CHAPTER 1

INTRODUCTION, DEFINITION AND PRELIMINARY RESULTS
CHAPTER - 1

PART - A

INTRODUCTION

1.1:- History and Motivation

The abstract tendency in analysis which developed into what is today known as “Functional Analysis” began at the turn of 20\textsuperscript{th} century with the work of Volterra, Fredholm, Hilbert, Riesz and Banach. To develop the concept of distance in abstract linear spaces, the idea of ‘norm’ was first formulated by F. Riesz in 1918. However, an abstract and full treatment to the subject was given by S. Banach in his book [18] which was published in 1932. This book was tremendously influential and signified the beginning of the systematic study on normed linear spaces. In last five decades, the research activities in this area grew tremendously. Sequences and in general functions of a real or complex variable were treated as elements of some abstract linear spaces. With a suitable norm defined on these linear abstract spaces to make them into Normed Linear Space (NLS), in which idea of convergence and continuity have been developed, to make them into better and beautiful spaces, known as Banach spaces [56] which become foundation stone for today’s Analysis.
As a result Banach space theory gained very much depth as well as scope in many branches of mathematical sciences and engineering. However, not enough work seems to have been done dealing with the inter-play between functional analysis and the theory of analytic functions of a complex variable the reason might be the functional analysis techniques which are essentially of real variable character. But there are part of the theory of analytic functions which blend beautifully with the concept and methods of functional analysis which lend clarity and elegance to the proof of classical theorems and there by making the results available on more general setting. A testimony to these facts are the books of Hille and Phillips [22], Taylor [27], Hoffman [34, 95], Wilnasky [40, 56], Porcelli [43], Branges [47], Duren [55], Maddox [57], Hille [64], Rudin [82], Limaye [89], Koosis [92], Kreszig [96], Rudin [98], Wojtaszczyk [97], Schechter [110] and Lax [113] etc.

In the theory of functions of a complex variable, those functions which are representable in the form of a power series or a Dirichlet series play an important role, due to its representation. Many problems of science and engineering modeled as Differential or Integral equations have solutions in the form of a power series or a Dirichlet series for the solution function analytic in a domain. So the class of such functions needs to be treated separately. Many mathematicians deal with such type of classes of functions
and study their properties as a normed linear spaces (NLS) or Banach spaces, ours is another attempt in this direction. In this work we have studied the various spaces and algebras of functions representable in the form of a Dirichlet series and their important properties.

1.2:- General Dirichlet Series

A series of the form

\[ \sum_{k=1}^{\infty} a_k e^{s \lambda_k} \]  \hspace{2cm} (1.2.1)

Where \( s = \sigma + it \) (\( \sigma, t \) reals) a complex variable, \( \{a_n\} \) is a complex sequence and \( \{\lambda_k\} \) is a sequence of strictly positive real numbers such that

\[ 0 \leq \lambda_1 < \lambda_2 < \lambda_3 ... ... ... < \lambda_k \rightarrow \infty \text{ as } k \rightarrow \infty \]  \hspace{2cm} (1.2.2)

is called a Dirichlet series [61, 62]. This Dirichlet series in its most original form \( \sum_{k=1}^{\infty} a_k k^{-s} \) was first introduced by Dirichlet for his studies in the number theory, is known as classical or ordinary Dirichlet series. Dirichlet and Dedekind considered the series only with real values of variable 's' and obtained many important results, contributing in the theory of numbers. The well-known zeta function \( \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \) and power series \( \sum_{k=1}^{\infty} a_k z^k \) are the particular cases of the Dirichlet series (1.2.1).
The first result involving the complex values of ‘s’ were obtained by E.Cahen in 1894, who determine the region of convergence of the generalized Dirichlet series (1.2.1). Later Littlewood [1] succeeded in showing that Dirichlet series (D.S.) was useful in the study of entire function, while Esterman [3] used D.S. in the study of meromorphic function and Doetsch [2] was first to study the growth aspect of a Dirichlet Series. Apostol [106] used the properties of D.S. to establish many result of prime number theory.

Further the theory of Dirichlet series were developed by the significant contribution made by Izumi [4], Sugimura [5], Mendelbrojt [7] and Hardy and Riesz [62]. But a vast enrichment to this field with new and fruitful ideas come in the wake of the works of Tanaka [16], Rahman [19, 21], Azpeitia [33, 39], Dagene [49] Kamthan and Gautam [63, 66, 67, 72, 73] etc. Some recent contribution for further development of the theory of the Dirichlet series were made by Srivastava [91], Juneja and Kapoor [93], Srivastava and Kumar [104], Sova [108] and Bayart [115, 116] etc. Very recently, a few years back, the idea of vector valued Dirichlet series [VVDS] was introduced by Srivastava [91].

