) is a constant.

Kiyono and Shimasaki [73] considered some dual trigonometric series equations which have application to microstrip transmission lines. Kelman [67] presented a perturbational analysis for dual trigonometric series and established the existence and uniqueness theorem for all the classic dual trigonometric equations.

Noble and Hussain [103] considered for solution some dual trigonometric series arising in certain indentation and inclusion problems. They obtained the closed form solution which
was not existing in the literature. Tranter [19] pointed out a relation between a set of dual cosine series and the series involving Legendre polynomials. The solution of cosine series was determined from the known solution of later equations. Das [23] also considered some dual trigonometric cosine series arising in potential problems.

Srivastav [17] also considered the set of equations

$$
\sum_{n=0}^{\infty} \frac{A_n}{r(\alpha+n+1) r(\beta+n+3/2)} p_n^{(\alpha,\beta)}(\cos \theta) = F(\theta), \quad 0 \leq \theta \leq \phi, \quad (1.6.7)
$$

$$
\sum_{n=0}^{\infty} \frac{A_n}{r(\alpha+n+1/2) r(\beta+n+1)} p_n^{(\alpha,\beta)}(\cos \theta) = G(\theta), \quad \phi \leq \theta \leq \pi, \quad (1.6.8)
$$

where $\alpha > -\frac{1}{2}$, $\beta > -1$ and $p_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial. He obtained the closed form solution following his own method.

The equations of the form below were considered by Srivastava [151]:

$$
\sum_{n=0}^{\infty} \frac{A_n}{r(\alpha+n+1)} I_n^{(\nu)}(x) = f(x), \quad 0 \leq x < y, \quad (1.6.9)
$$

$$
\sum_{n=0}^{\infty} \frac{A_n}{r(\beta+n+1)} I_n^{(\sigma)}(x) = g(x), \quad y < x \leq \infty, \quad (1.6.10)
$$

where $I_n^{(\sigma)}(x)$ is the generalised Laguerre polynomial and obtained the solution following Srivastav [17]. Particular case of these equations was considered by Lowndes [79].
More recently Srivastava [152] considered the dual series equations

\[ \sum_{n=0}^{\infty} \frac{A_n}{r(2\beta+\sigma+n+1)} K_2^{(\alpha+\sigma)}(x) = f(x), \ 0 \leq x < y, \quad (1.6.11) \]

\[ \sum_{n=0}^{\infty} \frac{A_n}{r(2\nu+\sigma+n+1)} K_2^{(\beta+\sigma)}(x) = g(x), \ y \leq x < \infty, \quad (1.6.12) \]

where \( \alpha+\sigma+1 > 0, \beta > \nu > \alpha - \frac{1}{2} m, \ 2\nu+\sigma+1 > 0, \sigma \) is a negative and \( K_\nu^{(\alpha)}(x) \) is the generalised Bateman function, by multiplying factor method. Special cases of the above equations were considered by Srivastava [154] and Dwivedi [36].

1.7 TRIPLE SERIES EQUATIONS

We know that the series solution of Laplace's equation to three part boundary value problems leads to triple series equations. Collin's [21] was the first investigator to introduce the triple series equations of the types

\[ \sum_{n=0}^{\infty} (2n+1) C_n P_n(\cos \theta) = 0, \ 0 \leq \theta < \alpha, \ \beta < \theta < \pi, \quad (1.7.1) \]

\[ \sum_{n=0}^{\infty} (1+H_n) C_n P_n(\cos \theta) = f(\theta), \ \alpha < \theta < \beta, \quad (1.7.2) \]

which he called as equations of the first kind and

\[ \sum_{n=0}^{\infty} (1+H_n) C_n P_n(\cos \theta) = f(\theta), \ 0 < \theta < \alpha, \ \beta < \theta < \pi, \quad (1.7.3) \]

\[ \sum_{n=0}^{\infty} C_n P_n(\cos \theta) = 0, \ \alpha < \theta < \beta, \quad (1.7.4) \]
as equations of the second kind. By following the method due to Gubenko and Mussakovaskii [62] he reduced the problem to two sets of dual series and then obtained the final solution by using Mehler-Dirichlet integrals for Legendre polynomials.

