In this chapter we have shown the applications of simultaneous triple integral equations to a mixed boundary value problem of elasticity. Here the considered problem is of determining the stress and displacements fields in the vicinity of a pair of Barenblatt Griffith cracks, located at the interface of two bounded dissimilar elastic half planes, when these cracks are subjected to internal pressure. Fourier transforms are employed in order to reduce the problem to that of solving a simultaneous set of triple integral equations containing a trigonometric kernel. The equivalence between this problem and that of a Riemann boundary value problem with closed form solution is demonstrated.  

7.1 Introduction 

The condition of finiteness of stresses at the end of a crack and smooth joining of the opposite sides of the crack were first proposed in hypothetical form by Khristianovich and proved, on the basis of the principle of virtual displacements, by Barenblatt [5]. Recent examples of such studies are provided by Sneddon [145], Burniston and Gurely [10], Thresher and Smith [182]. In all these investigations however, the crack is embeded in a homogeneous elastic medium.

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In this chapter we present a model of the pair of Coplanar Barenblatt cracks at the interface of two bounded dissimilar elastic half planes. We consider a pair of coplanar Barenblatt cracks $a \leq |x| \leq b$, $y = 0$, located at the interface of two bounded dissimilar half planes. We suppose that the upper half plane $y > 0$ is occupied by a medium with elastic constants, $\mu_1, k_1$ and the lower half plane $y < 0$ is occupied by a medium with elastic constants $\mu_2, k_2$ with $k_i = 3 - 4\nu_i(i=1,2)$ where $\nu_1$ and $\nu_2$ denote Poisson's ratio and rigidity moduli of two respective media.

Following Lowengrub and Sneddon [86], we shall require that

$$u_y(a^+, 0^+) = u_y(a^-, 0^-) = u_y(b^+, 0^+) = u_y(b^-, 0^-) = 0,$$
$$u_y(a^-, 0^+) = u_y(a^-, 0^-) = u_y(b^-, 0^+) = u_y(b^-, 0^-) = 0,$$

where $u_x, u_y$ are components of the displacement vector $\vec{u}$.

In addition, the components of the stresses $\sigma_{yy}$ and $\sigma_{xy}$ must satisfy the condition

$$\sigma_{yy}(x, 0) = o(x^{-1}), \quad \sigma_{xy}(x, 0) = o(x^{-1}) \text{ as } x \to \infty.$$

7.2 Reduction to A System of Integral Equations

We consider the problem of determining the stress field in the vicinity of Barenblatt Griffith cracks, described by a $\leq |x| \leq b$, $y = 0$, at the interface of two half planes, $y > 0$ which is occupied by elastic material with constants $\mu_1, k_1$ and $y < 0$ which is occupied by material with constants $\mu_2, k_2$. 
If we assume that the upper and lower surface of both cracks are subjected to a prescribed pressures \( p(x) \) and \( q(x) \), then inside the crack we have the conditions,

\[
\sigma_{yy}(x,0^+) = \sigma_{yy}(x,0^-) = -p(x), \quad -b \leq x \leq -a, \quad a \leq x \leq b,
\]

\[
\sigma_{xx}(x,0^+) = \sigma_{xy}(x,0^-) = -q(x), \quad -b \leq x \leq -a, \quad a \leq x \leq b,
\]

where \( p(x) \) and \( q(x) \) are the internal pressure and shear applied to the crack faces and that on the region of the interface outside of the cracks we have the continuity conditions:

\[
u_x(x,0^+) = u_x(x,0^-), \quad |x| < a, \quad \text{and} \quad |x| > b, \quad (7.2.3)
\]

\[
u_y(x,0^+) = u_y(x,0^-), \quad |x| < a, \quad \text{and} \quad |x| > b, \quad (7.2.4)
\]

\[
u_{yy}(x,0^+) = \nu_{yy}(x,0^-), \quad |x| < a, \quad \text{and} \quad |x| > b, \quad (7.2.5)
\]

\[
u_{xy}(x,0^+) = \nu_{xy}(x,0^-), \quad |x| < a, \quad \text{and} \quad |x| > b. \quad (7.2.6)
\]

Following Lowengrub and Sneddon [88], we take

\[
u_x(x,y) = \begin{cases}
\int_0^\infty \left[ A(\xi) + 2\xi B(\xi) \right] e^{-\xi y} \sin (\xi x) d\xi, & y > 0,