In fact in [91], he defined a vector valued Dirichlet series [VVDS] to be a series of type (1.2.1) in which the coefficient sequence \( \{a_k\} \) were chosen from a Banach space. He study VVDS for its various type of convergence, like weak
convergence, unconditional weak convergence, absolute and uniform convergence etc and establish a relationship among them. The work on WVDS was further extended by Srivastava & Sharma in [133].

Below we are going to give a brief theory of Dirichlet series, which are relevant to our further study. To have analogy with power series, we shall throughout our work consider the Dirichlet series with positive exponents rather than negative ones which usually appears in the mathematics literature and will take the liberty of interpreting the result of all those researches who have considered Dirichlet series with negative exponents in our terminology.

Consider the Dirichlet series, as described in (1.2.1), given below

\[ f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \]

About this series, it is known that if it converges at some point say \( s = s_o = \sigma_o + it_o \) of the complex plane so does it uniformly in the region of left half plane \( \sigma < \sigma_o \) and sum of the series \( f(s) \), is an holomorphic function [78] in its region of convergence \( \sigma < \sigma_o \).

The abscissa of ordinary convergence (\( \sigma_c \)) and absolute convergence (\( \sigma_a \)) of the series (1.2.1) are given as follows, see [69].

\[ \sigma_c = \sup\{\sigma \in \mathbb{R} : \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \text{ converges for } s = \sigma + it\} \]
The vertical line $\sigma = \sigma_c$ is called line of convergence. The left half plane $\sigma < \sigma_c$ of convergence is usually called the region of convergence of the series. There are two formulas to compute $\sigma_c$, depending on the convergence of the series $\sum_{k=1}^{\infty} a_k$, see [106]. If $\sum_{k=1}^{\infty} a_k$ is diverges or $\sigma_c \leq 0$ then it is given by,

$$\sigma_c = -\lim_{k \to \infty} \sup \frac{\log |\sum_{j=1}^{k} a_j|}{\lambda_k}$$

If $\sum_{k=1}^{\infty} a_k$ is convergent or $\sigma_c \geq 0$ then it is given by,

$$\sigma_c = -\lim_{k \to \infty} \sup \frac{\log |\sum_{j=k+1}^{\infty} a_j|}{\lambda_k}$$

The abscissa of absolute convergence ($\sigma_a$) can similarly be defined as follows,

$$\sigma_a = \sup \left\{ \sigma \in \mathbb{R} : \sum_{k=1}^{\infty} a_k e^{\sigma \lambda_k} \text{ converges absolutely for } s = \sigma + it \right\}$$

and it is given by the formulas [106],

$$\sigma_a = -\lim_{k \to \infty} \sup \frac{\log |\sum_{j=1}^{k} a_j|}{\lambda_k} \text{ if } \sum |a_k| \text{ is divergent and}$$

$$= -\lim_{k \to \infty} \sup \frac{\log |\sum_{j=k+1}^{\infty} a_j|}{\lambda_k} \text{ if } \sum |a_k| \text{ is convergent.}$$

In general $\sigma_c \neq \sigma_a$, so there may be a strip between the line of convergence and absolute convergence where the D.S. converges conditionally. The width of this strip is estimated by the inequality
0 \leq \sigma_c - \sigma_a \leq \lim_{k \to \infty} \sup \left( \frac{\log k}{\lambda_k} \right) = D \hspace{1cm} \text{(1.2.3)}

With \sigma_c and \sigma_a described as above.

Therefore if D is \textit{finite} and \sigma_c = \infty, \ f(s) represents an \textit{entire} function and by (1.2.3) \sigma_a = \infty so that the series (1.2.1) converges absolutely at every point of the finite complex plane.

Further if D = 0 then clearly abscissa of absolute convergence (\sigma_a) coincides with abscissa of ordinary convergence (\sigma_c) and it is given by a simple formula

\[ \sigma_c = \sigma_a = -\lim_{k \to \infty} \sup \frac{\log |a_k|}{\lambda_k} \hspace{1cm} \text{(1.2.4)} \]

The coefficients of the series (1.2.1) can be expressed in terms of its sum function f(s) by Hadmard’s formula,

\[ a_k = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(\sigma + it)e^{-\lambda_k(\sigma + it)} dt \]

Where \( \sigma < \sigma_c \) [77].

