CHAPTER VII

RHEOLOGICAL STUDY OF FLOW AND DIFFUSION IN CAPILLARIES WITH MILD STENOSIS

INTRODUCTION

An understanding of the role of blood flow dynamics, i.e., hemodynamics, in atherogenesis has been sought for years. The possible importance of fluid mechanics in atherosclerosis has produced a need for a more detailed knowledge of the properties of arterial blood flow. The abdominal aorta, the carotid arteries, the coronary arteries, and the peripheral arteries such as the iliacs and the femorals are favored sites for the development of atherosclerosis. Furthermore, it appears that it is regions of arterial branching and sharp curvature which have the greatest predilection for the disease. In the understanding of fluid mechanics as a factor in atherogenesis, it is not enough to identify the general regions of branching and sharp curvature. The flow patterns in such regions are so complex that any indictment of fluid mechanics as a disease factor which is based on the general pattern of the disease only raises additional questions relating to the detailed nature of both the flow and the disease pattern.
In addition to the questions emanating from our lack of knowledge about the detailed pattern. Of the disease, there is a different, but related and equally important set of questions which arise out of our limited understanding of the detailed fluid mechanics. Blood flow in the larger arteries cannot be described as laminar with a parabolic velocity profile, but instead is characterized by a variety of complexities. These include the presence of asymmetric velocity patterns due to vessel geometrics, turbulent or at least highly transitional flow, secondary fluid motions, and flow separation.

Specific sites of the arterial tree have an apparent predilection for the development of atherosclerotic lesions, and this has suggested that hemodynamic factor may play a role in atherogenesis [Caro (1973), Fry (1973), and Bergel et al. (1976)]. Depending upon the theory being investigated, both high [Fry (1973)] and low [Caro (1973)] blood flow shear rates have been suspected of enhancing the atherosclerotic process. These hemodynamic effects are thought to be mediated through either a direct mechanical injury to the wall or by negatively affecting the exchange of material between the wall and the luminal blood. There have been a considerable number of theoretical analyses of the possible blood flow patterns which may occur of these sites [Forrester and Young (1970), Lee and Fung (1970), Young and Tsai (1973), Morgan
and Young (1974), Deshpande et.al (1976), Fernandez et.al (1976), Kandarpa and Davis (1976)]. Many authors [Caro (1981), Dintenfass (1977), and Fry (1968)] have reported that the rheologic and fluid dynamic properties of blood and its flow behavior through non-uniform cross-section of the tube (stenosed tube, tapered tube etc.) could play an important role in the fundamental understanding diagnosis and treatment of many cardiovascular diseases. In view of the importance of the fluid dynamic factors in the understanding of blood flow, in the present chapter, a modest effort has been made to give a generalized model of blood flow and obtain some new information about the flow.

The steady flow of blood through a stenosed tube has been investigated by many researchers [Deshpande et.al (1976), Macdonald (1979), Shukla et.al (1980), Young (1968)]. Deshpande et.al (1976), Macdonald (1979) and Young (1968) have represented blood by a Newtonian fluid and Shukla et.al (1980) have represented blood by a non-Newtonian fluid. Ehrlich (1979) has analysed the hemodynamics of blood through an axially symmetric tube with stenosis or diameter reduction. The applicability of steady flow problem related to arterial blood. Flow is questioned since the actual blood flow is distinctly pulsatile with the same frequency as the heart beat.
Cassanova and Giddens (1978) and Young and Tsai (1973) have experimentally studied the effects of the pulsatility on the flow through a stenosed tube. In these investigations blood has been represented by a Newtonian fluid. The Newtonian behavior of blood is acceptable for flows through wide arteries, but for low shear rate \((0.1 \text{ sec}^{-1})\) flows in narrow tubes, in particular, in diseased states (e.g., patients with severe myocardial infarction, hypertension and clotting effects in small tubes), blood may be represented by a non-Newtonian fluid [Iida and Murata (1980)]. It is therefore, of interest to consider simultaneously the effect of stenosis and non-Newtonian behavior of blood on its flow.