\int_0^\infty \left[ (k_2 + 2\xi)A(\xi) + (k_1 k_2 - 1 + 2\xi k_1 y)B(\xi) \right] e^{\xi y} \sin (\xi x) d\xi, & y < 0,
\end{cases}
\]

\[
u_y(x,y) = \begin{cases}
\int_0^\infty \left[ A(\xi) + 2(k_1 y)B(\xi) \right] e^{-\xi y} \cos (\xi x) d\xi, & y > 0,

\int_0^\infty \left[ (k_2 - y)A(\xi) + (k_1 k_2 + 1 - 2k_1 y)B(\xi) \right] e^{\xi y} \cos (\xi x) d\xi, & y < 0,
\end{cases}
\]
where \( A \) and \( B \) are arbitrary functions and we have
\[
\mu = \frac{\mu_1}{\mu_2}.
\]
The displacement field (7.2.7)-(7.2.8) satisfies the equations of plane strain and the conditions (7.2.5) and (7.2.6).

The jump in the displacement components on the crack axis may be written
\[
\begin{align*}
u_x(x,0^+ - x,0^-) &= \int_0^\infty S(\xi) \sin (\xi x) d\xi, \\
u_y(x,0^+ - x,0^-) &= \int_0^\infty T(\xi) \cos (\xi x) d\xi,
\end{align*}
\]
where
\[
S(\xi) = (1+\mu k_2^2) A(\xi) + \mu (k_1 k_2 - 1) B(\xi),
\]
\[
T(\xi) = (1+\mu k_2^2) A(\xi) + (2k_1 + \mu k_1 k_2 + \mu) B(\xi).
\]

By using the stress displacement relations and equations (7.2.11) and (7.2.12) we can write the stress components on the bond line in the form
\[
\begin{align*}
\sigma_{yy}(x,0^+ - x,0^-) &= -\mu_1 \left\{(1+\mu k_2^2) (\mu + k_1)\right\}^{-1} \\
\int_0^\infty \left[ a_1 S(\xi) + a_2 T(\xi) \right] \cos (\xi x) d\xi,
\end{align*}
\]
\[
\begin{align*}
\sigma_{xy}(x,0^+ - x,0^-) &= -\mu_1 \left\{(1+\mu k_2^2) (\mu + k_1)\right\}^{-1} \\
\int_0^\infty \left[ a_1 T(\xi) + a_2 S(\xi) \right] \sin (\xi x) d\xi,
\end{align*}
\]
where
\[
a_1 = \mu + k_1^{-1} - \mu k_2, \quad a_2 = \mu + k_1 + \mu k_2.
\]
The above equations (7.2.9)-(7.2.15) together with the mixed boundary conditions (7.2.3), (7.2.4), (7.2.1) and (7.2.2) lead to the simultaneous triple integral equations

\[ \int_{-b}^{a} \left[ a_1 S(\xi) + a_2 T(\xi) \right] \cos(\xi x) d\xi = p(x), \quad \begin{cases} a_x < b, \\ b_x < a, \end{cases} \]  
\( (7.2.16) \)

\[ \int_{0}^{\infty} \left[ a_1 T(\xi) + a_2 S(\xi) \right] \sin(\xi x) d\xi = q(x), \quad \begin{cases} a_x < b, \end{cases} \]  
\( (7.2.17) \)

\[ \int_{0}^{\infty} S(\xi) \sin(\xi x) d\xi = 0, \quad |x| < a, \]  
\( (7.2.18) \)

\[ \int_{0}^{\infty} T(\xi) \cos(\xi x) d\xi = 0, \quad |x| > b, \]  
\( (7.2.19) \)

where

\[ S(\xi) = (1 + \mu k_2) A(\xi) + \mu (k_1 k_2 - 1) B(\xi), \]

\[ T(\xi) = (1 + \mu k_2) A(\xi) + (2k_1 + \mu k_1 k_2 + \mu) B(\xi), \]

\[ p(x) = p(x) \left( 1 + \mu k_2 \right) \frac{(\mu + k_1)}{\mu_1}, \]  
\( (7.2.20) \)

\[ q(x) = q(x) \left( 1 + \mu k_2 \right) \frac{(\mu + k_1)}{\mu_1}. \]  
\( (7.2.21) \)

