MATHEMATICAL FORMULATION TO STUDY THE GEOMETRY

DEPENDENCE OF SURFACE PLASMA OSCILLATIONS

6.1 Introduction: Magneto hydrodynamics (or MHD) as we have seen in the previous chapter, is the theory of macroscopic interaction of electrically conducting fluids with a magnetic field. In the case of viscous incompressible case, MHD flow is governed by the Navier- Stokes equations and the Maxwell’s equations of the magnetic field. Since the Maxwell’s equations transcend the region of conducting fluid and extend to all of space, this feature of the electromagnetic interaction of the electron gas or the fluid at the interface with either vacuum or different materials or with outside world, gives rise to challenging problems of mathematical analysis and numerical approximation.

An enormous amount of literature has accumulated during the past 25-30 years of investigations of the interaction of photons with surface plasmons. The surface plasmons at the interface of bulk metal and vacuum were first predicted by Ritchie (26). Powell and Swan (163) observed them experimentally. Stern and Ferell (27) found that the surface plasmons at the interface of a metal and its oxides could account for some of the perplexing peaks occurring in the inelastic scattering of fast electrons by metal foils. Bloch (46,47) proposed the hydrodynamical model to explain these oscillations. Ritchie- Wilems (48), Boardman et al, (49,50,55), Harsh –Agarwal (134), have made extensive studies using these models. Pines (85) applied the many body theory for explaining the bulk and other losses in the metals. Chen et al (176) have studied the SP (surface plasmons) and phonons in the space charge layer on the doped GaAs surfaces. Schmeits (177) has dealt with the surface plasmon coupling in cylindrical pores. Feibleman (178) has presented a simple physical
picture of the relationship between the SP dispersion and the associated induced surface charge. In most of the cases studies have been confined to rectangular boundaries.

Modern electron lithography, certain techniques developed in integrated optics, and other technological advances in sample preparation today enable fine control over the many types of surfaces fruitfully being studied via e.m interaction. Surface probing of sub-micron structures has also received some stimulus from the continuing development of lasers and synchrotron radiation sources. (High-intensity radiation is now available across a wide bandwidth for both elastic and inelastic scattering investigations).

Various geometrical structures with one or more sub-micron dimensions may be usefully and simply modeled in a theoretical framework, which uses non-retarded electrodynamics in appropriately, selected coordinate systems. Surface-plasmon dispersion relationships obtained within such a framework using local dielectric function was given by Ronveaux. This analysis leading to quantisation of surface waves is given by Ritchie (26), Ashley (67) and Ferrell (38).

In the following mathematical calculations, we would like to carry out calculations using the hydrodynamical model to see how surface plasmon dispersion relations are modified due to the various geometries.

The table below gives some coordinate systems in which Laplace's equation is simply separable.

The normal modes of oscillations of polarization charge induced on a surface may be discrete for any surface of finite area. On the other hand, if one or more dimension of the surface are infinite, the modes have continuous component. These facts are reflected in the linear superpositions which form such physical quantities as electric scalar potential, the surface charge density, the current density and the total energy of the oscillations.
Some Coordinate Systems of Revolution in which the Laplace's Equation is simply separable.

<table>
<thead>
<tr>
<th>System</th>
<th>Coordinates</th>
<th>Scale factors</th>
<th>Geometries</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Cylindrical</td>
<td>(R, z, φ)</td>
<td>h_R = h_z = 1, h_φ = R</td>
<td>a. cylinder (R = a)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>b. plane (z = 0)</td>
</tr>
<tr>
<td>2. Spherical</td>
<td>(r, θ, φ)</td>
<td>h_r = 1, h_θ = r</td>
<td>a. sphere</td>
</tr>
<tr>
<td></td>
<td></td>
<td>h_φ = r sin θ</td>
<td>b. cone</td>
</tr>
<tr>
<td>3. Prolate</td>
<td>(β = cos θ, φ)</td>
<td>h_β = h_α</td>
<td>a. prolate spheroid</td>
</tr>
<tr>
<td>spheroidal</td>
<td>α = cosh β, φ</td>
<td>= a (sinh² α + sin² θ)⁴ / 2</td>
<td>b. two sheeted hyperboloid</td>
</tr>
<tr>
<td></td>
<td></td>
<td>h_φ = a sinh β sin θ</td>
<td></td>
</tr>
<tr>
<td>4. Oblate</td>
<td>β = cos θ,</td>
<td>h_β = h_α</td>
<td>a. oblate spheroid</td>
</tr>
<tr>
<td>spheroidal</td>
<td>α = cosh β, φ</td>
<td>= a (sinh² α + cos² θ)⁴ / 2</td>
<td>b. single sheeted hyperboloid</td>
</tr>
<tr>
<td></td>
<td></td>
<td>h_φ = a cosh β sin θ</td>
<td></td>
</tr>
<tr>
<td>5. Toroid</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The most commonly used toroidal coordinate system, generally called the local or quasitoroidal system, has the coordinates (r, θ, φ). Because of its close resemblance to the cylindrical system, this affords easy comparison of the toroidal solutions to those of a straight cylinder. We therefore start with the description of the cylindrical geometry and first obtain the surface modes for cylinders. We subsequently describe the toroidal system and find the surface modes in this geometry.
6.2 CYLINDERS

Literature on light scattering in and from cylinders extend back over many years. The original calculations of the scattering of electromagnetic waves incident perpendicular to the axis of a transparent, homogeneous, circular cylinder was given in 1881 by Lord Rayleigh. More recently, interest in reflection and transmission of light from and in cylindrical structures has been stimulated by the applications in the field of fibreoptics. Characterisation of fibres used in textile industry has also contributed to this development.

The possibility of observing surface plasmon modes on a cylinder in a light scattering experiment was suggested by Miziumski. He reported studies with a Helium–neon laser beam reflected from an Al-quartz cylinder at near grazing incidence. The total light reflected from the fibre was found to fluctuate in intensity as the angle of incidence was varied. These fluctuations were attributed to excitation of surface waves, and a simple theory based on a guided surface-plasma-wave-model was involved to explain their angle of occurrence.

Extensive theoretical and experimental studies are available on the surface plasma modes in media with plane interfaces. Theoretical interest in the surface modes for cylindrical interface has developed only recently.

Below is a brief description of the cylindrical coordinate system. The metric tensor for polar cylindrical coordinates is also given so that a model may be obtained in future which will show the surface modes on the cylindrical surface.

Polar Cylindrical Coordinates

After Cartesian coordinates, Polar coordinates are the most frequently used coordinate system.
They easy represent problems with circular symmetry in a flat space. Polar cylindrical coordinates combine circular symmetry with cylindrical symmetry, in a third dimension.

Analytical Coordinate Transformations

By convention, the symbol $r$ is used for the radial coordinate, $\theta$ for the angular coordinate, and $z$ for the direction of cylindrical symmetry.

\[
\text{polar cylindrical}
\]

\[
[[\cos(\phi) \cdot r, \sin(\phi) \cdot r, z], r, \phi, z]
\]

Plot of Coordinate Surfaces

One can use the clipping slider to see the inside of this graphic.

We can see the circular cylinders at constant values of $r$, the horizontal planes at constant values of $z$, and the vertical planes constant values of $\theta$.

\[
\text{block([title:"Polar Cylindrical Coordinates"],}
\]

\[
\text{plot3d\_coords(polarcylindrical,[-\pi,-\pi],[0,\pi,1],[-8,8,4],[3,3,3])}$
\]
The Metric Tensor for Polar Cylindrical Coordinates

The metric tensor of polar coordinates displays the basic symmetries of the coordinate system. The cylindrical dimension $z$ is independent of the other directions, which is reflected in the fact that the metric entries $(r,z)$, $(z,r)$, $(\theta,z)$ and $(\theta,q)$ are zero, while the $z$-coordinate itself is normalized (that is, the metric's $(z,z)$ component is 1).