MATHEMATICAL ANALYSIS

Consider an axially symmetric, and fully developed flow of blood (assumed to be incompressible) in the z-direction through an artery with mild stenosis as shown in Figure 7.1. We shall take cylindrical coordinate system \((r,z)\) whose origin is located on the vessel (stenosed artery) axis. It can be shown that the radial velocity is negligibly small in its magnitude and may be neglected for a low mean Reynolds number flow problem with mild stenosis. With the above considerations, momentum equations are

\[
0 = -\frac{\partial P}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} [r \cdot \tau] \tag{7.1}
\]
Fig. 7.1 Geometry of the artery with stenosis.
\[ 0 = \frac{dP}{dr} \]  

(7.2)

where for a Casson fluid (blood) \( \tau \) is given by

\[ \tau^{1/2} = \tau_0^{1/2} + \sqrt{\mu} \left( \frac{dU}{d\tau} \right)^{1/2} \quad \text{if} \quad \tau \geq \tau_0 \]  

(7.3)

\[ \frac{dU}{d\tau} = 0 \quad ; \quad \tau \leq \tau_0 \]  

(7.4)

where

\[ \tau = - \frac{r}{2} \frac{dP}{dz} \quad \text{and} \quad \tau_0 = - \frac{R_C}{2} \frac{dP}{dz} \]  

(7.5)

where \( U \) is the axial component of velocity, \( P \) is the pressure, \( \mu \) the Casson's blood viscosity, \( \tau \) the shear stress and \( \tau_0 \) is the yield stress.

The geometry of the stenosis is given by \( R(z) = R_0 \) in the normal artery region and for \( d \leq z \leq d + L_0 \)

\[ R(z) = R_0 - \frac{\delta_s}{2} \left[ 1 + \cos \frac{2\pi}{L_0} \left( z-d-\frac{L_0}{2} \right) \right] \]  

(7.6)

where \( R(z) \) is the radius of the artery in the stenotic region, \( R_0 \) is the radius of the normal artery, \( L_0 \) is the length of the stenosis, \( d \) indicates its location and \( \delta_s \) is the maximum height of the stenosis such that \( \delta_s/R \ll 1 \).
The concentration equation for the solute is expressed as follows,

\[
\frac{\partial C_1}{\partial t} + U \frac{\partial C_1}{\partial z} = D_1 \left( \frac{\partial^2 C_1}{\partial r^2} + \frac{1}{r} \frac{\partial C_1}{\partial r} \right) \tag{7.7}
\]

where, \( C_1 \) represents the concentration of the solute, \( U \) is the axial velocity and \( D_1 \) the diffusion coefficient for the solute under consideration in the blood.

To solve the above system of equation, the following boundary conditions are prescribed.

\[
\begin{align*}
\delta U \bigg|_{r=0} &= 0 \\
U &= 0 \quad \text{at} \quad r = R_c \\
P &= P_0 \quad \text{at} \quad z = 0 \\
P &= P_L \quad \text{at} \quad z = L \\
\delta C_1 \bigg|_{r=0} &= 0 \\
D_1 \delta C_1 \bigg|_{r=R} &= VNC_1
\end{align*}
\tag{7.8}
\]

Solving for \( \frac{dU}{dr} \) from equation (7.3) we have

\[
\frac{dU}{dr} = \frac{1}{\mu} \left[ - \frac{r}{2} \frac{dP}{dz} - \sqrt{\frac{r}{T_o}} \right] \tag{7.9}
\]

Integrating this equation and using the boundary condition (7.7), we have
\[
\frac{R}{r} \frac{dU}{dr} \cdot dr = U|_R - U|_r
\]
\[
= \frac{1}{\mu} \int_{r}^{R} \left( \frac{r}{2} \frac{dP}{dz} - \sqrt{\tau_o} \right)^2 \, dr
\]
or
\[
U = \frac{1}{4\mu} \frac{dP}{dz} \left[ R^2 - r^2 - \frac{8}{3} \mu \frac{1}{2} \left( R^{3/2} - r^{3/2} \right) + 2R_c(R-r) \right] \quad \text{for } R_C \leq r \leq R
\]
(7.10)