Mandelbrojt [61] has also proved that there are constants \( d, c_1, c_2 \) depending only on \{\lambda_k\} satisfying

\[ |a_k| < c_2 M(s_1)e^{\lambda_k(c_1 - \sigma_1)} \text{ for each } k \hspace{1cm} \text{(1.2.5)} \]

Where \( M(s_1) = \text{lub}_{|s - s_1| \leq d}|f(s)| \) for arbitrary \( s_1 \in \mathbb{C} \) (set of complex numbers).
1.3: Spaces and Algebras of Entire and Analytic Functions

Iyer in 1948, introduced, on the linear space $\Gamma$, of all entire functions, a topology, in which convergence was equivalent to uniform convergence on compact sets. Iyer in a series of papers [9, 11, 13, 20, 26] obtained a number of useful results for the space $\Gamma$, including those on the bases etc.

Iyer's concept of proper bases was modified by Arsove [23, 24] who obtained a characterization of such bases in the space $\Gamma$. Markushevich [35], on the other hand considered the class $F_R$ of all single valued analytic functions in the disk $|z| < R$ (R is finite) and defined an invariant metric on it, so that $F_R$ becomes a linear metric space.

Krishna Murthy [31] made a systematic study of proper bases, continuous linear functionals, continuous linear transformations etc on different subspaces of entire functions. Ekblaw [70] studied the several subspaces of a space of entire functions for their many important properties. Sisarcick [75] gave a characterization for scalar homomorphism and numerically bounded linear functionals on the space $\Gamma$. Because of the vast importance of the space $\Gamma$, Patwardhan [85] studied its bornological aspects. P.D. Srivastava in his dissertation [84] considered a linear metric space $(X, d)$ with a schauder base which includes a number of known results on different
spaces, as a special case, the known results on different spaces and subspaces of entire functions considered by earlier workers.

The study of the spaces and subspaces of entire functions represented by Dirichlet series was initiated by H. Hussain and Kamthan [48] and pursued by Kamthan and Gautam [66, 67, 72, 73] in a series of papers. In fact this study runs parallel to the study of Iyer, Arsove and Krishna Murthy for the corresponding spaces of entire functions defined by Taylor series.

We shall state below some of their results since they are relevant to our subsequent studies. Let \( X \) be the space of all function \( f \), given by Dirichlet series of the form,

\[
f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k}, s = \sigma + it \quad (\sigma, t \text{ reals})
\]

and \( \{ \lambda_k \} \) is a fixed sequence of positive real numbers satisfying the conditions

\[
0 < \lambda_1 < \lambda_2 \ldots \ldots < \lambda_k < \lambda_{k+1} \to \infty \quad \text{as} \quad (k \to \infty)
\]

with

\[
\lim_{k \to \infty} \sup \frac{\log k}{\lambda_k} = D = 0
\]

and \( \{ a_n \} \) a sequence in \( \mathbb{C} \) (set of complex numbers), satisfying the condition

\[
\sigma_c = \sigma_a = - \lim_{k \to \infty} \sup \frac{\log |a_k|}{\lambda_k} = \infty
\]
From (1.3.3), it is clear that $X$ consists of entire functions only. It is easy to see that $X$ is a linear space with respect to (w.r.t.) the usual addition of the function and multiplication by a complex number. For $f \in X$ and $\sigma \in \mathbb{R}$ (the set of real numbers), let

$$M(\sigma, f) = \sup_{|t|<\infty} |f(\sigma + it)|$$ \hspace{1cm} (1.3.4)

Then \{\(M(\sigma_1, \sigma_2, \ldots, f), \sigma \in \mathbb{R}\)\} defines a family of semi norms (in fact norms) on $X$. If $T_1$ be the locally-convex hausdorff topology on $X$, generated by this family of semi-norms, then it can be easily seen that this topology is equivalent to the topology generated by the metric

$$d_1(f, g) = \sum_{k=1}^{\infty} \frac{M(\sigma_k, f - g)}{1 + M(\sigma_k, f - g)}$$

Where $\sigma_1 < \sigma_2 < \sigma_3 < \ldots < \sigma_k \to \infty$ as $k \to \infty$ and $f, g \in X$

Now for each $f \in X$, define,

$$\rho(\sigma, f) = \sum_{k=1}^{\infty} |a_k|e^{\sigma \lambda_k}$$

Then \{\(\rho(\sigma_1, \sigma_2, \ldots, f), \sigma \in \mathbb{R}\)\} again defines a family of semi-norms (indeed norms) on $X$. Which generates another locally convex hausdorff topology $T_2$ on $X$, which can also be given by an invariant metric $d_2$ defined by
\[ d_2(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\rho(\sigma_k, f - g)}{1 + \rho(\sigma_k, f - g)} \]

Where \( \sigma_1 < \sigma_2 < \sigma_3 \ldots \ldots < \sigma_k \to \infty \text{ as } k \to \infty \).