Williams [203] presented a method by reducing the triple series equations to triple integral equations. Srivastava [155] obtained the solution of more general triple equations which included Collin's solution as particular case. He considered the triple series

\[ \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)\Gamma(n+\alpha+1/2)} p_n^{(\alpha,\beta)}(\cos \theta) = 0, \]
\[ b < \theta < \pi, 0 \leq \theta < a, \quad (1.7.5) \]

\[ \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+1)\Gamma(n+\beta+3/2)} p_n^{(\alpha,\beta)}(\cos \theta) = g(\theta), \]
\[ a < \theta < b, \quad (1.7.6) \]

for \( \alpha > -1, \beta > -1/2 \) and similarly equations of the second kind. As a special case of these equations Srivastava [155] also gave the solution of triple series equations involving ultra-spherical polynomials.

Lowndes [80] considered triple series equations which are extensions of those considered earlier by Noble [102]. He reduced his solution to Fredholm integral equation of second kind. In another paper Lowndes [31] considered some
triple series involving Laguerre polynomials and obtained the solution in the form of Fredholm integral equations. A closed form solution of triple series which are special cases of those considered earlier by Srivastava [156] was given by Tranter [190].

Parihar [109] gave closed form solutions of certain triple series with trigonometric kernels. Beside this various authors such as Dwivedi [37] Dwivedi and Trivedi ([40], [42]), gave the solution of triple series involving various orthogonal polynomials.

In chapter four certain triple series equations involving Laguerre polynomial have been studied.

1.8 QUADRUPLE SERIES EQUATIONS

Quadruple series relations have so far attracted little attention apart from an important investigation of Cooke [17] on quadruple series involving Fourier-Bessel series. Recently Dwivedi and Trivedi published a series of research papers on quadruple series equations involving orthogonal functions (e.g. see [38], [39], [41]). More recently Dwivedi and Singh [45] have obtained the solution of some quadruple series equations involving Jacobi polynomials which are extensions of that of triple series considered earlier by Srivastava [156].
In chapter four, we have considered some further quadruple series equations involving ultra-spherical polynomials.

APPLICATIONS TO ELASTICITY

Integral and series equations are responsible for many developments in the mixed boundary value problems of elasticity, since their birth.

In the following lines we shall discuss systematically the use of dual and triple integral and series equations and also the corresponding simultaneous equations to the mathematical theory of elasticity during the last one decade.

1.9 DUAL INTEGRAL EQUATIONS

Dual integral equations have been responsible for many investigations in the mathematical theory of elasticity and in particular to crack problems. Sneddon was the first worker who initiated the use of such equations to elasticity. Since then various investigators employed this theory for solving many mixed boundary value problems of elasticity. Recently Srivastav and Narain [179] considered a mixed boundary value problem of elastostatics for an isotropic elastic cylinder containing a strip crack opened by internal pressure. They reduced the above problem first to dual integral equations and then finally to Fredholm integral equations. Expression for crack energy was determined numerically.
Srivastava and Singh [170] considered application of dual integral equations to find the effect of penny-shaped crack on the distribution of stress in a semi-infinite solid. The quantities of physical interest were determined. Further Srivastava and Palaiya [168] considered the distribution of thermal stress in a semi-infinite elastic solid containing a penny-shaped crack. They reduced the problem first to dual integral equations and finally in the form of simultaneous Fredholm integral equations. Expressions for stress intensity factor, crack shape and crack energy were determined. Srivastava and Mahajan [166] considered linear thermoelastic problem for the thermal stress and displacement fields in an elastic solid of infinite extent containing a spherical cavity and an external crack occupying the space outside of a concentric circular region. The crack faces are heated by maintaining them at a certain temperature. The curved surface of the cavity is kept at zero temperature. The problem is reduced to that of solving dual integral equations and then to Fredholm integral equations of second kind which are solved by iterative method. The solutions are used for determining expressions for various quantities of physical interest. Srivastava and Gupta [157] considered the problem of pressurized cruciform crack in a thin circular plate and determined quantities of physical interest.
Watanabe and Atsumi [197] discussed the problem of the infinite parallel row of cracks under the assumptions of plain strain by the use of a suitable form of the Neuber-Papkovich representation. Two types of boundary conditions on the side edges of the strip are considered: (1) the edges are confined between fixed smooth rigid planes, (2) the edges are free of stresses. The problem is reduced to that of solving dual integral equations and then to Fredholm integral equation of second kind. The solution of this equation is obtained by iteration method and numerically by the use of Gaussian quadrature formulas. The effect of the crack array distance and the width of the strip on the stress intensity factor is shown graphically.