7.3 Solution of the System of Triple Equations

We shall assume that \( p(x) \) and \( q(x) \) are even functions. It then follows that equations (7.2.16)-(7.2.19) reduce to

\[ \int_{0}^{\infty} S(\xi) \sin(\xi x) d\xi = 0, \quad 0 < x < a, \]  
\( (7.3.1) \)

\[ \int_{0}^{\infty} T(\xi) \cos(\xi x) d\xi = 0, \quad 0 < x < a, \]  
\( (7.3.2) \)
\[
\int_{0}^{\infty} \left[ a_1 S(\xi) + a_2 T(\xi) \right] \cos \xi \, x \, d\xi = P(x), \quad a \leq x \leq b, \quad (7.3.3)
\]
\[
\int_{0}^{\infty} \left[ a_1 T(\xi) + a_2 S(\xi) \right] \sin \xi \, x \, d\xi = Q(x), \quad a \leq x \leq b, \quad (7.3.4)
\]
\[
\int_{0}^{\infty} S(\xi) \sin (\xi x) \, d\xi = 0, \quad x > b, \quad (7.3.5)
\]
\[
\int_{0}^{\infty} T(\xi) \cos (\xi x) \, d\xi = 0, \quad x > b. \quad (7.3.6)
\]

We proceed as in \([88]\), we define
\[
\int_{0}^{\infty} T(\xi) \cos (\xi x) \, d\xi = \begin{cases} r_1(x), & a \leq x \leq b, \\ 0, & 0 < x < a, \quad x > b, \end{cases} \quad (7.3.7)
\]
\[
\int_{0}^{\infty} S(\xi) \sin (\xi x) \, d\xi = \begin{cases} s_1(x), & a \leq x \leq b, \\ 0, & 0 < x < a, \quad x > b. \end{cases} \quad (7.3.8)
\]

It is easily shown that if we make extensions \( r(u) \) and \( s(u) \) of \( r_1(u) \) and \( s_1(u) \) to \( -b \leq x \leq -a \) as follows
\[
r(u) = \begin{cases} r_1(u), & a \leq u \leq b, \\ r_1(-u), & -b \leq u \leq -a, \end{cases}
\]
\[
s(u) = \begin{cases} s_1(u), & a \leq u \leq b, \\ -s_1(-u), & -b \leq u \leq -a. \end{cases}
\]
then
\[
\int_{0}^{\infty} T(\xi) \sin (\xi x) \, d\xi = \frac{1}{n} \int_{L} \frac{r(w) \, dw}{w-x}, \quad (7.3.9)
\]
and \[ \int_0^\infty S(\xi) \cos (\xi x) d\xi = -\frac{1}{\pi} \int_L \frac{s(u) du}{x-u}, \] (7.3.10)

where

\[ L = \left[ (-b, -a) \cup (a, b) \right]. \]

In like manner it is a simple matter to verify that

\[ \int_0^\infty S(\xi) \cos (\xi x) d\xi = \begin{cases} S_1(x), & a \leq x \leq b, \\ 0, & 0 < x < a, \ x > b, \end{cases} \] (7.3.11)

where

\[ S_1(x) = \int_x^b s(u) du, \]

and

\[ \int_0^\infty T(\xi) \cos (\xi x) d\xi = \frac{1}{\pi} \int_a^b \int_L \frac{r(u) du}{u-w}. \] (7.3.12)

If we substitute (7.3.7), (7.3.8), (7.3.9) and (7.3.10) in (7.3.3) and (7.3.4), we see that \( r \) and \( s \) must be solutions to the set of singular integral equations,

\[ a_1 r(x) + a_1 \int_L \frac{s(u) du}{x-u} = P(x), \ a \leq |x| \leq b, \] (7.3.13)

\[ a_1 s(x) + a_1 \int_L \frac{r(u) du}{x-u} = Q(x), \ a \leq |x| \leq b, \] (7.3.14)

where we recall that \( P(x) \) and \( Q(x) \) are even functions defined on \( L \).

The substitution

\[ \lambda(u) = s(u) + ir(u), \] (7.3.15)
reduces the pair of equations (7.3.13) and (7.3.14) to the single integral equation,

\[ a_2(x) + \frac{a_1}{\pi i} \int_{\mathbb{L}} \frac{\Lambda(u) du}{x-u} = \Omega(x), \quad x \in \mathbb{L}, \quad (7.3.16) \]

where

\[ \mathbb{L} = [-b, -a] \cup [a, b], \]
\[ \Omega(x) = \text{iP}(x) + \text{Q}(x). \]