Because of the spaces \((X, T_1)\) and \((X, T_2)\) being non-normable, attempts have been made to define norm topologies on certain subspaces of \(X\). Thus R.K. Srivastava [80] have studied a class \( \mathcal{D} \) given as,

\[ \mathcal{D} = \left\{ f, f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k} : |a_k| e^{k \lambda_k} < \infty, k \geq 1 \right\} \]

This class \( \mathcal{D} \), with point wise linear operations and scalar multiplication and norm defined by the following way

\[ \|f\| = \sup |a_k| e^{k \lambda_k} \]

becomes a Banach space. He also characterized the multiplier for \( H^p \) and \( l_p \) into \( \mathcal{D} \) in his work [81].

The study of spaces and subspaces of analytic function in a half plane represented by Dirichlet series was initiated by M.G. Khaplanov in his work [30, 77]. In fact Khaplanov in this work has considered the space of analytic functions represented by Dirichlet series of the form,
\[ f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}, s = \sigma + it \ (\sigma, t \text{ reals}) \] where the exponents \( \lambda_k \) are strictly increasing sequence of real numbers which tend to infinity as \( k \to \infty \). That is

\[ 0 < \lambda_1 < \lambda_2 \ldots \ldots < \lambda_k \to \infty \ (k \to \infty) \]

and satisfy the condition \( \lim_{k \to \infty} \frac{\log k}{\lambda_k} = 0 \)

It is known that in this case the Dirichlet series has no strip of the conditional convergence and that the abscissa of convergence \( \sigma_c \) is defined by

\[ \limsup_{k \to \infty} \frac{1}{\lambda_k} = e^{-\sigma_c} \]

Which is equivalently expressed in (1.2.4). The series converges uniformly in each half-plane \( \sigma < \sigma_c - \varepsilon, \ \varepsilon > 0 \), hence, the sum function \( f(s) \) of the series is an analytic function in the plane \( \sigma < \sigma_c \) and it is an entire function if \( \sigma_c = \infty \).

Let \( v_{\sigma_c} \) and \( \overline{v}_{\sigma_c} \) be the spaces of analytic function in the half planes \( \sigma < \sigma_c \) and \( \sigma < \sigma' (\sigma' > \sigma_c) \) respectively. Further if \( \mathcal{D}_{\sigma_c} \) and \( \overline{\mathcal{D}}_{\sigma_c} \) denote the spaces of sequences \( \{\xi_1, \xi_2, \xi_3 \ldots \ldots \xi_k \ldots \ldots \} \) satisfying the condition

\[ (i) \lim_{k \to \infty} \sup |\xi_k|^{\frac{1}{\lambda_k}} \leq e^{\sigma_c} \text{ and} \]

\[ (ii) \lim_{k \to \infty} \sup |\xi_k|^{\frac{1}{\lambda_k}} < e^{\sigma_c} \text{ respectively,} \]
then it can easily be seen that all these spaces are linear spaces and $\nu_{\sigma_c}$ and $D_{\sigma_c}$ (also $\nu_{\sigma_c}$ & $\overline{D}_{\sigma_c}$) are algebraically isomorphic.

He has also shown that by defining suitable topologies on these spaces they can be made topologically isomorphic too.

Juneja and Srivastava [88] considered a class $\Omega_o$ of all those functions $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ in which $\left|\frac{a_k}{\alpha_k}\right| \to 0$ for $\{\alpha_k\}$ being a sequence of non-zero complex number satisfying certain conditions.

They have shown that $\Omega_o$ is a non-uniformly convex Banach space which is separable also. They have also characterize the matrix transformation from $\Omega_o$ to itself.

In the year 2001, a deep study of the space of Dirichlet series was made by F. Bayart [112]. In this paper he investigated new spaces of Dirichlet series and their composition operators.

Growth study of Dirichlet series of two complex variable and space of analytic functions represented by Dirichlet series of two complex variables has been initiated by H.S. Behnam and G.S. Srivastava in his paper [107] in 1998.