We can now find the rate of volume flow by an integration,
\[
Q = 2\pi \int_{0}^{R} U \, r \, dr
\]
(7.11)
we obtain
\[
Q = \frac{\pi R^4}{8\mu} \left[ \frac{dP}{dz} - \frac{16}{7} \left( \frac{2 \tau_o}{R} \right)^{1/2} \left( \frac{dP}{dz} \right)^{1/2} \right.
\]
\[+ \left. \frac{2}{3} \left( \frac{\tau_o}{R} \right)^{3/2} \right] ; \quad (7.12)
\]
if \( \frac{dP}{dz} > \frac{2 \tau_o}{R} \)

whereas \( Q = 0 \); if \( \frac{dP}{dz} < \frac{2 \tau_o}{R} \)
(7.13)

If we introduce the notation
\[
\bar{\xi} = \left( \frac{2 \tau_o}{R} \right) \left( \frac{dP}{dz} \right)^{-1} \quad (7.14)
\]
Then the equation (6.11) can be written as

\[ Q = \frac{\pi R^4}{8\mu} \cdot \frac{dP}{dz} \cdot F(\bar{\xi}) \quad (7.15) \]

where

\[ F(\bar{\xi}) = 1 - \frac{16}{7} \bar{\xi}^{1/2} + \frac{4}{3} \bar{\xi} - \frac{1}{2} \bar{\xi}^4 \quad (7.16) \]

The expression for pressure gradient can be written, following Bird et al. (1960) and Whitmore (1968), the volumetric flow rate \( Q \), can be also be written in the form of a Robinowitsch equation as

\[ Q = \frac{\pi R^3}{3\tau_R} \int_0^{\tau_R} \tau^2 f(\tau) d\tau \quad (7.17) \]

where \( f(\tau) = (- \frac{du}{d\tau}) \) and \( \tau_R = -\frac{R}{2} \frac{dP}{dz} \quad (7.18) \]

hence

\[ \frac{dP}{dz} = -\frac{2R}{\tau_R} \left[ \frac{8}{7} \sqrt{\tau_o} + 2\mu \sqrt{\frac{Q}{\pi R^3}} \right]^2 \quad (7.19) \]

for small \( \tau_o / \tau_R \ll 1 \).

where \( Q \) is independent of \( z \).

Integrating equation (7.18) and using the conditions (7.7), we get
The resistance to flow, $\lambda$, is defined as follows [Burton (1968), Young (1968)]:

$$\lambda = \frac{P_o - P_L}{Q}$$ (7.21)

which gives,

$$\lambda = \frac{2}{R_o \pi} \left[ \int_0^L \frac{d}{dz} \left[ \frac{2}{7} \sqrt{\tau_o} + 2\mu \sqrt{\frac{Q}{\pi(R/R_o)^3}} \right]^2 dz \right]$$

$$+ \int_{d+L_0}^L \frac{L}{dz} dz$$ (7.22)

$$\lambda = \frac{2}{R_o \pi} \left[ \frac{L-L_0}{I_0} + \int_0^{d+L_0} \frac{dz}{f(\xi)} \right]$$ (7.23)

where $I_0 = \int_0^{R_0} \frac{dr}{f(\xi)}$.

Following Tandon et al. (1979) the apparent viscosity $\mu_1/\mu$, is defined as follows:

$$\frac{\mu_1}{\mu} = \frac{1}{(R_0)^4 \left( F(\xi) \right)}$$ (7.24)

The shearing stress at the wall is defined similar to equations (6.18) and (6.19), i.e.,

$$P_o - P_L = -\frac{2}{R_o} \int_0^L \left[ \frac{2}{7} \sqrt{\tau_o} + 2\mu \sqrt{\frac{Q}{\pi(R/R_o)^3}} \right]^2 dz$$ (7.20)
\[ \tau_R = \left[ -\mu \frac{dU}{dr} \right]_{r=R(z)} \]  
(7.25)

or

\[ \tau_s = \left[ \frac{R(z),}{\pi \cdot F(\xi)} \right] \]  
(7.26)

We solve equation (7.7), using the following non-dimensional quantities,

\[ t_1 = \frac{t}{t}, \quad \bar{t} = \frac{L}{U}, \quad \xi = \frac{z-Ut}{L}, \quad \eta = \frac{r}{R_0} \]  
(7.27)

\[ \bar{C}_1 = \frac{C_1}{C_0}, \quad R' = \frac{R}{R_0} \]

hence, the equation (7.7) takes the form

\[ \frac{1}{t} \left[ \frac{\delta \bar{C}_1}{\delta t_1} + \frac{\nu}{L} \frac{\delta C_1}{\delta \xi} \right] = \frac{D_1}{R_0^2} \left( \frac{\delta^2 \bar{C}_1}{\delta \eta^2} + \frac{1}{\eta} \frac{\delta \bar{C}_1}{\delta \eta} \right) \]  
(7.28)