Polar cylindrical coordinates exhibit a complication which is typical of curvilinear coordinates. The coordinate unit vectors in the $\theta$ direction is not of unit length. However, the off-diagonal element of the metric are still zero, which means that the coordinate unit vectors are orthogonal to one another. While all the coordinate systems in this notebook are orthogonal coordinate systems, this constraint can be relaxed in Macsyma's
tensor calculus packages for computing with generalized smooth coordinate systems.

To study the dependence of SP on geometries, we initially take up the case of surface plasma oscillations for semi infinite cylindrical bounded electron gas. The hydrodynamical model of Bloch (46,47) is used. We consider an uniformly positive neutralizing background for the electron gas in a cylindrical bounded region of thickness along +ve z-axis.

The cylindrical bounded region has been taken in such a way that the particle density is \( n_0 \) in the region \( 0 < z < d \) and is zero beyond it. The surface plasmon dispersion relation derived in the present work shows good agreement with other models(26,27,85,179,180) and also with experiments. At low value of the wavevector the present result is also close to the hydrodynamical surface plasmon dispersion relation (181) of rectangular bounded inhomogeneous charge density medium.

6.3 **Calculation of surface plasmon dispersion relation for a cylindrical bounded electron gas using hydrodynamic model of Bloch(46,47) and Ritchie-Wilems(48).**

We have a uniform +ve neutralizing background for our electron gas in a cylindrical boundary and the cylinder is symmetrically disposed about the origin. The electrostatic potential \( \varphi \) and the velocity \( v \) of a hydrodynamic fluid satisfy,

\[
m \frac{dv}{dt} = e \nabla \varphi - \nabla \cdot \mathbf{J} \frac{dp(n')}{p(n')}
\]

\[
\nabla^2 \varphi = 4\pi e \left[n(r,\theta,z,t)-D_i(r,\theta,z)\right] \tag{6.1}
\]

\[
D_i(r,\theta,z) \text{ is the ion density, } P(n') \text{ is the fermi pressure, } n \text{ and } n' \text{ are electron concentrations,}
\]

\[
d/dt \text{ is the comoving time derivative.}
\]

\[
P_p(n') = (3 \pi^2)^{2/3} \sqrt{5} m \ h^2 \ n^{3/2} \text{ is the Fermi pressure}
\]
It is convenient to introduce a velocity potential $\psi(r,t)$, such that $v = \nabla \psi(r,t)$

The equation of hydrodynamics after substitution and simplification become

$$\frac{d}{dt}(\psi) = \frac{1}{2}(-\nabla \psi)^2 - e/m \varphi + \frac{1}{2} \int_0 dp(n')/n'$$

(6.3)

$$\nabla^2 \varphi = 4\pi e n$$

(6.4)

Along with the equation of continuity,

$$\frac{\partial n}{\partial t} = -\nabla \cdot (n v)$$

$$= \nabla \cdot [n \nabla \psi]$$

(6.5)

The process of linearization is a convenient tool to deal with this problem. Therefore,

$$n(r,t) = n_0(r) + \lambda n_1(r,t) + \lambda^2 n_2(r,t)$$

$$\varphi(r,t) = \varphi_0(r) + \lambda \varphi_1(r,t) + \lambda^2 \varphi_1(r,t)$$

$$\psi(r,t) = \psi_0(r) + \lambda \psi_1(r,t) + \lambda^2 \psi_2(r,t)$$

(6.7)

Where the parameter for smallness is $\lambda$, is to be set equal to unity in the final result. Substituting these expansions into the above equations, we equate coefficients of various powers of $\lambda$ to zero separately.

Here it is assumed that $n_0 >> n_1 >> n_2$. As the fluid spreads in the medium, particle density decreases.

Collecting the coefficients of zeroth order, 1st order etc, we obtain for the zeroth order.

Substituting eqn’s (6.7) in eqn’s (6.1), (6.2), (6.4), (6.5) and equating coefficients of variables.

$$\nabla^2(\varphi_0 + \lambda \varphi_1 + \lambda^2 \varphi_2) = 4\pi e \{n_0 + \lambda n_1 + \lambda^2 n_2\} - 1$$

$$\frac{\partial}{\partial t}(n_0 + \lambda n_1 + \lambda^2 n) = \nabla \cdot \{n_0 + \lambda n_1 + \lambda^2 n_2\} \{\nabla (\lambda \psi_1 + \lambda^2 \psi_2)\}$$
For zeroth order terms

\[ \frac{e}{m} \phi_0 = \frac{5}{2} P n_0^{2/3} \]  

(6.8)

From First order terms

\[ \frac{\partial \psi_1}{\partial t} = -\frac{e}{m} \phi + (5P/3m) n_0^{-1/3} n \]  

(6.9)

\[ \nabla^2 \phi_1 = n_1 e \]  

(6.10)

Equation of continuity gives,

\[ \frac{\partial}{\partial t} n_1 = \nabla \cdot n_0 \nabla \psi_1 \]  

(6.11)

For an impenetrable barrier at the plane \( z = 0 \),

The disturbances in the electron gas is represented by the linear equation

\[ n(r,t) = n_0(r) + \lambda n_1(r,t) + \lambda^2 n_2(r,t) + \ldots \]

\( n_1 \) and \( \psi_1 \) are to be taken as zero in the region \( z < 0 \).

Our problem is a cylindrical boundary problem, where the particle or fluid density varies according to the condition,

\[ n_0(r) = \begin{cases} n_0 & \text{for } r < a \\ 0 & \text{outside} \end{cases} \]  

(6.12)

The equation (6.9) may be rewritten as,

\[ \frac{\partial \psi_1}{\partial t} = -\frac{e}{m} \phi_1 + (v_f^2/3n_0) n_1 \]  

(6.13)

Where at temperature \( T = 0 \),
Equation (6.10) and (6.11) may be rewritten as,

\[ \nabla^2 \psi_1 = n_1 e \quad (6.14) \]

\[ \frac{\partial n_1}{\partial t} = n_0 \nabla^2 \psi_1 \quad (6.15) \]

Eliminating \( \psi_1 \) between equations (6.13) - (6.15), we get,

\[ (\frac{\partial^2}{\partial t^2} + \alpha_{\psi_0}^2 - \beta^2 V^2 + \lambda n_1(r, t) = 0 \quad (6.16) \]

where \( \omega_{\psi_0}^2 = \frac{\omega_{\psi_0}}{e}, \omega_{\psi_0}^2 = 4 \pi n e^2/m \) and \( \beta = v_f / \sqrt{3} \)

Equation (6.16) is the condition for volume plasma oscillations. It contains only space and time in fluid density \( n_1 \). Due to cylindrical symmetry, the space and time can be separated as follows.

\[ n_1(r, t) = n_1(r) e^{i\omega t} \quad (6.17) \]

\( n_1(r) \) can be expanded for the interior,

\[ n_\ell(r) = \sum R\ell(r) F_{\ell m}(\theta, z) \quad (6.18) \]

Substituting equation (6.18) in (6.16), we get for \( R\ell(r) \)

\[ d^2 R\ell(r)/dr^2 + 1/r \ d R\ell(r)/dr + (\ell^2 - \ell^2/r^2) R\ell(r) = 0 \quad (6.19) \]

where \( \ell^2 = (\omega_e^2 - \omega_{\psi_0}^2)/\beta^2 \), \( \ell \) is a constant and \( \ell^2 \) is a wave vector.

The solution of (6.19) is given by,

\[ R\ell(r) = C\ell J_{\ell}(\ell^2 r) \quad (6.20) \]
C₁ is a constant and \( J_\ell (\ell r) \) is Bessel's function.