Very recently a space $\Omega_u$, defined as follows;
\[ \Omega_u = \left\{ f, f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k} : \frac{a_k}{\alpha_k} \text{ is bounded} \right\} \]

Where \( u(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k} \) being a fixed Dirichlet series with none of the coefficients \( a_k \) being zero, have been studied by Srivastava and Abhay Raj [127] for its various properties. In fact this work has been a part of this study and it is included in chapter 2, of the present thesis.

Attempts have also been made to study the spaces of functions

\[ F, F(s) = \int_0^{\infty} f(t)e^{st} dt \]

converges, being a Laplace transformation of a real-valued function \( f(t) \). But a very few works seems to have been done in this direction see [100]. Theory of Laplace transformation which is continuous analogue of Dirichlet series, may be found to work with in [6, 32, 58, 100] etc.

vector-valued form of Laplace transformation which is in fact continuous form of vector-valued Dirichlet series [118] may be found in [117].

**1.4:- Hilbert spaces of Dirichlet series**

Classes of functions, represented by Dirichlet series have been considered also as Hilbert spaces. Hardy like spaces of ordinary Dirichlet series, called Dirichlet Hardy spaces \( H_p \) \( (p \geq 1) \) have been the focus of increasing interest among researchers, specially the space \( H_2 \). Thus H. Hedenmalm [105] consider a Hilbert space.
And find dialates as Riesz basis and the translates of Riemann-Zeta functions as the reproducing kernels. F. Bayart [115] studied the Hardy spaces of Dirichlet series and their composition operators and in [116] he further described the compact composition operators on some Hilbert Spaces of Dirichlet series while J.E. McCarthy [121] has considered a weighted Hilbert space of Dirichlet series and found the multipliers operators on this space. Local interpolation property in the Hilbert spaces of Dirichlet series have been the subject of research of J.F. Olsen and K. Seip [128], while Olsen and Saksman in a recent paper [130] have studied some local properties of functions in the Hilbert spaces of Dirichlet series.

1.5:- **Algebras of Entire and Analytic Functions**

Attempts have also been made by mathematicians to embed the space of entire functions or its subspaces with additional algebraic structures. Thus Iyer [20] defines multiplication in space $\Gamma$ by two ways and showed that it is a commutative topological algebra w.r.t. both of these multiplication. Henriksen [14, 15] worked out ideal structure in the space $\Gamma$. Sen [54] considered subspace.
\[ R_1^0 = \left\{ f, f(z) = \sum_{n=1}^{\infty} a_n z^n : n! |a_n| \text{ bounded} \right\} \]

of \( R \) and defined a multiplication in it, thus making it a commutative Banach algebra with identity \( e^z \). Field and Sisarcick [74] made a deeper study of \( R_1^0 \) and characterize scalar homomorphism in it.

Analogously, subspaces of entire Dirichlet series have also been studied by Chakraborthy [76]. In fact he introduced a class \( \Omega \) of all Dirichlet series \( \sum_{n=1}^{\infty} a_n e^{s \lambda_n} \) having the exponents \( \{\lambda_n\} \) which satisfies \( 0 < \lambda_1 < \lambda_2 \ldots \lambda_n < \lambda_{n+1} \to \infty \) \( (n \to \infty) \). After defining addition (\( + \)) and multiplication by scalars in \( \Omega \) in the usual way, he also defined (\( \ast \)) multiplication in \( \Omega \) and showed that \( \Omega \) is a commutative Banach algebra without identity. Srivastava [118] has generalized the results of Chakraborthy [76] for a class of vector-valued Dirichlet series (VVDS).

Further interesting results studying the various aspects of theory of Banach algebra, B*-algebra, Frechet algebra and related topics may be found in [65], [71], [83], [86], and [90].

1.6:- Sequence Spaces

Many of the spaces of entire or analytic functions defined by power series or Dirichlet series may be thought of as sequence space since the
topologies in these spaces are being generated with the help of the sequences of the coefficients in the series defining the functions. Thus, any development in the theory of sequence spaces has deeply affected the study of spaces of analytic functions and vice-versa.

Today the theory of sequence spaces which has its origin in the early work of Hardy [10], Cooke [12] stands as separate discipline to work with. The books by Kothe [51], Maddox [57], Lindenstrauss and Tzafrin [79] and the research monograph of Kamthan and Gupta [87] bear an eloquent testimony to this fact.