If we now define

\[ \Lambda(z) = \frac{1}{2\pi i} \int_{\mathbb{L}} \frac{\Lambda(u) du}{u-z}, \quad (7.3.17) \]

then the Plemelj formulae,

\[ \Lambda^+(x) - \Lambda^-(x) = \lambda(x), \quad \Lambda^+(x) + \Lambda^-(x) = \frac{1}{\pi i} \int_{\mathbb{L}} \frac{\Lambda(u) du}{u-x}, \]

show that (7.3.16) is equivalent to the condition that

\[ \Lambda^+(x) = -m \Lambda^-(x) - (a_1-a_2)^{-1} \Omega(x), \quad x \in \mathbb{L}, \quad (7.3.18) \]

where

\[ m = \frac{a_1 + a_2}{a_1 - a_2} > 0. \]

Thus, we must find a sectionally holomorphic function \( \Lambda(z) \), vanishing at infinity and satisfying the condition (7.3.18). The solution to this problem is well known \([97, p. 952]\) and is given by

\[ \Lambda(z) = \frac{\text{-i}(z)}{2\pi i(a_1-a_2)} \left[ \int_{\mathbb{L}} \frac{f(t) dt}{x^+(t)(t-z)} \right] \left[ c_1z + c_2 \right] X(z), \quad (7.3.19) \]
where $c_1$ and $c_2$ are complex constants and $X(z)$ is the solution to the homogeneous Riemann problem,

$$X^+(t) = -m X^-(t), \ t \in L.$$  \hfill (7.3.20)

The homogeneous Riemann problem is known to have a solution $[97, p. 450]$ given by

$$X(z) = \left[ (z-a)(z+b) \right]^{i\beta - \frac{1}{2}} \left[ (z+a)(z-b) \right]^{-i\beta - \frac{1}{2}},$$ \hfill (7.3.21)

where

$$\beta = (2\pi)^{-1} \log m.$$ \hfill (7.3.22)

In the case in which $\Omega$ is a polynomial,

$$\int \frac{\Omega(t)dt}{L X^+(t)(t-z)} = \frac{2\pi i}{L+m} \left[ \frac{\Omega(z)}{X(z)} - L(z) \right],$$

where

$$L(z) = \lim_{R \to \infty} \frac{2\pi}{2\pi} \int_0^{2\pi} \frac{(\text{Re} e^{i\theta})\text{Re} e^{i\theta} \text{d}\theta}{X(\text{Re} e^{i\theta})(\text{Re} e^{i\theta} - z)},$$ \hfill (7.3.23)

and

$$\Lambda(z) = \frac{1}{2a_1} \left[ \Omega(z) - X(z)\Lambda(z) \right] + (c_1 z + c_2) X(z).$$ \hfill (7.3.24)

It follows from (7.2.13), (7.3.6) and (7.3.10) that for $0 < x < a$ and $x > b$,

$$\sigma_{yy}(x,0^+) = \sigma_{yy}(x,0^-) = -2a_3 \Im \Lambda^+(z),$$ \hfill (7.3.25)

and from (7.2.14)

$$\sigma_{xy}(x,0^+) = \sigma_{xy}(x,0^-) = 2a_3 \Re \Lambda^+(z),$$ \hfill (7.3.26)
where

\[ a_3 = \frac{(\mu + k_1 - 1 - \mu k_2) \mu_1}{(1 + \mu k_2)(\mu + k_1)} \]

7.4 The Case of Constant Internal Pressure

We now consider, as a specific example the case in which the cracks are opened by constant normal and shearing pressure say \( p(x) = q(x) = p_0 \) so that,

\[ \Omega(x) = p_0 | \mu_1 (1 + \mu k_2)(\mu + k_1) \{1+i\} = \Omega(0). \]

Thus, if, \( Y = -i\beta + \frac{1}{\varepsilon} \), then

\[
\int_0^{2\pi} \frac{Re^i\theta}{x(Re^i\theta)(Re^i\theta - z)} \Omega(0) \{ z^2 + 2Ybz - 2Yaz - bz + az - (Y-1) \left( \frac{5b^2 + 4a^2}{2} \right) + (4Y^2 + 4Y - 1)0(R^{-1}) \}
\]

So that

\[ L(z) = \Omega(0) \left[ z^2 + (2Y-1)(b-a)z - Y(Y-1)(\frac{5b^2 + 4a^2}{2}) + \ldots \right]. \quad (7.4.1) \]

It follows from (7.3.24) that

\[ \Lambda(z) = \frac{i\Omega(0)}{2a_1} \left[ 1 - \{ z^2 + c_1 z + c_2 X(z) \} \right], \quad (7.4.2) \]

where \( X(z) \) is defined by (7.3.21) and \( c_1, c_2 \) are arbitrary complex constants.