Since we have the boundary condition of finiteness of \( R \ell \) at the origin, \( \varphi \ell (r,t) \) given by equation (6.14) can also be separated as,

\[
\varphi (r,t) = \varphi_1 (r) e^{io_1 t}
\]  

(6.21)

And, \( \varphi_1 (r,\theta, z) = \sum \varphi \ell (r) F_{\ell m}(\theta,z) \)  

(6.22)

Using (6.14) and (6.22), the equation for \( \varphi \ell (r) \) is,

\[
d^2 \varphi \ell (r) / dr^2 + 1/r \ d \varphi \ell (r) / dr + (\ell^2/r^2) \varphi \ell (r) = C_2 J_\ell (\ell^2 r)
\]  

(6.23)

Adding \( (\ell^2 \varphi_1) \) to both sides of (6.23),

\[
d^2 \varphi \ell (r) / dr^2 + 1/r \ d \varphi \ell (r) / dr + (\ell^2 - \ell^2/r^2) \varphi \ell (r) = C_2 J_\ell (\ell^2 r) + \ell^2 \varphi \ell (r)
\]  

(6.24)

The interior solution of (6.24) may be written as,

\[
\varphi \ell (r) = -C_2 / C_1 J_1(\ell r) + C_3 \ r^\ell , \quad r < R
\]  

(6.25)

for the exterior, there are no real charges \( (n_1 = 0) \) and the equation for \( \varphi \ell (r) \) becomes

\[
\nabla^2 \varphi \ell = 0 ,
\]

This has a solution,

\[
\varphi \ell (r) = C_4 / r^\ell , \quad r > R
\]  

(6.26)

\( C_3, C_4 \) are constants and \( R \) is the radius of the cylinder. Imposing the boundary condition \( r = R, \)
\[ \Phi e^{\text{int}}(r) = \Phi e^{\text{ext}}(r) \]  \hspace{1cm} (6.27)

And, \( \partial / \partial r [\Phi e^{\text{int}}(r)] = \partial / \partial r [\Phi e^{\text{ext}}(r)] \)

\[ C_3 = C_2 / R^\ell \ (1/(2 \ell^\prime)^2 \ [J_\ell (\ell^\prime R) + R J_\ell^\prime (\ell^\prime R) / \ell]) \]  \hspace{1cm} (6.28)

Here, \( J_\ell^\prime (\ell^\prime R) = (d/dr J_\ell (\ell^\prime r)) \mid_{r=R} \)

Thus the complete solution for the interior is given by,

\[ \Phi e(r) = \Sigma e C_i / \ell^2 \ {-J_\ell (\ell^\prime r) + (\ell +1)/(2 \ell +1) [J_\ell (\ell^\prime R) + R / (\ell +1) J_\ell^\prime (\ell^\prime R)]} \quad (r/R)^i \ F m(\theta, z) \]  \hspace{1cm} (6.29)

and \( n^\prime e(r) = \Sigma R e(r) F m(\theta, z) \)  \hspace{1cm} (6.30)

Use of the usual hydrodynamical conditions that electronic velocity normal to the surface must vanish, gives the surface plasmon dispersion relation. Since the normal velocity \( v_1 \) and acceleration \( v_{1,1} \) are given by,

\[ v = v_1 e^{j \omega t} \]  \hspace{1cm} (6.31)

\[ v = e/m \ \nabla \phi_1 - \beta^2/n_0 \ \nabla n_1 \]  \hspace{1cm} (6.32)

From equation (6.29), (6.30) and (6.32)

\[ J_1 (\ell^\prime r) / 2R = J_1^\prime (\ell^\prime R) \ [1/(2 \ell) + \beta^2 \ell^2 / \ell \ * \omega_p^2] \]  \hspace{1cm} (6.33)

This is the required surface plasmon dispersion relation for the cylindrical bounded electron gas.

This equation is valid when the medium in which the cylinder is placed is vacuum.
If on the other hand, the dielectric constant of the medium is $\varepsilon_r$, in which the slab is embedded, the dispersion relation becomes

$$J_\ell (\ell' r) / 2R = J'_\ell (\ell' R) \left[ 1 / \ell' \left( 1 + \beta^2 \ell'^2 / \omega_p^2 \right) \right] (\varepsilon_\infty \ell + \ell / \varepsilon_r)$$

(6.34)

$\varepsilon_\infty$ is the dielectric constant of medium of the dielectric embedded in vacuum. If $\varepsilon_\infty = 1$ in the above equation, then the dispersion relation for a metallic cylinder may be obtained.

### 6.4 Surface plasmon dispersion relation for polar semiconductor cylindrical bounded electron gas.

In order to obtain this relation, one has to impose conditions of continuity of $\phi$ ($r$) and $\varepsilon \partial \phi / \partial t$ (here $\varepsilon$ is dielectric constant of semiconductor cylinder) at the surface and the hydrodynamic condition for the surface waves to exist i.e. the normal component of the velocity $v$ vanishes at the cylindrical boundary $r = R$. (6.25) and (6.26) may be written as,

$$C_2 / \ell'^2 J_1 (\ell' r) + C_2 r\ell = C_4 / r\ell$$

(i)

$$\varepsilon C_3 \ell r^{\ell - 1} - \varepsilon_\ell / \ell' J'_1 (\ell' r) = - \varepsilon_\infty C_4 r^{- (\ell + 1)}$$

(ii)

Here $\varepsilon_\infty$ is the dielectric constant of the medium in which the cylinder is embedded and $\varepsilon$ is given by $(\varepsilon_0 + \varepsilon_\infty) / 2$, where $\varepsilon_0$ and $\varepsilon_\infty$ are the low and high frequency dielectric constants respectively and $\varepsilon_\ell$ (lattice dielectric constant) may be written as,

$$\varepsilon_\ell (\omega) = \varepsilon_\infty \omega^2 - \varepsilon_0 \omega^2 / \omega^2 - \omega^2$$

(iii)
\( \omega_t \) is the transverse optical phonon frequency. Adopting the same procedure as we have done earlier and using the above equation \((6.35,36,37)\), the dispersion relation for the polar semiconductor may be approximated as,

\[
\left\{ \varepsilon_+ \left( \omega_t^2 - \omega_t^2 / \omega_t^2 \right) / \left( \omega_{t_0}^2 / \omega_t^2 \right) + \left[ \varepsilon_0 \omega_t^2 / \omega_t^2 - \varepsilon_0 / \omega_t^2 \right] \left( \omega_t^2 / \omega_t^2 \right) \right\} J_2(\ell^R R) / J_0(\ell^R R)
\]

\[
= \varepsilon \left[ (\omega_t^2 - \omega_t^2 / \omega_t^2) / \left( \omega_{t_0}^2 / \omega_t^2 \right) - \varepsilon_0 \left( \omega_t^2 / \omega_t^2 \right) - \varepsilon_0 \right] \left( \omega_t^2 / \omega_t^2 \right)
\]

Here it has been assumed that \( \varepsilon = 0 \) and \( \ell = 1 \).

(a) Discussion and comparison of results of Surface plasmon dispersion relation.

(i) Equation (6.33) or (6.34) reduce into well known Stern and Ferrell (27) results if we put \( \varepsilon_+ = +1 \), \( \ell^R \rightarrow \infty \) and \( \ell = \infty \), that is \( \omega_t = \omega_{t_0} / 2^{1/2} \), which is the surface loss. It is also supported by Ritchie (26, 189) for planer interface.

(ii) The condition \( \varepsilon_- = -1 \) can also be applied to general volume plasmon dispersion relation (85) for free electron gas as

\[
\varepsilon_-(\omega) = 1 - \omega_{t_0}^2 / \omega_t^2 = -1
\]

which again gives \( \omega = \omega_{t_0} / \sqrt{2} \).

(iii) Equation (6.34) represents the general characteristic of surface plasma oscillations at \( \varepsilon_+ = +1 \) and can be transformed into (6.33). The second term of (6.34) is an interaction term. If it is zero then there is a free surface plasmon oscillation which may be expressed by (6.35).

The result (6.34) & (6.35) are analogous to volume plasmon dispersion relation given by Pines (85).
(iv) For fixed electron density, one can vary the wave vector \( \ell' \) and observe the change in the surface plasmon dependence of momentum. Calculations have been performed using (6.34) For Mg \( (n = 8.6 \times 10^{22} \text{ electron/cm}^3) \), we give the calculated curves because experimental data exists for it. Figure 6.2 shows the square of the reduced surface plasmon frequency, \((\omega/\omega_p)^2\) versus \( \ell' \) for values of \( \ell' \leq 0.5 k_F \). In general, the lower equilibrium density in the surface region results in lower values for the resonant frequencies. For \( \ell'=0 \), the surface mode approaches \( \omega_p/\sqrt{2} \). The dispersion curve for \( \ell=1 \) exhibits the homogeneous material behaviour. The curve is monotonically increasing and is fitted well by Ritchie’s (189), Harsh-Agarwal (181) and Kunz’s(190) results.(Fig.6.2)

(b) Comparison of volume plasmon dispersion relation.