Matrix transformation on several known spaces of functions have been a subject of central investigation by many mathematician like Brundo [8], Ramanujan [41], Raphael [50] and Rao [53] etc. Mention must also be made of the special work of Maddox [57], who in a series of papers has carefully studied the matrix transformation on sequence spaces. Lascarides [60] carried over the work of Maddox and studied various other aspects of sequence spaces like $C_0(p), l_\infty(p), l(p)$ etc of Maddox. Maddox [57] further studied the space $W_p$ of all strongly Cesaro summable complex sequences of order 1 and index p, about which he observed that $W_p$ is a complete normed linear space if $p \geq 1$, while for $0 < p < 1$, it is a complete p-normed space.
Hence, we see that the study of spaces and algebras of holomorphic functions has been vastly enriched in different direction by a number of mathematicians who have made important and significant contribution to it, by their fruitful and new ideas. However in this present introductory chapter we confined ourselves to only those aspects of the theory in which we have attempted to make our contribution. Thus, keeping in view the set trend in mathematics, of generalization of ideas, which play prominent role, with regard to solution of many problems in more general setting, especially in analysis, the present thesis is devoted to the study of spaces and algebras of holomorphic functions representable by Dirichlet series.

The present thesis consists of six chapters including the first introductory chapter.

In chapter 2, we study a Banach space of a class of Dirichlet series $\Omega_u$, which consists of all those function $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k}$ being a Dirichlet series such that $\{\frac{a_k}{\lambda_k}\}$ is bounded where $u(s) = \sum_{k=1}^{\infty} \alpha_k e^{s \lambda_k}$ is a given fixed Dirichlet series with none of the $\alpha_k$ being zero. $\Omega_u$ has been shown in theorem 2.3.1 & 2.3.2 that it is a non-uniformely convex Banach space which is non-separable also. Further it can not be converted into a Hilbert space, has been proved in section 2.4.2. Another class $\Omega_u(1)$ of function
\[ f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k} \] such that \( \sum |a_k b_k| < \infty \) has also been introduced and it was found that \( \Omega^*_u \) (dual of \( \Omega_u \)) contain a proper subspace which is linearly isometric to \( \Omega_{u(1)} \) (theorem 2.5.1). In order to obtain all the continuous linear transformation on the space \( \Omega_u \), the matrix transformation on \( \Omega_u \) has been defined and bounded linear transformation on \( \Omega_u \) has been characterized in theorem 2.6.1. All the bounded linear transformation from the space \( \Omega_u \) to \( \Omega_0 \) has been obtained in theorem 2.6.2. Finally in the last section an application of solving an operator equation for an unique solution in the space \( \Omega_u \) has been done. Contents of this chapter has been published in JPAS Vol (14) 2008, 106-113.

In chapter 3, a class \( \Omega^2_u \) of function represented by Dirichlet series

\[ f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k} \] with \( \sum |a_k|^2 < \infty \) has been studied as an Hilbert space. In theorem 3.1.1, the class \( \Omega^2_u \) has been shown to be an Hilbert space under a suitable inner product. In section 3.2, an orthonormal basis in \( \Omega^2_u \) and a direct sum of it were obtained. Isometric image of \( \Omega^2_u \) has been obtained in theorem 3.3.1. Bounded linear compact operators on \( \Omega^2_u \) has been characterized in theorem 3.4.1. Multiplier class \( (\Omega^2_u, \Omega^2_0) \) has been obtained in theorem (3.5.1) of section 3.5.
Chapter 4 deals with some special operators on the Hilbert space $\Omega^2_u$. A diagonal operator $T$ has been introduced in article 4.2 and its conjugate $T^*$ has been obtained. The norm of the diagonal operator $T$ has been calculated in theorem 4.2.1. Theorem 4.2.2 gives respectively condition under which this operator becomes self-adjoint, positive and unitary. Condition of being compact operator, Hilbert Schmidt operator, and Nuclear operator has also been obtained in Theorem 4.2.3. Spectrum of the operator has been studied in theorem 4.3.1. Orthonormal projection and resolution of Identity has been dealt in article 4.4. Some application of solving few operator equations in $\Omega^2_u$ has been made in section 4.5. Contents of this chapter has been accepted for publication in BPAM Vol (5) Nov/Dec 2011.