We obtain from relation (7.3.21), the expressions for \( X^+ \) and \( X^- \) as follows:
\[ X^+(x) = -\text{im}^{1/2} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left[ \cos \beta + i \sin \beta \right], \]
\[ X^-(x) = \text{im}^{-1/2} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left[ \cos \beta + i \sin \beta \right], \]

where
\[ \theta = \log \left( \frac{x - a}{x + a} \right) \left( \frac{b + x}{b - x} \right), \]

while for \(-b < x < -a,\)
\[ X^+(x) = \text{im}^{1/2} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left[ \cos \beta + i \sin \beta \right], \]
\[ X^-(x) = \text{im}^{-1/2} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left[ \cos \beta + i \sin \beta \right]. \]

(ii) for \(0 < x < a,\)
\[ X^+(x) = X^-(x) = \left[ (a^2 - x^2)(b^2 - x^2) \right]^{-1/2} \left[ \cos \beta_1 + i \sin \beta_1 \right], \]

where
\[ \theta_1 = \log \left( \frac{x - a}{a + x} \right) \left( \frac{b + x}{b - x} \right) \]

(iii) for \(x > b,\)
\[ X^+(x) = X^-(x) = \left[ (x^2 - a^2)(x^2 - b^2) \right]^{-1/2} \left[ \cos \beta_2 + i \sin \beta_2 \right], \]

where
\[ \theta_2 = \log \left( \frac{x - a}{x + a} \right) \left( \frac{x + b}{x - b} \right) \]

Hence for \(a < x < b,\) we find that if we let \(c_1 = c_1^1 + ic_1^2, c_2 = c_2^1 + ic_2^2\) where \(c_i^j, i, j = 1, 2\) are real arbitrary constants, then,
\[
\Lambda^+(x) - \Lambda^-(x) = \frac{\Omega(0)}{\sqrt{a_1^2 - a_2^2}} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left\{ (x + c_1 x + c_2) \cos \beta \theta \right.
\]

\[-(c_1^2 x + c_2^2) \sin \beta \theta \} + i \left[ \frac{\Omega(0)}{a_1} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left[ (x + c_1 x + c_2) \cos \beta \theta \right.
\]

\[+ (x^2 + c_1^2 x + c_2^2) \sin \beta \theta \} \right]\]

\[\text{(7.4.7)}\]

while

\[
\Lambda^+(x) + \Lambda^-(x) = \frac{\Omega(0) a_2}{a_1 \sqrt{a_1^2 - a_2^2}} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left[ (x + c_1 x + c_2) \cos \beta \theta \right.
\]

\[-(c_1^2 x + c_2^2) \sin \beta \theta \} + i \left[ \frac{\Omega(0) a_2}{a_1 \sqrt{a_1^2 - a_2^2}} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left[ (x + c_1 x + c_2) \sin \beta \theta \right.
\]

\[+ (x^2 + c_1^2 x + c_2^2) \cos \beta \theta \} \right]\]

\[\text{(7.4.8)}\]

The Plemelj relations then yield on \( a < x < b \),

\[
s_1(x) = \frac{-\Omega(0)}{\sqrt{a_1^2 - a_2^2}} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left[ (x + c_1 x + c_2) \cos \beta \theta \right.
\]

\[-(c_1^2 x + c_2^2) \sin \beta \theta \}, \text{(7.4.9)}\]

\[
r_1(x) = \frac{-\Omega(0)}{\sqrt{a_1^2 - a_2^2}} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left[ (c_1^2 x + c_2^2) \cos \beta \theta \right.
\]

\[+ (x^2 + c_1^2 x + c_2^2) \sin \beta \theta \} \right]. \text{(7.4.10)}\]

We may also note that on \( -b < x < -a \),

\[
s(x) = \frac{-\Omega(0)}{\sqrt{a_1^2 + a_2^2}} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{-1/2} \left[ (x^2 + c_1 x + c_2) \cos \beta \theta \right.
\]

\[-(c_1^2 x + c_2^2) \sin \beta \theta \} \right],
\[ r(x) = \frac{\eta(0)}{\sqrt{(a_1 - a_2)(b_2 - x^2)}} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{1/2} \left[ (x^2 + c_1^2 x + c_2^2) \sin \beta \theta + (c_1^2 x + c_2^2) \cos \beta \theta \right]. \]

Hence, in order that the relation \( s(x) = -s_1(-x) \) and \( r(x) = r_1(-x) \) on \(-b < x < -a\) be satisfied, we must choose \( c_1^2 \) and \( c_2^2 \) so that \( c_1^2 = c_2^2 = 0 \).