From (6.16) and (6.18)

\[
\omega_l^2 = \omega_p^2 + v_F^2 \frac{l^2}{3}
\]

(6.36)

where \( \omega_l \) is the frequency of outer fluid or incident radiations. This dispersion relation closely resembles the Pines formula (85,181).

\[
\omega_k^2 = \omega_p^2 + 3/5v_F^2 k^2
\]

(6.37)

Critical wave vector

The critical wave vector, which is the maximum value of the wave vector to excite the plasmon oscillations, may be determined by the intersection of the dispersion relations given in (6.36), (6.37) and the following expansion for the maximum energy to excite individual electron
E(k) = (\( \hbar k_F \) + \( \hbar k \))^2/2m - (hk_F)^2/2m \tag{6.38}

Here \( \hbar k_F \) is the momentum of an electron at the Fermi surface. We can see the value of critical wave vector for the X-Ray emission spectra is less than that of the hydrodynamical flow value calculated from the hydrodynamic model, i.e. the intersection of the (6.36) and (6.37).

**FIGURE CAPTIONS (fig.6.2)**

Plot of \( (\omega/\omega_p)^2 \) as a function of K (wave vector)


B : Volume plasmon dispersion curve according to Bohm and Pines (85).

C : Surface Plasmon dispersion curve using hydrodynamical model in plane bounded Electron gas (179).

D : The continuous line is the surface plasmon dispersion curve using hydrodynamical model, in cylindrical bounded electron gas (6.33). The solid points are the experimental data of Kunz (190).
Fig 6.2

A: Volume Plasmon Dispersion Curve
B: Volume Plasmon Dispersion Curve (Bohm and Liles; ref. 2326)
C: Surface Plasmon Dispersion Curve (Plane bounded electron gas)
D: S.D. Dispersion Curve (Cylindrical bounded Gas).

Solid Points are experimental data for Hugz.
FIGURE CAPTIONS (fig.6.3)

Plot of \((\omega/\omega_p)^2\) as a function of \(k\) (wave vector)


B : Volume plasmon dispersion curve according to Bohm and Pines (23).

C : Surface Plasmon dispersion curve for tangential modes using hydrodynamical model in plane bounded Electron gas (3).

D : Surface plasmon dispersion curve for normal modes obtained, using hydrodynamical model, in cylindrical bounded electron gas (64). The solid points are the experimental data of Kunz (190).
Fig. 6.3

A: Volume Plasmon Dispersion Curve (6.36)
B: Volume Plasmon Dispersion (Bohm and Ames: Ref. 28-25)
C: Surface Plasmon Dispersion Curve for Tangential modes in a plane (3)
D: S.P. Dispersion Curve in Cylindrical Bounded Electron Gas (6.33)

Solid Points are Experimental Data of Kung (190)
INTERFACE PLASMAS

According to Stern and Ferrel (27) the surface plasmon dispersion relation for a plasma with dielectric constant $\varepsilon_A$, bounded by a dielectric medium of dielectric constant $\varepsilon_B$ is given by,

$$\varepsilon_A + \varepsilon_B = 0$$ (6.39)

Similarly consider the cylindrical interface at $r = 0$ between metal 1 at $r > 0$ and metal 2 at $R r < 0$. With $\omega_{p1}$ and $\omega_{p2}$ as the bulk plasmon frequencies of metal 1 and 2 and $\varepsilon_+$ and $\varepsilon_-$ as the dielectric constants of the medium and the cylinder, (6.35) and (6.39) give for the hydrodynamic interface surface plasmons,

$$P_1 - \ell_1 - 1/\ell_1 (1-P_1) + P_2 - \ell_2 - 1/\ell_2 (1-P_2) = 0$$ (6.40)

Where, $P_1 = J_1 (\ell' R) \omega_{p1}^2 / R J_1' (\ell' R) \beta_1^2 \ell_1$ and

$$P_2 = J_2 (\ell' R) \omega_{p2}^2 / R J_2' (\ell' R) \beta_2^2 \ell_2$$

$\ell_1$ and $\ell_2$ are the wave vectors for metal 1 and 2, respectively. If we neglect the higher order terms in (6.40), it becomes

$$\omega_{p12} = \sqrt{\frac{1}{2} (\omega_{p1}^2 + \omega_{p2}^2)}$$ (6.41)

Such plasmons have been observed at the Al/Mg interface by Kunz (190).

6.5 SURFACE PLASMA DISPERSION RELATION FOR A TOROIDAL BOUNDED ELECTRON GAS USING HYDRODYNAMICAL MODEL OF BLOCH.

Information on surface properties can be obtained mainly by studying the dielectric function and the dispersion relation, but the dispersion relation and the frequency of surface waves depends
upon the nature and geometry of the surface and the bounding media. The dispersion properties of
surface plasma oscillations in metals have been extensively studied by several workers in a number
of geometries such as semi-infinite plane surfaces (24,6), thin films (16) spheres (17) and cylinders
(18). These studies have been made using Bloch’s hydrodynamic model, which provides a way for
introducing spatial dispersion i.e. non-local effects.

In this section the Bloch model has been developed to study the surface modes in toroidal
geometry for a metal-vacuum interface.

**Introduction**

Initiated by pioneering work of Ritchie (26) in 1957, surface plasmons have been the subject of
intense theoretical and experimental investigations during the past few decades. The existence of
surfaces leads to the creation of new modes, which are absent in bulk. Surface plasmon modes
have frequencies and fields (polarization fields), which are dependent on the geometrical shape of
the surface. In this context we would like to study the surface plasmon modes in a toroidal cavity.
These are of fundamental interest in the radio-frequency (RF) heating of tokamak plasmas,
besides having other interesting applications in the microwave circuit theory, in the physics of
fibre optics and in storage rings.

The difficulties that occur in the determination of the modes are tied up with the fact that the wave
equation describing the electron density variations and also the potentials in a torus is not separable
in any form from the toroidal coordinate system.

Brambilla & Finzi (182) and Mishustin & Sheherbakov (75) calculated the electromagnetic
eigenmodes of a toroidal cavity by using the coupled 2nd order partial differential equation in
toroidal coordinates for the electric and magnetic field components and expanding solutions in
terms of inverse aspect ratio. Cap and Deutsch (77) derived solutions to Maxwell’s equations for an empty torus by representing the electromagnetic fields in terms of the Hertz vector. Several works have appeared describing the eigenmodes in toroidal systems containing plasma (183) (186). In most of the above work the terms of the series representing the solutions are Bessel’s functions of various orders.

Before embarking in finding a solution of the Bloch hydrodynamic equations, it would be quite in context to make a description of the toroidal coordinate system.

A. The toroidal coordinate system

The most commonly used toroidal coordinate system, generally called the local or quasi-toroidal system has the coordinates \((r, \theta, \phi)\). Because of its close resemblance to the cylindrical coordinate system, this affords easy comparison of the toroidal coordinates to those of a straight cylinder. The system of toroidal coordinates used here have the coordinates \((\sigma, \psi, \Phi)\); here \(s = \cosh \sigma\) and \(\psi\) and \(\Phi\) are angle variables.

![Fig. (6.4)](image1)

![Fig.(6.5)](image2)
The above figures show the toroidal coordinate system for coordinates \((u,v,\Phi)\) instead of \((\sigma,\psi,\Phi)\).

The toroidal coordinates have two directions of circular symmetry: a conventional rotational symmetry about the cylindrical axis, and a direction of rotation through the hole of the donut. Toroidal coordinates are old friends of plasma physicists who work on toroidal shaped plasma confinement devices in the quest for controlled fusion energy.