Srivastava, Palaiya and Chaudhury [169] gave an analysis of the distribution of thermal stress in an elastic layer bonded to half-spaces along its plane surfaces and contains a penny-shaped crack parallel to the interfaces. The crack is situated in the mid plane of the layer. The thermal and elastic properties of the layer and of the half-spaces are assumed to be different. The problem is first reduced to dual integral equations. These equations are further reduced to Fredholm integral equations of the second kind which are solved iteratively. Expressions for quantities of physical interest are derived.
A problem on the thermal stresses in an infinite homogeneous isotropic medium containing an external crack has been considered by Maiti [33]. By using integral transform techniques and the theory of dual integral equations he obtained the solution for two cases viz. (i) when the surfaces of the crack are subjected to a prescribed temperature and (ii) when the surfaces of the crack are subjected to a prescribed heat-flux.

Using the theory of dual integral equations Dhanwan ([30], [32]) considered the problems of (i) asymmetric distribution of thermal stress in a thick plate containing a penny-shaped crack and (ii) distribution of thermal stress in the vicinity of an external crack in an infinite thick plate respectively.

Fu and Keer [58] considered the application of simultaneous pairs of dual integral equations to discuss the stress distribution in an infinite elastic solid in the presence of coplanar circular cracks under shear loading. Olesiak and Sneddon [106] considered the distribution of surface stress necessary to produce a penny-shaped crack of prescribed shape. By using Hankel transforms and theory of dual integral equations the solution was found out.

Dhaliwal and Rau [29] considered axisymmetric Boussinesq problem for a thick elastic layer under a punch of arbitrary profile and obtained the solution using the theory of dual integral equations.

Srivastava and Dwivedi [159] considered the effect of penny-shaped crack on the distribution of stress in an elastic sphere. Three problems were considered and using the transform methods and theory of dual integral equations each problem was reduced to Fredholm integral equations of second kind whose numerical solution was obtained.

The problem of determining the stress and displacement fields in an elastic half-plane containing an edge crack normal to the free surface when the crack faces are subjected to normal pressure was discussed by Sneddon and Das [146]. The theory of dual integral equations was used to reduce the boundary value problem to simultaneous Fredholm integral equations. Numerical results for physical quantities of interest were also obtained.

Beside the above mentioned authors Erdogan and Gupta [54], Singh [135], Dhaliwal and Singh [28], Srivastava and Gupta [158], Srivastav and Lee [176] and Tweed and Rooke [195] considered various problems of the mathematical theory of elasticity and found their solution by using the theory of dual integral equations.
1.10 SIMULTANEOUS DUAL INTEGRAL EQUATIONS

Simultaneous dual integral equations are also helpful in the study of many crack problems of elasticity. We mention below some references in which such equations have been employed for solution.

Srivastava and Kumar [160] investigated the elastic equilibrium of a semi-infinite two dimensional medium containing a Griffith crack situated parallel to the free boundary. By assuming that the equations of classical theory of elasticity to hold they have considered two problems. In first problem it is assumed that the free edge is stress-free while in the second the free edge is assumed to be rigidly clamped. In the case of an axisymmetric loading the problems are first reduced to a system of simultaneous dual integral equations involving trigonometric kernels and then to simultaneous Fredholm integral equations of the second kind. Finally analytical expressions for stress intensity factor, shape of the crack and the crack energy were derived.