In chapter 5, we study $\Omega_u$ as a Banach algebra under a suitable product. In Lemma 5.2.1, $\Omega_u$ has been shown to be a Banach algebra with identity ‘u’ $u(s) = \sum_{k=1}^{\infty} \alpha_k e^{s\lambda_k}$ being the fixed Dirichlet series. Characterization of regular/singular elements and topological zero divisor (TZD) has been obtained in theorem 5.3.1 and 5.3.2 respectively. Spectrum $\sigma(f)$ of an element $f$ belonging to the Banach-algebra $\Omega_u$, has been obtained in theorem 5.3.3. Spectral radius $r(f) = \|f\|$ has been established in theorem 5.3.4. $\Omega_u$ is not division algebra has been proved in theorem 5.4.1. Two ideals $I_1$ and $I_2$ of $\Omega_u$ were obtained such that $\Omega_u = I_1 \oplus I_2$ in theorem 5.5.1. Theorem 5.6.1 study
\( \Omega_u \) as a B*-algebra. Characterization of self adjoint, unitary and normal element in \( \Omega_u \) were made in theorem 5.6.2. Theorem 5.7.1 deals with the *-homomorphism on \( \Omega_u \). In theorem 5.8.1, \( \Omega_u \) has been studied as a Banach Lattice under an order relation \((\prec)\). Further \((\Omega_u, \prec)\) has been shown to be an l-ring in theorem 5.8.2. Theorem 5.8.4 shows that \((\Omega_u, \prec)\) is an AM-space while theorem 5.8.5 proves that it is an Abstract Lebesgue-space (AL-space).

Chapter 6 is a brief study of certain sub-algebras \( \Omega_o \) and \( \Omega_1 \) of \( \Omega_u \). For these sub-algebras which are without identity characterization of quasi-regular / singular elements has been obtained in theorem 6.3.2. Every element of the two subalgebras \( \Omega_o \) and \( \Omega_1 \) has been shown to be topological zero divisor (TZD) in theorem 6.3.4. In theorem 6.4.1, \( \Omega_o \) has been shown to be a B*-algebra under an involution \((\ast)\). While \( \Omega_1 \) is not a B*-algebra under this involution has been given in theorem 6.4.2. In theorem 6.4.4, \( \Omega_o \) and \( \Omega_1 \) has been shown to be symmetric subset of \( \Omega_u \). Theorem 6.5.1 study \( \Omega_1 \) as a two norm space. Concluding last two theorems 6.5.4 and 6.5.5 shows that two norm space \( \Omega_1 \) is a Sack space.
Part- B

In this part, we collect certain definitions and preliminary results concerned with Dirichlet Series, Banach space, Hilbert space, operators’ theory, Banach algebra and Banach lattice and their properties which will be used latter in our work.

Definitions

1.1 [61]:- A series of the form \( \sum_{n=1}^{\infty} a_n e^{s \lambda_n} \) is called a Dirichlet series where \( \{a_n\} \) is a complex sequence, \( s = \sigma + it \) (\( \sigma, t \) are real variable) is the complex variable and \( \{\lambda_n\} \) is an increasing sequence non-negative real numbers such that

\[
0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \ldots \ldots \ldots \ldots \ldots \ldots \lambda_n < \lambda_{n+1} \to \infty \text{ as } n \to \infty
\]

1.2 [94, p-100]:- A linear (vector) space \( X \) is said to be a normed linear space if to every \( x \in X \) there is associated a non-negative real number \( \|x\| \), called the norm of \( x \), in such a way that –

(i) \( \|x\| \geq 0 \) and \( \|x\| = 0 \iff x = 0 \),

(ii) \( \|x + y\| \leq \|x\| + \|y\| \) for all \( x \) and \( y \) in \( X \),

(iii) \( \|ax\| = |a|\|x\| \) if \( x \in X \) and \( a \in \mathbb{C} \)
Every Normed Linear Space (NLS) is always a metric space with the metric defined as

\[ d(x, y) = \|x - y\| \quad \text{for all } x, y \in X \]

**Remark: 1.1 [96, p-21]** - A metric space is separable if it has a countable dense subset in it.

**1.3 Cauchy's Sequence in a NLS [111, p-71]** - Let \( X \) be a normed linear space (NLS). A sequence \( \{x_n\} \) in \( X \) is called a Cauchy's Sequence if for any given \( \epsilon > 0 \) there exist a positive integer \( N \) such that-

\[ \|x_m - x_n\| < \epsilon \quad \text{for } m, n \geq N \]

**1.4 Convergent Sequence [96, p-67]** - A sequence \( \{x_n\} \) of a NLS \( X \) is said to be a Convergent Sequence if there exist a \( x_0 \in X \) such that-

\[ \|x_n - x_0\| \to 0 \text{ as } n \to \infty \]

A NLS \( X \) is called Complete if every Cauchy's Sequence in it is Convergent.