Below is exhibited the 3-D orthogonal coordinate system in Macsyma’s Vect_ort and CoordSys packages which contains the metric tensor information. It is a combination of algebraic coordinates transformation, geometric information and interactive graphics.

\[
\frac{(e \cdot \cos(\phi) \cdot \sinh(v))}{(\cosh(v) - \cos(u))}, \frac{(e \cdot \sin(\phi) \cdot \sinh(v))}{(\cosh(v) - \cos(u))}, \frac{(e \cdot \sin(u))}{(\cosh(v) - \cos(u))}, \text{ u, v, phi}
\]

Plot of the toroidal coordinate system as given in the Macsyma notebook.

The surfaces of constant \(v\) are the torus-shaped surfaces which are most easily visible in the plot below. The surfaces of constant \(u\) are like spokes of a wheel inside the tori, rotated around the torus. Surfaces of constant values of \(\Phi\) are planes which converge on a common vertical line on the \(z\) axis (though there is no \(z\) coordinate in toroidal coordinates).

The coordinate \(v\) plays a role similar in many respects to an inverted radius. Large values of \(v\) yield points which approach the circle.
\[ x^2 + y^2 = 1 \] in the x-y plane. Small values of \( v \) yield points on larger tori which converge at the centerline along the z axis.

The Metric Tensor for Toroidal Coordinates

\[
\text{(ct_coordsys('toroidal), disp('lg=lg))} \quad \text{lg = } \left[
\begin{array}{ccc}
\left(\frac{e}{2}/((\cosh(v) - \cos(u))^2)\right), & 0, & 0, \\
0, & \left(\frac{e}{2}/((\cosh(v) - \cos(u))^2)\right), & 0, \\
0, & 0, & \left(\frac{e}{2} \cdot \sinh^2(v)/((\cosh(v) - \cos(u))^2)\right)
\end{array}
\right]
\]
AB is a straight line of length 2a through the origin O, lying in the xy-plane and making an angle \( \phi \) with the x-axis. The coordinates of a point P are defined by:

\[
\begin{align*}
\sigma &= \log \frac{PB}{PA} \\
\psi &= \angle BPA \quad \text{corresponding to the poloidal angle} \\
\phi &= \angle BOX \quad \text{corresponding to the toroidal angle}
\end{align*}
\]

To find \( \sigma \) and \( \phi \), with \( \psi \) varying from \(-\pi\) to \(\pi\), the point P generates a circle (minor cross-section of the torus whose plane is the x-y plane and center is at O').

As the angle \( \phi \) increases from 0 to \(2\pi\), keeping \( \sigma \) fixed, a torus is generated with P as point on the surface of the torus. The circle generated by AB rotating about Z axis is the limiting circle to which the torus collapses when the minor radius is reduced to zero.

The coordinate transformation is given by,

\[
\begin{align*}
x &= a \sinh \sigma \cos \phi / \cosh \sigma - \cos \psi \\
y &= a \sinh \sigma \sin \phi / \cosh \sigma - \cos \psi \\
z &= a \sin \psi / \cosh \sigma - \cos \psi
\end{align*}
\]
Surfaces of constant $\sigma$ are given by the toroids,
\[ x^2 + y^2 + z^2 + a^2 = 2a \left( x^2 + y^2 \right)^{1/2} \coth \sigma. \]

The major radius $R$, and minor radius $r$ of the torus (OO' and O'P) are given by
\[ R = OO' = a \coth \sigma \]
\[ r = O'P = a \cosech \sigma. \]

Minor cross section of different tori do not form a system of concentric circles (fig 6.3, 6.4, 6.5, 6.6). The center of the minor cross-section moves towards $B$ with decrease of $r$. For a fixed torus the length ‘$a$’ is related to the major radius $R$ and the minor radius $r$ by,
\[ a^2 = R^2 - r^2 \]
and the aspect ratio $R/r$ is given by
\[ s = \cosh \sigma = R/r. \]

Surfaces of constant $\psi$ are given by spherical bowls
\[ x^2 + y^2 + (z - a \cot \sigma)^2 = a^2 / \sin^2 \sigma \]

Surfaces of constant $\phi$ are given by
\[ \tan \phi = y/x. \]

**B. MATHEMATICAL FORMULATION**

The Bloch hydrodynamical equation have been simplified in chapter V, we obtained equations (5.1)–(5.3),

\[ m \frac{dv}{dt} = e V\phi - \nabla \int dp(n')/ (n') \]  (6.42)

\[ \partial n/\partial t = - \nabla (n \nabla \psi) \]  (6.43)

\[ = \nabla\left[ n \nabla \psi \right] \]  (6.44)
\[ \nabla^2 \varphi = 4\pi e \left[ n(r,\theta,z,t) - D_i (r,\theta,z) \right] \]  

(6.45)

By process of linearization by expanding \( n \), \( \varphi \) and \( \psi \) as

\[
n(r,t) = n_0(r) + \lambda n_1(r,t) + \lambda^2 n_2(r,t) + \ldots
\]

\[
\varphi(r,t) = \varphi_0(r) + \lambda \varphi_1(r,t) + \lambda^2 \varphi_2(r,t) + \ldots
\]

\[
\psi(r,t) = \psi_0(r) + \lambda \psi_1(r,t) + \lambda^2 \psi_2(r,t) + \ldots
\]

(6.46)

Here \( n_0 >> n_1 >> n_2 \), so first order perturbation terms are sufficient to describe the system.

\[
\frac{\partial \psi_1}{\partial t} = -e/m \varphi_1 + \left( \frac{v_f^2}{3n_0} \right) n_1
\]

(6.47)

\[
\frac{\partial n_1}{\partial t} = n_0 \nabla^2 \psi_1
\]

(6.48)

\[
\nabla^2 \varphi_1 = \left( 4\pi e/\varepsilon \right) n_1
\]

(6.49)

By eliminating \( \psi_1 \) and \( \varphi_1 \) from these equations, we obtain the wave equation for bulk plasma oscillations.

\[
\left[ \frac{\partial^2}{\partial t^2} + \omega_p^2 - \beta^2 \nabla^2 \right] n_1(r,t) = 0
\]

(6.50)

\[
\omega_p^2 = \left[ 4\pi n_0 e^2 / \varepsilon m \right]^{1/2} \quad \text{and} \quad \beta = v_f / \sqrt{3}
\]

(6.51)

By taking the exponential time dependence, the above equation simplifies to,

\[
\left[ \nabla^2 - k^2 \right] n_1(r) = 0
\]

(6.52)

\[
k = 1/\beta \left[ \omega_p^2 - \omega^2 \right]^{1/2}
\]

(6.53)

(Solution of the equation is \( \omega^2 = \omega_p^2 + \beta^2 k^2 \))

(6.54)