Another use of the Plemelj relations yields

\[
\frac{1}{\pi} \int \frac{r(u)du}{u-x} = \frac{\eta_0 a_2^2}{\eta_0 a_1 \sqrt{(a_1 - a_2)}} \left[ (x^2 - a^2)(b^2 - x^2) \right]^{1/2} \left[ (x^2 + c_2^2) \cos \beta \theta \right.
\left. - c_1^2 x \sin \beta \theta \right], \tag{7.4.11}
\]

\[
\frac{1}{\pi} \int \frac{s(u)du}{u-x} = \frac{\eta_0 a_2^2}{\eta_0 a_1} \left[ 1 + \frac{a_2}{\sqrt{(a_1 - a_2)}} \left[ (x^2 - a^2)(x^2 - b^2) \right]^{1/2} \left[ (x^2 + c_2^2) \sin \beta \theta \right.
\left. + c_1^2 x \cos \beta \theta \right] \right]. \tag{7.4.12}
\]

We can deduce from (7.4.11) and (7.4.9) that

\[
\frac{1}{\pi} \int \frac{r(u)du}{u-x} = \frac{a_2}{a_1} s_1(x), \quad a < x < b. \tag{7.4.13}
\]

It is a simple matter to show that for \( x > b \)

\[ A^+(x) = \frac{\eta_0 a_1}{2a_1} - \frac{\eta_0 a_2^2}{2a_1} \left[ (x^2 - a^2)(x^2 - b^2) \right]^{1/2} \left[ (x^2 + c_2^2) \cos \beta \theta_2 - c_1^2 x \sin \beta \theta_2 \right.
\left. - (x^2 + c_2^2) \sin \beta \theta_2 + c_1^2 x \cos \beta \theta_2 \right], \tag{7.4.14}
\]

and hence, for \( x > b \)

\[
\sigma_{yy}(x) = -\rho(1+i) \left[ 1 - \left[ (x^2 - a^2)(x^2 - b^2) \right]^{1/2} \left[ (x^2 + c_2^2) \cos \beta \theta_2 - c_1^2 x \sin \beta \theta_2 \right] \right] \tag{7.4.15}
\]
\[
\sigma_{xy}(x,0^+) = p_0(1+i) \left[ (x^2-a^2)(x^2-b^2)^{-1/2} (x^2+c_2^1) \sin \beta_2 + c_2^1 x \cos \beta_2 \right], \quad (7.4.16)
\]

where

\[
\beta_2 = \log \left( \frac{x-a}{x+b} \right) .
\]

We see from (7.4.15) and (7.4.16) that as \( x \to \infty \),

\[
\sigma_{yy}(x,0^+) = p_0(1+i) 0(x^{-1}),
\]

and

\[
\sigma_{xy}(x,0^+) = p_0(1+i) \left\{ 2\beta(b-1) + c_1^2 \right\} x^{-1} + o(x^{-1}).
\]

Hence, it follows that the condition \( \sigma_{yy}(x,0^+) = 0(x^{-1}) \) as \( x \to \infty \) is automatically satisfied while that the remaining \( \sigma_{xy}(x,0^+) = 0(x^{-1}) \) as \( x \to \infty \) will only be satisfied if we choose \( c_2^1 = -2\beta(b-a) \). Thus it only remains to determine the constant \( c_2^1 \). This is determined from the condition that at the end point \( x = a \), \( u_y(a,0^+) = 0 = u_y(a,0^-) \). From (7.2.8), (7.2.11), (7.2.12), (7.3.11), (7.3.12), (7.4.9) and (7.4.13), we see that

\[
u_y(x,0^+) = \frac{S_1(x) C}{a_1(k_1^1+u)^2(1+\mu k_2)}, \quad a < x < b, \quad (7.4.17)
\]

\[
u_y(x,0^-) = \frac{D S_1(x)}{2a_1(k_1^1+u)^2(1+\mu k_2)}, \quad a < x < b, \quad (7.4.18)
\]

where
\[ c = \left[ -\mu k_1 - 2\mu - 2\mu^2 k_2^2 - 2\mu k_1 k_2 - 2\mu k_2 - 2k_1 - 2k_2 \right], \]
\[ d = \left[ (\mu + k_1 - 1) (\mu k_2) (-2\mu k_1 k_2 - 2\mu - 2k_1 k_2 + k_1 k_2 + 1) \right. \]
\[ + (\mu + k_1 + 1) (\mu k_1 k_2 - \mu + k_1 k_2 + 1) \].