Singh [136] put an analysis, for a class of asymmetric mixed boundary value problems for a semi-infinite solid containing a penny-shaped crack situated parallel to the free boundary. Using an established integral solution form, the problems are reduced to three pairs of simultaneous dual integral equations which are finally reduced to three simultaneous Fredholm
integral equations of second kind. An iterative solution of these integral equations is presented. This solution is used for deriving expressions for stress intensity factor and normal components of displacements.

Lowengrub and Sneddon [89] considered the problem of determining the displacement field in the vicinity of a penny-shaped crack situated at the interface of two half-space of different elastic materials bonded together along their common plane boundary. The deformation in two half-spaces is a result of the application of a symmetrically distributed pressure to the faces of the crack. The representation of the displacement field in the form of Hankel transforms leads to a set of simultaneous dual integral equations for two unknown functions. Finally these equations are transformed in turn to singular integral equation. An explicit formula is obtained for the crack energy in the case in which the applied pressure is constant, it is also indicated how simple integral expressions may be obtained for the components of stress along the interface in this case.

Recently Parihar and Garg [110] presented a model of the Barenblatt crack at the interface of two bonded dissimilar half planes where in the fitness of stresses and smooth closure of opposite sides at the crack tips is insured. Some numerical results are presented graphically.
In the discussion of stress distribution due to a Griffith crack at the interface of an elastic half plane and a rigid foundation Lowengrub [85] used the theory of simultaneous dual integral equations with trigonometric kernels and in another paper Lowengrub and Sneddon [88] used the same theory for finding the stress field near a Griffith crack at the interface of two bonded dissimilar elastic half planes.

1.11 TRIPLE INTEGRAL EQUATIONS

Triple integral equations are helpful in finding the solution of three part mixed boundary value problems of elasticity. Lowengrub and Srivastava ([96], [97]) considered the problem of stress distribution in the presence of a pair of Griffith cracks firstly in an infinite elastic medium and later in an infinitely long strip. The solution of these problems was obtained by reducing them to triple integral equations which were solved by using Hilbert transform technique.

Srivastava and Kumar [161] discussed the problem of determining the distribution of stress in a long isotropic cylinder containing two strip cracks situated symmetrically on a diametral plane. Two types of problems were considered. Firstly the lateral surface of the cylinder is stress free while secondly, the curved surface of the cylinder is supposed to be
rigidly clamped. In both the problems the cracks are opened by a constant internal pressure \( p_0 \). First the problems have been reduced to triple integral equations with cosine kernel. These equations are solved by using finite Hilbert transform technique. Analytical expressions for stress intensity factors, shape of the deformed cracks and the energy required to open the cracks are derived.

In another paper Srivastava and Kumar [162] considered the distribution of stress in an infinite solid containing a cylindrical cavity and two strip cracks and using the same method found the solution of the problem.

Srivastava, Kumar and Nath [164] discussed the thermal stresses in an infinite cylinder containing two strip cracks. Here also they reduced the problem to set of triple integral equations which was solved by known methods. In another paper, by Srivastava and his students (Srivastava and Nath [167], Srivastava, Kumar and Jha [163]) problems of stress distribution were discussed and solution were obtained with the help of triple integral equations.

Dhaliwal [27] considered the problem of determining the stress intensity factors and the crack energy in an infinitely long elastic strip containing two coplanar Griffith cracks. He assumed that the strip is bonded to semi-infinite elastic planes on either side and that the cracks are opened by constant
internal pressure. By the use of Fourier transforms he reduced the problem to solving a set of triple integral equations with cosine kernel and a weight function. These equations are solved using finite Hilbert transform techniques. Analytical expressions up to the order of $\delta^{-10}$ where $2\delta$ denotes the thickness of the strip and $\delta$ is much greater than 1 are derived for the stress intensity factors and crack energy.

Using the theory of triple integral equations, Dhawan [31] considered the distribution of thermal stress in an infinitely long elastic strip containing two coplanar Griffith cracks. The faces of the cracks are heated by maintaining them at certain constant temperature. Quantities of physical interest viz. temperature, displacement and stress in the neighbourhood of cracks were determined.