Since we wish to obtain solutions for the surface plasmon modes in the toroidal system, the advantage of chosen coordinate system \( (s, \psi, \Phi) \) is that in this system the Laplace’s equation (corresponding to the static limit) is separable and the solution are given in terms of toroidal harmonics. The wave equation is however not separable in the \( (s, \psi, \Phi) \) coordinates. Therefore before expanding \( n(r,t) \) and \( \nabla^2 \) in toroidal coordinates we first reduce the vector Helmholtz (6.52)
to a scalar one by introducing the Hertz vector $\Pi$. The equation satisfied by $\Pi$ is the wave equation, 
\[ \nabla^2 \Pi + k^2 \Pi = 0 \]  
\hspace{1.5cm} (a)
this equation for Cartesian components is
\[ \nabla^2 \Pi_j + k^2 \Pi_j = 0 \]  
\hspace{1.5cm} (b)
Here $j = x, y, z$ and $k^2$ is given by equation (6.52)
$\Pi_x$, $\Pi_y$, and $\Pi_z$ are all functions of $\sigma$, $\psi$, and $\Phi$.
The toroidal components of $\Pi$: $\Pi_\sigma$, $\Pi_\psi$, and $\Pi_\phi$ are obtained from the Cartesian components as,
$\Pi_\sigma = \frac{1 - \cosh \sigma \cos \psi}{\cosh \sigma - \cos \psi} \cos \Phi \Pi_x + \frac{1 - \cosh \sigma \cos \psi}{\cosh \sigma - \cos \psi} \sin \Phi \Pi_y - \sinh \sigma \sin \psi / (\cosh \sigma - \cos \psi) \Pi_z$.
$\Pi_\psi = \frac{-\sinh \sigma \sin \psi}{\cosh \sigma - \cos \psi} \cos \Phi \Pi_x + \frac{-\sinh \sigma \sin \psi}{\cosh \sigma - \cos \psi} \sin \Phi \Pi_y - (1 - \cosh \sigma \cos \psi) / (\cosh \sigma - \cos \psi) \Pi_z$.
$\Pi_\phi = -\sin \Phi \Pi_x + \cos \Phi \Pi_y$  
\hspace{1.5cm} (6.55)
Now we can rewrite equation (b), the scalar Helmholtz equation in toroidal coordinates.
Substituting $s = \cosh \sigma$, $t = \cos \psi$, and $\Pi_0 = \sqrt{(s - \tau)} P_j$  
\hspace{1.5cm} (6.56)
(b) can be put, after some algebra, as
\[ \frac{\partial}{\partial s} \left[ (s^2 - 1) \frac{\partial}{\partial s} P_j \right] + \frac{\partial}{\partial \psi} \frac{\partial^2}{\partial \psi^2} P_j + \frac{P_j}{4(1/s^2 - 1)} \frac{\partial^2}{\partial \Phi^2} P_j + k^2 a^2/(s - \tau)^2 P_j = 0 \]  
\hspace{1.5cm} (6.57)

**Dispersion relation**

We consider a metal torus of aspect ratio $s$, and $\psi$ and $\Phi$ as the poloidal angle and toroidal angle.
The torus is bounded by vacuum.
Let the electron density be expressed as,
\[ n_1 (r,t) = P(s,\psi) e^{i m \Phi} e^{i \omega t} \]  
\hspace{1.5cm} (6.58)
since P(s, ψ) is the r-dependent part,

\[ n_1(r) = P(s, ψ) e^{im\psi} \]  \hspace{1cm} (6.59)

The differential equation (6) can then be expanded according to the equation

\[ [\nabla^2 - K^2]n_1(r) = 0 \]  \hspace{1cm} (6.60)

\[ [\nabla^2 - K^2] P(s, ψ, \Phi) = 0 \]  \hspace{1cm} (6.61)

\[ [\nabla^2 - K^2] P(s, ψ) e^{im\psi} = 0 \]  \hspace{1cm} (6.62)

We can make the following expansions,

\[ (s - r)^2 = 1/s^2 \left[ 1 + 2\cos\psi/s + 3\cos^2\psi/s^2 + \ldots \right] \]  \hspace{1cm} (6.63)

\[ P = P_0 + P_1 + P_2 + \ldots \]  \hspace{1cm} (6.64)

Using (6.63) and (6.64), (6.57) can be written as

\[ \frac{\partial}{\partial s} (s^2 - 1) \frac{\partial P_0}{\partial s} + \frac{\partial^2}{\partial \psi^2} P_0 + \left( \frac{1}{4} - m^2/(s^2 - 1) \right) P_0 + \frac{k^2a^2}{s^2} P_0 = 0 \]  \hspace{1cm} (6.65)

\[ \frac{\partial}{\partial s} (s^2 - 1) \frac{\partial P_1}{\partial s} + \frac{\partial^2}{\partial \psi^2} P_1 + \left( \frac{1}{4} - m^2/(s^2 - 1) \right) P_1 + \frac{k^2a^2}{s^2} P_1 = -\frac{2k^2a^2}{s^3} \cos\psi P_0 \]  \hspace{1cm} (6.66)

\[ \frac{\partial}{\partial s} (s^2 - 1) \frac{\partial P_2}{\partial s} + \frac{\partial^2}{\partial \psi^2} P_2 + \left( \frac{1}{4} - m^2/(s^2 - 1) \right) P_2 + \frac{k^2a^2}{s^2} P_2 = -\frac{2k^2a^2}{s^3} \cos\psi P_1 - \frac{3k^2a^2}{s^4} \cos^2\psi P_0 \]  \hspace{1cm} (6.67)

In the expansion of (6.64) we have assumed that the successive terms decrease by powers of \( k^2a^2/s^2 \). The zero order equation can be separated by expanding P as a Fourier series in \( \psi \).

\[ P_0(s, \psi) = \Sigma P_{0n}(s) \cos n\psi \]  \hspace{1cm} (6.68)

Equation (13) is then transformed to,

\[ \frac{d}{ds} (s^2 - 1) \frac{dP_{0n}}{ds} - \left( n^2 - 1/4 - m^2/(s^2 - 1) \right) P_{0n} - \frac{k^2a^2}{s^2} P_{0n} = 0 \]  \hspace{1cm} (6.69)

The above equation is not in hypergeometric form because it has singularities at \( s = 0, +1 \) and \( \infty \).

It can be put in hypergeometric form by making the following transformations.

\[ y = 1/s^2 \]

\[ P_{0n} = (s^2 - 1)^{m/2} s^{-(m + n + 1/2)} F \]  \hspace{1cm} (6.70)
After some algebra we get,
\[ y (y-1) \frac{d^2}{dy^2} F + [ (\alpha + \beta + \gamma ) y -1 ] \frac{d}{dy} F + \alpha \beta F = 0 \] (6.71)

where, \( \alpha = \frac{1}{2} [ m + n + 1 + \chi ] \)
\[ \beta = \frac{1}{2} [ m + n + 1 - \chi ] \]
\[ \gamma = n + 1 \]
\[ \chi = \sqrt{\left( \frac{1}{4} + k^2 a^2 \right)} \] (6.72 a,b,c)

Equation (6.71) is the Gauss Hypergeometric equation and its solution are given in terms of hypergeometric function \( F (a, b, c; z) \) or by generalized hypergeometric function \( _2F_1 (a,b,c;z) \). Further the derivative of \( F \) w.r.t. \( z \) is
\[ \frac{d}{dz} F(a,b,c;z) = ab/c F(a+1, b+1, c+1; z) \] (6.73)

Expressing in terms of the old variable \( s \), the solution of (6.69), valid in the region within the torus
\( s_0 < s < \infty \), where \( s = s_0 \) defines the surface of the torus, can be given by,
\[ P_{0n} = A_{mn} s^{-(m+n+1/2)} (s^2 -1)^{m/2} _2F_1 (a, \beta, \gamma; 1/s^2 ) \] (6.74)

\( A_{mn} \) is a constant depending on \( m \) and \( n \).

It can be easily verified that in the static limit \( (k \rightarrow 0) \), the above solution is the toroidal Harmonics \( Q^{n}_{m-1/2} (\cosh \sigma ) \) which is valid in the space interior to the torus where the \( Q \) function is defined by,
\[ Q^{m}_{n-1/2} (s) = s^{-(m+n+1/2)} (s^2 -1)^{m/2} _2F_1 [(m+n+3/2)/2, (m+n+1/2)/2, n+1, 1/s^2] \] (6.75)

\( P_0(s) \) gives the \( s \)-dependence. Expressing in terms of \( r \), the minor radius, taking the major radius \( R \) as unity,
\[ P_{0n} = A_{mn} r^{(m+n+1/2)} (1/r^2 -1)^{m/2} _2F_1 (a, \beta, \gamma; r^2) \] (6.76)
\[ P_{0n}(ks) = A_{mn} r^{(2m+n+1/2)} (1- r^2)^{m/2} _2F_1 (a, \beta, \gamma; r^2) \] (6.77)

Thus the electronic concentration can be given as,
\[ n_1 (r,t) = \sum A_{nm} r^{(2m+n+1/2)} (1-r^2)^{n/2} {}_2F_1 (\alpha, \beta, \gamma; r^2) \cos\psi e^{im\phi} e^{i\omega t} \]

for \( s < s_0 \)

\[ = 0 \]

for \( s > s_0 \)

(6.78)

This may be rewritten as,

\[ n_1 (r,t) = \sum A_{nm} P_{on} (k s) e^{i\omega t} e^{im\phi} \cos\psi \]

for \( s < s_0 \)

\[ = 0 \]

for \( s > s_0 \)

(6.79)

On solving equation (6.49) for the potential perturbation function \( \varphi_1 (r,t) \), one obtains for the interior region,

\[ \varphi^{\text{int}} (s,t) = \sum B s^n + A_1 P_{on} (ks)/k^2 \ e^{i\omega t} e^{im\phi} \cos\psi \]

(6.80)

Here, the coefficient \( A_1 = 4\pi e A l / \varepsilon \), \( \varepsilon \) is the dielectric constant of medium.