To solve the above equation, we have the following non-dimensional boundary conditions,

\[ \frac{\delta \bar{C}_1}{\delta \eta} = 0 \quad \text{at} \ \eta = 0 \]  
(7.29)

\[ \bar{B}_1 \frac{\delta \bar{C}_1}{\delta \eta} = V \cdot \bar{N} \cdot \bar{C}_1 \quad \text{at} \ \eta = R/R_0 \]

As before, from Chapter VI (equation 6.24) equation (7.28) takes the form,
\[
\frac{\delta^2 C_1}{\delta \eta^2} + \frac{1}{\eta} \frac{\delta C_1}{\delta \eta} = \nu \frac{R_0^2}{D_1 L}, \frac{\delta C_1}{\delta \xi}
\] (7.30)

where
\[
\nu = U - \bar{U}
\] (7.31)

and
\[
\bar{U} = \frac{2}{(R')^2} \int_0^{R'} \eta U d\eta
\]
\[
\bar{U} = \frac{R_o^2}{2\mu L} \frac{dP}{d\xi} \left[ \frac{1}{4} \frac{\eta_c}{3} R' - \frac{4}{7} \nu \eta_c (R')^{3/2} \right]
\] (7.32)

To solve equation (7.30), we use the boundary conditions (7.29), we obtain,
\[
\bar{C}_1 = \frac{R_0^4}{2(R')^2 \mu L^2 D_1} \frac{dP}{d\xi} \frac{\delta C_1}{\delta \xi} \left[ \frac{(R')^2}{4} \frac{\eta_c^2}{4} - \frac{\eta_c^4}{16} \right.
\]
\[
- \frac{8}{3} \eta_c^{1/2} \left( \frac{(R')^{3/2}}{4} \eta_c^2 - \frac{4\eta_c^{7/2}}{49} \right) + 2\eta_c \left( \frac{R' \eta_c^2}{4} - \frac{\eta_c^3}{9} \right)
\]
\[
- \frac{R_0^4 \bar{U} \eta_c^2}{2D_1 L} \frac{\delta C_1}{\delta \xi} + K_1
\] (7.33)

where
\[
K_1 = \left\{ \frac{R_0^3}{2 \mu L^2} \frac{dP}{d\xi} \left[ \frac{R_1}{4} (\frac{D_1}{VN} - \frac{3}{4} \frac{R' R_0}{D_1}) \right.
\]
\[
+ \frac{2}{7} \nu \eta_c \nu R' \left( \frac{2D_1}{VN} - \frac{11}{7} \frac{R' R_0}{D_1} \right) + \frac{\eta_c}{3} \left( \frac{\bar{D}_1}{VN} - \frac{5R' R_0}{6D_1} \right) \right\}
\]
The volumetric rate and axial or Taylor-diffusion coefficients are obtained in the similar manner as shown in Chapter VI, equations (6.29), (6.32) and (6.35). Hence, the volumetric rate $M$ is obtained as follow

$$M = \frac{R_0 \eta_c}{D_1 L^2} \left[ \frac{R_0}{8\mu} \cdot (R')^6 - \frac{\eta_c}{4\mu} \cdot (R')^3 \right] \frac{\delta C_1}{\delta \xi}$$

(7.34)

and the axial-diffusion coefficients

$$D^* = \frac{R_0 \bar{U}}{D_1} F_0$$

where

$$F_0 = \frac{1}{L^2} \left[ \frac{R_0 \eta_c (R')^6}{8\mu} - \frac{\eta_c (R')^3}{4\mu} \right]$$

(7.35)

and

$$D^* = \frac{dP}{d\xi} \frac{R_0^9}{D_1} F_1$$

where

$$F_1 = \frac{(R')^2}{8\mu L^3} \left[ \frac{R_0 \eta_c (R')^6}{8\mu} - \frac{\eta_c (R')^3}{4\mu} \right]$$