Therefore, we have the condition
\[ S_1(a) = \int_a^b s_1(u) \, du = 0. \]

This gives
\[ C_2 = \frac{-L_2 + 2\beta (b-a) I_1}{L_0}, \quad (7.4.19) \]

where
\[ L_2 = \int_a^b \frac{u^2 \cos \beta \theta \, du}{\sqrt{(u^2-a^2)(b^2-u^2)}}, \]
\[ L_1 = \int_a^b \frac{u \sin \beta \theta \, du}{\sqrt{(u^2-a^2)(b^2-u^2)}}, \]
\[ L_0 = \int_a^b \frac{\cos \beta \theta \, du}{\sqrt{(u^2-a^2)(b^2-u^2)}}. \]

These integrals are obtained by separating the real and imaginary parts of the following integrals, which have been evaluated by making the substitution \( u = a \cos^2 \phi + b \sin^2 \phi \) and using binomial expansion.
\[
\int_{a}^{b} \frac{e^{i\beta \theta}}{\sqrt{(u^2 - a^2)(u^2 - b^2)}} \, du = \int_{a}^{b} \frac{(u-a)^{-1/2 + i\beta}(u+a)^{-1/2 - i\beta}}{\sqrt{(u^2 - a^2)(u^2 - b^2)}} \, du
\]

\[
(b-u)^{-1/2 - i\beta}(b+u)^{-1/2 + i\beta} \, du = \frac{\pi}{(a+b)\cos \beta} \, F_3\left(\frac{1}{2} + i\beta, \frac{1}{2} - i\beta, \frac{1}{2}; 1; z, -z\right),
\]

(7.4.20)

\[
\int_{a}^{b} \frac{ue^{i\beta \theta}}{\sqrt{(u^2 - a^2)(u^2 - b^2)}} \, du = z \, \Gamma\left(\frac{1}{2}, -i\beta\right) \, \Gamma\left(\frac{3}{2} + i\beta\right) \, F_3\left(\frac{1}{2} + i\beta, \frac{1}{2} - i\beta, 1; \frac{1}{2} + i\beta; 1; z, -z\right),
\]

(7.4.21)

\[
\int_{a}^{b} \frac{u^2 e^{i\beta \theta}}{\sqrt{(u^2 - a^2)(u^2 - b^2)}} \, du = \frac{\pi a^2}{(a+b)\cos \beta} \, F_3\left(\frac{1}{2} + i\beta, \frac{1}{2} - i\beta, \frac{1}{2}; 1; z, -z\right)
\]

+ \left(\frac{b-a}{a+b}\right) \, \Gamma\left(\frac{1}{2}, -i\beta\right) \, \Gamma\left(\frac{3}{2} + i\beta\right) \, F_3\left(\frac{1}{2} + i\beta, \frac{1}{2} - i\beta, 1; \frac{3}{2} + i\beta; 2; z, -z\right)
\]

+ \left(\frac{b-a}{b}\right) \, \Gamma\left(\frac{3}{2} - i\beta\right) \, \Gamma\left(\frac{3}{2} + i\beta\right) \, F_3\left(\frac{1}{2} + i\beta, \frac{1}{2} - i\beta, \frac{3}{2} - i\beta, \frac{3}{2} + i\beta; 3; z, -z\right),
\]

(7.4.22)

where \( z = (b-a)(b+a) \) and \( F_3 \) is hypergeometric function of two variables defined in [27, p. 27]. By separating into real and imaginary parts, we get
\[ L_0 = \frac{\pi}{(a+b) \cosh \frac{\pi b}{a}} \, _2F_1 \left( \frac{1}{2} + i\beta, \frac{1}{2} - i\beta; 1; \frac{x^2}{b^2} \right), \]  
(7.4.23)

\[ L_1 = \frac{\pi^2 z}{\cosh \frac{\pi b}{a}} \, _2F_1 \left( \frac{1}{2} + i\beta, \frac{1}{2} - i\beta, \frac{1}{2} + i\beta; \frac{x^2}{b^2} \right), \]  
(7.4.24)

\[ L_2 = \frac{\pi^2}{(a+b) \cosh \frac{\pi b}{a}} \, _2F_1 \left( \frac{1}{2} - i\beta, \frac{1}{2} + i\beta, \frac{1}{2} - i\beta, \frac{x^2}{b^2} \right) + \frac{\pi(b-a)}{2 \cosh \frac{\pi b}{a}} \, _2F_1 \left( \frac{1}{2} - i\beta, \frac{1}{2} + i\beta, \frac{1}{2} - i\beta, \frac{x^2}{b^2} \right) \]

\[ - \frac{\pi(b-a)^2}{4(b+a) \cosh \frac{\pi b}{a}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(1-i\beta)(1+i\beta)_q(1+i\beta)_p}{3! q! (3)_p q} \times (2q+1)(q-p)(-z)^q. \]  
(7.4.25)

In [88] Lowengrub and Sneddon has mentioned that the constant \( c_2^1 \) in (7.4.19) has to be calculated numerically. We shall, however, like to point out here that by virtue of the results (7.4.23-7.4.25), \( c_2^1 \) can be expressed in closed form.