In the exterior the potential perturbation function is given by,

\[ \varphi^{\text{ext}} (s,t) = \sum C s^{-(n+1)} e^{i\omega t} e^{im\phi} \cos\psi \]

(6.81)

To get the dispersion relation for surface waves, we impose the condition of continuity of potential function \( \varphi \) and the first derivative of the potential function \( \partial / \partial s \varphi \) at the surface.

\[ B s^n + A_1 P_{on} (ks)/k^2 = C s^{-(n+1)} \]

(6.82)

At \( s = s_0 \)

\[ B s_0^n + A_1 P_{on} (ks_0)/k^2 = C s_0^{-(n+1)} \]

(6.83)

From the 2nd condition of continuity of partial derivatives at the boundary of the two dielectric materials with dielectric constant \( \varepsilon \) and bounded by nondispersive medium of dielectric constant \( \varepsilon' \)

\[ \varepsilon - \varepsilon_\perp \partial / \partial s ( B s^n + A_1 P_{on} (ks)/k^2 ) = \varepsilon' \partial / \partial s [ C s^{-(n+1)} ] \]

(6.84)

\[ \varepsilon_\parallel \partial / \partial s ( B s^n + \varepsilon_\perp A_1 P_{on} (ks)/k^2 ) = \varepsilon' \partial / \partial s [ C s^{-(n+1)} ] \]

(6.85)
Differentiating and applying boundary condition,
\[ \varepsilon_n \left( B_{so}^{n-1} \right) + \varepsilon A_1 P'_{on} \left( ks_0 \right) / k = -\varepsilon'(n+1) \left[ C s_0^{-\left(n+2\right)} \right] \]  (6.86)
\[ \varepsilon_n \left( B_{so}^{n-1} \right) + \varepsilon A_1 P'_{on} \left( ks_0 \right) / k = -\varepsilon'(n+1) s_0 B s_0^{n} + A_1 P_{on} \left( ks_0 \right) / k^2 \]  (6.87)

Solving for the special case of a metal- vacuum interface, the dielectric constants \( \varepsilon_n = \varepsilon_{n+1} \)
\[ n \left( B_{so}^{n-1} \right) + A_1 P'_{on} \left( ks_0 \right) / k = -(n+1) s_0^{-1} \left[ B s_0^{n} + A_1 P_{on} \left( ks_0 \right) / k^2 \right] \]
\[ = -(n+1) s_0^{-1} \left[ B s_0^{n} - A_1 P_{on} \left( ks_0 \right) / k^2 \right] \]  (6.88)
\[ (2n+1) B s_0^{n+1} + A_1 \left[ P_{on} \left( ks_0 \right) / k + (n+1) s_0^{-1} P_{on} \left( ks_0 \right) / k^2 \right] = 0 \]  (6.89)
\[ B = -\frac{A_1}{(2n+1) s_0^{n+1}} \left[ P_{on} \left( ks_0 \right) / k + (n+1) s_0^{-1} P_{on} \left( ks_0 \right) / k^2 \right] \]  (6.90)

Substituting the value of B and rewriting the potential perturbation in the interior of the torus,
\[ \varphi^i(s,t) = -\frac{A_1}{(2n+1) s_0^{n+1}} \left[ P_{on} \left( ks_0 \right) / k + (n+1) s_0^{-1} P_{on} \left( ks_0 \right) / k^2 \right] s^n + A_1 P_{on} \left( ks_0 \right) / k^2 \]  (6.91)

The hydrodynamic condition for surface wave to exist is that the normal component of velocity vanishes at the boundary \( s = s_0 \).
\[ i.e. \; v = e/m \nabla \varphi - \beta^2/n_0 \nabla n_1 = 0 \]  (6.92)

Substituting for electron density function and potential perturbation functions,
\[ e/m \left( \partial / \partial s \left( \varphi^{i}(s) \right) \right)_{s=s_0} = \beta^2/n_0 \left( \partial / \partial s \left( n_1 \right) \right)_{s=s_0} \]  (6.93)
\[ e/m(-A_1) \left[ P_{on} \left( ks_0 \right) / k + (n+1) s_0^{-1} P_{on} \left( ks_0 \right) / k^2 \right] n s_0^{n-1} + A_1 P_{on} \left( ks_0 \right) / k \]
\[ = \beta^2/n_0 A k P_{on} \left( ks_0 \right) \]  (6.94)
\[-\frac{\epsilon}{n^{m+n-1}} \left\{ \frac{(4\pi e n s_{0}^{n-1} [ P_{0m}(s_{0})/k + (n+1) s_{0}^{-1} P_{0m}(s_{0}) / k^2 ] + A_{1} P_{0m}(s_{0}) / k)}{e (2n+1)s_{0}^{n-1}} \right\}
\]

\[= \frac{\beta^2}{n_{0}} \frac{k P_{0m}(s_{0})}{k} \]

or, \[\omega_{p}^{2} \left\{ \frac{n}{(2n+1)} P_{0m}/k + n(n+1)/(2n+1) s_{0}^{-1} P_{0m}/k^2 \right\} = \beta^2 k P_{0m} \]

or, \[\omega_{p}^{2}/k^2 \left\{ k (n+1)/(2n+1) P_{0m} + n(n+1)/(2n+1) s_{0}^{-1} P_{0m} \right\} = \beta^2 k^2 / \omega_{p}^2 \frac{k P_{0m}}{k} \]

or, \[P_{0m}/k s_{0} = (2n+1)/n(n+1) \left\{ \beta^2 k^2 / \omega_{p}^2 - (n+1)/(2n+1) \right\} P_{0m} \]

or, \[1/k s_{0} P_{0m} / P_{0m} = (2n+1)/n(n+1) \left\{ \omega_{p}^2 - \omega_{p}^2/\omega_{p}^2 - (n+1)/(2n+1) \right\} \]

This is the dispersion relation.

Here, \[P_{0m}(s) = A_{mn} s^{-(m+n+1/2)} (s^2 - 1)^{m/2} _2 F_1(\alpha, \beta, \gamma, 1/s^2) \]

Then, \[P_{0m}' = \partial/\partial s P_{0m}(s) - A_{mn}(m+n+1/2) s^{-(m+n+1/2)-1} (s^2 - 1)^{m/2} _2 F_1(\alpha, \beta, \gamma, 1/s^2) \]

\[+ A_{mn} s^{-(m+n+1/2)} m/2 \times 2s (s^2 - 1)^{m/2-1} _2 F_1(\alpha, \beta, \gamma, 1/s^2) \]

\[+ A_{mn} s^{-(m+n+1/2)} (s^2 - 1)^{m/2-1} (\alpha \beta / \gamma ) F(\alpha + 1, \beta + 1, \gamma + 1; 1/s^2) \]

At the boundary, \[s = s_{0} \]

\[P_{0m}'(s_{0}) = - A_{mn}(m+n+1/2) s_{0}^{-(m+n+1/2)-1} (s_{0}^2 - 1)^{m/2} _2 F_1(\alpha, \beta, \gamma, 1/s_{0}^2) \]

\[+ A_{mn} s_{0}^{-(m+n+1/2)} m/2 \times 2s_0 (s_0^2 - 1)^{m/2-1} _2 F_1(\alpha, \beta, \gamma, 1/s_0^2) \]

\[+ A_{mn} s_{0}^{-(m+n+1/2)} (s_0^2 - 1)^{m/2-1} (\alpha \beta / \gamma ) F(\alpha + 1, \beta + 1, \gamma + 1; 1/s_0^2) \]