(7.36)
Fig. 7.2 Variation of apparent viscosity \((\mu/\mu_0)\) with \(z/L_0\) for different values of \(\delta_s/R_0\).
RESULTS AND DISCUSSIONS

In the usual study of cardiovascular diseases, doctors, commonly are concerned with cholesterol levels, the abnormalities of blood pressure, the formation of atherosclerotic plaques and so forth. Most of these measures are to fight the heart disease, the number one killer, that is, once it occurs, how to cure it. Even after all these efforts, the killer is still at large— even now 55% of all annual deaths from all causes are due to these diseases. To tame the killer, perhaps, the strategy has to be changed. We have to adopt preventive measures, that is diagnose these diseases much before their clinical symptoms appear and interpret the cause of disease.

Therefore in this chapter an attempt has been made, to locate the various arterial diseases just at the developing stages. We have obtained the Apparent viscosity, Resistance to flow and the wall shearing stresses for blood flow through growing stenosis.

The Apparent viscosity has been represented in Figures 7.2 and 7.3 for different parametric values. It has been observed that the apparent viscosity increases as the stenosis grows and remains constant outside the stenotic regions. It
Fig. 7.3 Variation of apparent viscosity ($\mu/\mu_0$) with $\delta_s/R_0$ for different $\tau_0$. 

$\frac{z}{L_0} = 0.5$

\begin{align*}
\tau_0 &= 0.0 \\
0.1 \\
0.2 \\
0.3 \\
0.4 \\
0.5 \\
\end{align*}
Fig. 7.4 Variation of flow resistance ($\lambda$) with $L_0/L$ for different $\delta_s/R_0$. 
is seen from the figure 7.3 that increase in the yield stress $\tau_o$ decreases the apparent viscosity as the stenosis develops.

By plotting equation (7.23) it is noted from Figure 7.4 that resistance to flow $\lambda$ increases as the stenosis grows, and is larger in the stenotic regions as compared to the non-stenotic regions. It may be observed from figure 7.5 that resistance to flow $\lambda$ decreases as the yield stress $\tau_o$ increases.

Variation in wall shearing stress with the developing stenosis for different values of the parameters have been brought out in Figures 7.6 and 7.7. As the stenosis grows the wall shearing stress increases in the stenotic regions. The results for increasing values of $\tau_o$ are similar to those for resistance to flow.

It is noted from Figures 7.2, 7.4 and 7.6, that, the curves are symmetric in the stenotic region about an axis along which the constriction of the arterial lumen is maximum. The apparent viscosity, resistance to flow and wall shearing stress become higher and higher till the stenosis is maximum i.e., at $d + \frac{L^0}{2}$ and gradually diminishes towards the end of the stenotic region. The symmetry in the curves in the stenotic region may be attributed to be due to the assumed symmetry in the geometrical configuration of the
Fig. 7.5 Variation of flow resistance (\( \lambda \)) with \( \delta_s/R_0 \) for different \( \tau_0 \).
Fig. 7.6 Variation of wall shearing stress ($\tau_s$) with $z/L_0$ for different $\delta_s/R_0$. 
stenosis. These results are similar to those obtained in Chapter VI.

In this chapter, we have studied the effects of stenosis in an artery by considering the blood as Casson-model fluid. It can be concluded that apparent viscosity, resistance to flow and wall shearing stress increases as the size of the stenosis increases for a given non-Newtonian model of the blood, but these increases are, however, smaller due to non-Newtonian behaviour of the blood. Thus it appears that the non-Newtonian behaviour of the blood is helpful in the functioning of diseased arterial circulation.

Taylor-diffusion coefficient (Axial diffusivity) for the solute particles as described by the equations (7.35) and (7.36) has been observed to depend on the yield stress \( \tau_0 \). The variations of the functions \( F_0 \) and \( F_1 \) has been presented in Table 7.1. It has been observed that as \( \tau_0 \) increases the values of \( F_0 \) and \( F_1 \) decrease very rapidly with yield stress \( \tau_0 \).
Fig. 7.7 Variation of wall shearing stress ($\tau_s$) with $\delta_s/R_0$ for different $\tau_0$. 
Table 7.1

Variations of the Axial Diffusivity Coefficients with yield stress $\tau_0$

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<th>$F_0 \times 10^2$</th>
<th>$F_1 \times 10^3$</th>
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