Shibuya, Nakahara and Kozumi [129] analyzed the axisymmetric distribution of stress in an infinite elastic solid containing a flat annular crack under internal pressure which is a one of three-part mixed boundary value problems. Assuming that the deformation on the crack surface is continuous, the problem is reduced to a set of triple integral equations which have been solved numerically. In other paper [130] they considered the axisymmetric stress distribution in an infinite elastic solid containing a flat annular crack under torsion. First they reduced the problem
to triple integral equations and finally to system of algebraic equations which were solved numerically.

Tweed and Rooker [196] considered the problem of determining the stress intensity factors and crack energy of a radial system of line cracks in an infinite elastic solid reduced to the solution of a singular integral equation via triple integral equations. The equation is solved numerically for special case in which the cracks are opened by a constant pressure.

Tweed and Longmuir [194] considered some crack problems by reducing them to triple integral equations of inverse Mellin type transforms.

1.12 SIMULTANEOUS TRIPLE INTEGRAL EQUATIONS

Simultaneous triple integral equations have also been used recently for the solution of some problem of fracture mechanics. Recently Lowengrub [86] considered the problem of stress distribution for the two bonded dissimilar elastic half-planes containing a pair of coplanar cracks at the interface line. Fourier transforms are employed in order to reduce the problem to that of simultaneous triple integral equations with trigonometric kernels. The solution of these set of equations was obtained by reducing them to singular integral equations and in turn to Riemann boundary value problem whose solution is well known. A thorough discussion of the case of constant internal pressure is also included.
1.13 QUADRUPLE INTEGRAL EQUATIONS

Quadruple integral equations have also been employed for solving mixed boundary value problems. Singh [137] considered the plane strain problem of determining the stress distribution in the vicinity of a Griffith crack in an infinite elastic solid. The crack is opened by two symmetrical rigid inclusions. The geometrical shapes of the inclusions are such as to permit the frictionless contact with the surrounding medium. Depending upon the shapes of the inclusions he obtained quadruple integral equations. Finally various quantities of physical interest are obtained.

1.14. SERIES RELATIONS

Series relations such as dual series, triple series etc. have been proved to be very useful tool for finding solutions to various mixed boundary value problems of elasticity [144]. Recently Parihar and Kushwaha [113] made use of dual series relations to study a crack problem for a strip containing Barenblatt crack. They presented a model for the deformation of a crack under constant pressure with Barenblatt type tensile forces acting in the immediate vicinity of the crack tips. The crack is placed symmetrically in a strip which is confined between two fixed smooth rigid planes. The boundary conditions on the side edges of the strip, in
this case, can be considered as the symmetry condition and model is equivalent to that of an infinite row of collinear Barenblatt cracks in an infinite elastic solid.

In other paper Parihar and Garg [11] considered the problem of determining the stress and displacement field in the vicinity of an infinite row of collinear Griffith cracks, located at the interface of two bonded dissimilar elastic half planes, when these cracks are subjected to internal pressure. By making use of finite Fourier transforms the problem is reduced to a set of simultaneous dual series relations which in turn is shown to be equivalent to a Hilbert problem. In the special case of uniform internal pressure, closed formulae and some diagrams are given for the resulting shape of cracks and for the stresses in their neighbourhood.

More recently Parihar and Garg [112] studied plane strain problem of the bonded dissimilar isotropic elastic strips. It is assumed that the composite strip containing an infinite row of interface cracks located symmetrically on the central line. Here also by using Fourier transform, they derived a set of simultaneous dual series relations. Final solution was obtained by reducing this set to Hilbert problem.
TRIPLE SERIES EQUATIONS

Triple series equations have been used for finding solution of crack problems. Parihar and Kushwaha [114] considered the problem of distribution of stress in an infinite strip containing two Griffith cracks under the action of body forces. By using finite Fourier transforms, they reduced the above problem to triple series equations with trigonometric kernels and completed the problem by finding the closed form solution. More problems of this type can be found in a recent work of Kushwaha [75].

In chapter five, six and seven of the present thesis, we have considered four problems to show some more applications of dual, triple and simultaneous triple integral equations in the mathematical theory of elasticity.