In order to obtain the dispersion relation, we try to estimate the zeros of the hypergeometric function, and its dependence on \(m, n\) and \(k a\) for fixed values of \(s\), the numerical work of finding the zeros of the hypergeometric function has been done by using the following algorithm which has been developed to solve the problem since the values of the function are not available directly. The following program has been developed by us to help in computing the results and achieving higher order of accuracy in the results.
function hyper (a,b,c,z)
imPLICIT real * B (a-h, o-z)
real * B hyper
term 2 = 1
hyp = 0.0
do in = 1,20
   i fail = 1
   gamma_a = s14aaf (a + in , i fail)
   i fail = 1
   gamma_b = s14aaf (b + in, i fail)
   i fail = 1
   gamma_c = s14aaf (c + in , i fail)
   term 1 = gamma_a*gamma_b /gamma_c
   term 2 = term2*z/in
   hyp = hyp + term1*term2
end do
hyper = hyp
return
end
implicit real * B (a-h, o-z)
real *B z

external s14aaf

write(6,*)' M N S PO PO'' w/wp w/wp sqr

do 10, m=0,2,1

do 10, n=0,4,1

do 10, s=1.1,3.5,0.1

z = 1.0/(s*s)

a = 0.5* (m+n+1+1.18)

b = 0.5* (m+n+1-1.18)

c = n+1

HypF = hyper (a,b,c,z)

t1 = ((s*s-1)**(m/2.0))

t2 = (s**(-(m+n+0.5)))0

PO = hypf * t1 * t2

t2p = ((m+n+0.5)) * t2) / s

t1p = m*s*(s*s-1)**(m*0.5-1)

Hypfp = hyper (a+1, b+1,c+1, z )*a*b / c

POp = - t2p * t1*hypf + t2*t1p*hypf + t2*t1*hypf

x = (PO / (s*POp))

if (x.LT.0.0) then

    write (6,9999) m, n, s, PO, POp, 0.0, (1/x)

else

    write (6,9999) m, n, s, PO, POp, x**(0.5), (1/x)

141
The solutions for arbitrary values of m and n i.e., the toroidal and poloidal mode numbers respectively, have a pronounced 3-D character. A toroidally uniform mode that is zero toroidal mode number possesses some distinct characteristics.

These solutions should have very interesting property that in the static limit they reduce to toroidal harmonics (ring functions), which are exact solutions of Laplace’s equation in toroidal coordinates. Toroidal coordinates are most elegantly suited to incorporate toroidal geometric effects. Thus the solutions obtained should incorporate toroidicity effects even in the zeroth order.

We have evaluated the dispersion relation for arbitrary toroidal and poloidal mode numbers by imposing the conditions that the field components should vanish at the surface of the torus. Some of the zeros of the solution with low poloidal and toroidal mode numbers have been estimated numerically by using the above algorithm. The dependence of the frequencies on the aspect ratio of the torus can also be studied. The results are as follows.
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We make the following observations from the tables:

For m = 0, i.e. toroidally uniform modes it is seen that when n = 0, i.e. poloidally uniform, the plasmon peaks are absent.

For n = 1, bulk plasmons appear, there are single as well as double plasmon modes.

For higher values of n i.e, n = 2, we start observing single plasmon modes.

For n = 3, we are finally able to observe surface plasmon modes for frequencies $\omega_p/\sqrt{2}$ and $\omega_p/\sqrt{3}$.

Hence we may conclude that for toroidally uniform modes we are able to observe bulk as well as surface plasmon modes.

When we move from toroidally uniform modes at m = 0 to other modes at m = 1, we make the following observations. For n = 1, we not only observe surface plasmon peak at $\omega_p/\sqrt{3}$, but in addition there appear other peaks also at $\omega_p/\sqrt{5}$, $\omega_p/\sqrt{9}$ etc.
As the values of \(m\) and \(n\) are arbitrarily increased the dispersion relation yields peaks of various kinds, \(\omega_p/\sqrt{10}, \omega_p/\sqrt{11}, \omega_p/\sqrt{13}, \omega_p/\sqrt{30}\) etc.

This mixture of modes appears to impart the complex behaviour to the plasma. An effort may be made to explain the occurrence of such a mixture. On the basis of some experimental and theoretical information available an attempt may be made to provide some explanation of the modes which occur in a toroid. The geometry of the surface indicates that the fields within and on the surface are complex.

Describing the conduction electrons by the Hydrodynamical equations of Bloch, it has been shown by Ritchie (26) that for very thin idealized foils, energy loss may occur at a value less than the plasma energy. This loss has been attributed to the finite thickness of the foil, and as the foil thickness decreases below \(\sqrt{\omega_p/\omega_p}\), the loss at the plasma energy becomes less than that predicted by the more conventional theories. The net result is an increase in the energy loss per unit thickness as the foil thickness is decreased. These predicted losses at subplasma energies correspond to some of the low lying energy losses which have been observed experimentally by using thin films.

Ritchie (26) has shown that the effect of the boundary is to cause a decrease in loss at the plasma frequency and an additional loss at \(\omega = \omega_p/\sqrt{2}\). The shift in the resonance frequency is due to the depolarizing effect of the surfaces of the foil. Jensen (65) has shown that the resonance frequency of plasma oscillations of a small sphere is less than the value appropriate to an infinite plane by a factor of \(1/\sqrt{3}\) and this shift is due to the depolarizing effect of the surface charge on the sphere.

An analogous effect occurs for the plane foil.

The sub-plasma frequency losses may be identified with the low lying losses observed by experimenters using thin foils (164). It is also seen that the observed values of the losses are not
1/\sqrt{2} times the 'Characteristic' losses observed in the same metals. It should be noted that thin metallic films have a strong granular structure. The strong variation of the grain structure with the substrate composition, rate and amount of condensation etc. of thin evaporated metallic films have been discussed by Heavens (187).

In this reference are given electron micrographs which clearly show that the transition from small grain size to a state in which the grains merge to form a nearly uniform films the amount of material deposited is increased in a series of films. The surface depolarizing effect will certainly be larger for a small grain of average dimension than for a semi infinite plane foil of thickness 'a'. Thus one would expect the 'lowered' losses in an actual foil to lie closer to the value \( \hbar \omega_p/ \sqrt{3} \), appropriate to a spherical grain, than to the value \( \hbar \omega_p/ \sqrt{2} \).

Metals or semiconductors exhibit a self-consistent electron density profile or some form of manipulated or depletes or accumulated layer. This may be represented as an inhomogeneous electron density in the vicinity of the surface. Surface waves on such dielectric media possessing some kind of surface structure are therefore of great interest. The earliest hydrodynamic model of inhomogeniety originates from Bennett (43) who used a collisionless jellium model to make a numerical study of a linear surface electron density profile on spatially dispersive electrostatic surface plasmon modes.

Eguiluz and Quinn (188) have studied the effect of the electron density profile at the surface of a metal on the surface plasma modes. It is found that for sufficiently diffuse surface, higher multipole excitations can exist in addition to the usual surface plasmons.

A hydrodynamic model based on a quasistatic generalization of the density function for is used and approximate analytical results of Bennett (43) and those obtained in RPA.
The analysis leads to the rather satisfying intuitive classification of multipole i.e. dipole, quadupole, octupole etc. charge fluctuations for the new modes, as distinguished from the regular surface plasmon (which is a monopole fluctuation). In the $q=0$ limit, frequency of the higher multipoles depends quite sensitively on the electron density profile. Some of the terms $\omega_p/\sqrt{2}$, $\omega_p/\sqrt{12}$, $\omega_p/\sqrt{30}$ etc may be explained on the basis of the multipole expansions.

Thus, on the basis of observations made by Ritchie (26), Egiluz (188) and Boardman (50) an attempt has been made to explain some of the modes which occur in the generalized dispersion relation for arbitrary values of toroidal and poloidal numbers and aspect ratio.

These higher surface modes discussed here could be very useful tool in the experimental study of the surface electron density, especially of chemisorbed species on metallic surfaces.