We find that for \( x > b \) stresses are given by

\[ \sigma_{yy}(x,0^+) = \sigma_{yy}(x,0^-) = -p_0 (1+i) \left[ (x^2 - a^2)(x^2 - b^2)^{-1/2} \right. \]

\[ \left. (x^2 + c_2) \cos \beta \theta_2 + 2\beta (b-a)x \sin \beta \theta_2 \right], \]  
(7.4.26)

\[ \sigma_{xy}(x,0^+) = \sigma_{xy}(x,0^-) = p_0 (1+i) \left[ (x^2 - a^2)(x^2 - b^2)^{-1/2} \right. \]

\[ \left. (x^2 + c_2) \sin \beta \theta_2 - 2\beta (b-a) \cos \beta \theta_2 \right]. \]  
(7.4.27)

The stresses for \( 0 < x < a \) are
\[ \sigma_{yy}(x,0^+) = \sigma_{yy}(x,0^-) = -p_c (1+i) \frac{1}{\sqrt{(a^2-x^2)(b^2-x^2)}} \{ (x_2^2 + c_2^1) \cos \beta_{\theta_1} + 2 \beta (b-a) \sin \beta_{\theta_1} \}, \] (7.4.28)

\[ \sigma_{xy}(x,0^+) = \sigma_{xy}(x,0^-) = \frac{p_c (1+i)}{\sqrt{(a^2-x^2)(b^2-x^2)}} \left[ (x_2^2 + c_2^1) \sin \beta_{\theta_1} \right. \left. - 2 \beta (b-a) x \cos \beta_{\theta_2} \right], \]

where

\[ \theta_1 = \log \left( \frac{a-x}{a+x} \right) \left/ \left( \frac{b-x}{b+x} \right) \right. \]

and \( c_2^1 \) is obtained from (7.4.19).

### 7.5 The Stress Intensity Factors

If we define the normal and shear stress intensity factors \( N_{1b} \) and \( N_{2b} \) at the edge \( x = b \) by the relation

\[ N_{1b} = \lim_{x \to b} \left[ (x-b)^{1/2} \sigma_{yy}(x,0^+) \right], \] (7.5.1)

\[ N_{2b} = \lim_{x \to b} \left[ (x-b)^{1/2} \sigma_{xy}(x,0^+) \right], \] (7.5.2)

then from (7.4.26) and (7.4.27), we get

\[ N_{1b} = \frac{-p_c (1+i)}{2b(b^2-a^2)^{1/2}} \lim_{x \to b} \left\{ (x_2^2 + c_2^1) \cos \beta_{\theta_2} + 2 \beta (b-a) x \sin \beta_{\theta_2} \right\}, \]

\[ N_{2b} = \frac{p_c (1+i)}{2b(b^2-a^2)^{1/2}} \lim_{x \to b} \left\{ (x_2^2 + c_2^1) \sin \beta_{\theta_2} - 2 \beta (b-a) x \cos \beta_{\theta_2} \right\}. \]

From this we get
\[ N_{1b} + iN_{2b} = \frac{P_0^{(1+i)}}{2b(b^2 - a^2)^{1/2}} \ln \{(x^2 + c_2^1 + 2ib(b-a)x)e^{-2\theta_2}, \] (7.5.3)

and

\[ N_{1b}^2 + N_{2b}^2 = \frac{P_0^2 (1+i)^2}{2b(b^2 - a^2)} \{ (x^2 + c_2^1 + 4b^2 (b-a)^2) \}. \] (7.5.4)

Similarly at \( x=a \) the stress intensity factors can be shown to be

\[ N_{1a}^2 + N_{2a}^2 = \frac{P_0^2 (1+i)^2}{2b(b^2 - a^2)} \{ (x^2 + c_2^1 + 4b^2 (b-a)^2) \}, \] (7.5.5)

where \( \theta_2 = \log \{(x-a)(x+b)/(x+a)(x-b)\} \),

and \( c_2^1 \) is given by (7.4.19).