A Banach Space is a normed linear space which is complete as a metric space. Equivalently, a NLS \( X \) is a Banach space if every Cauchy's Sequence in it is Convergent.
1.5 (a) **Schauder Basis** [96, p-68] - A sequence \( \{e_n\} \) of a normed linear space \( X \) is called a *Schauder basis* if for every \( x \in X \) there exists an unique sequence of scalars \( \{\alpha_n\} \) such that

\[
\left\| x - \sum_{k=1}^{n} \alpha_k e_k \right\| \to 0 \quad \text{as} \quad n \to \infty
\]

Further if a normed space \( X \) has a *Schauder basis*, then \( X \) is *separable*.

(b) [96, p-55] - A subset \( B \) of a normed linear space (in fact linear space) \( X \) is called a *Hamel basis* (or simply *basis*) if

(i) \( B \) is linearly independent and

(ii) \( \text{span } (B) = X \), and in this case every element of \( X \) has a unique representation as a linear combination of the elements of \( B \) with non-zero scalars and coefficients.

1.6 [51, p-6] - A set \( G \) of a NLS \( X \) is said to be *Convex* set if

\[
aG + (1 - a)G \subseteq G \quad (0 \leq a \leq 1)
\]

A set \( H \subseteq X \) is said to be *balanced* if \( aH \subseteq H \) for every \( a \in \mathbb{C} \) with \( |a| \leq 1 \).

1.7 [82, p-24] - A convex set \( G \) of \( X \) is absorbing in the sense that every \( x \in X \) lies in \( tG \) for some.
1.8 [82, p-7] - If a topology $\tau$ on a linear space $X$ is induced by a metric $d$, we say that $d$ and $\tau$ are *compatible* and space is metrizable.

1.9 [82, p-8] - A metric $d$ on a vector space $X$ is called *invariant* if

$$d(x + z, y + z) = d(x, y) \text{ for all } x, y, z \in X$$

1.10 [82, p-7] - Suppose $\tau$ is a topology on a vector space $X$ such that

(a) Every point of $X$ is closed set, and

(b) The vector space operations are continuous with respect to $\tau$.

Under these conditions, $\tau$ is said to be a vector- Topology on $X$, and $X$ is a Topological Vector Space written in short as TVS.

**Remark: 1.2** - Condition (a) and (b) together imply that $\tau$ is a Hausdorff topology on $X$.

1.11 [82, p-24] - A semi-norm on a vector space $X$ is a real valued function $p$ on $X$ such that

(a) $p(x + y) \leq p(x) + p(y)$

(b) $p(ax) = |a|p(x)$ for all $x$ and $y$ in $X$ and $a \in \mathbb{C}$.

A family $p$ of seminorms on $X$ is said to be separating if to each $x \neq 0$, there corresponds at least one $p \in P$ with $p(x) \neq 0$. 
Remark: 1.3 [51] - In every locally convex space there exists a separating family of continuous seminorm. Conversely, if $p$ is a separating family of seminorm on a vector pace $X$, then $p$ can be used to define a locally convex topology on $X$ with the property that every $p \in P$ is continuous.

1.12 [98, p-55] - The dual space of a TVS $X$ is a vector space $X^*$ whose elements are the continuous linear functional on $X$.

1.13 [51, p-353] - A normed space $(X, \| \cdot \|)$ is said to be uniformly convex if for each $\varepsilon$ with $0 < \varepsilon \leq 2$ there exist $a\delta(\varepsilon) > 0$ for which it always follows from $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ that $\left\| \frac{1}{2}(x + y) \right\| \leq 1 - \delta(\varepsilon)$

Theorem: 1.1 [51, p-353] - A normed space $(X, \| \cdot \|)$ is uniformly convex if and only if it always follows from $\|x_n\| \leq 1, \|y_n\| \leq 1$ and $\lim_{n \to \infty} \left\| \frac{1}{2}(x_n + y_n) \right\| = 1$ that $\lim_{n \to \infty} \left\| (x_n - y_n) \right\| = 0$

1.14 [124, p-88] - Let $X$ be a normed linear space over the field $\mathbb{C}$ (complex numbers field). A mapping $(;): X \times X \to \mathbb{C}$ which takes each ordered pair $(x, y) \in X \times X$ into the number $(x, y) \in \mathbb{C}$ is called an inner-product in $X$ if it satisfies,

\begin{align*}
(i) & \quad (x, y) = (\overline{y}, x) \\
(ii) & \quad (x + y, z) = (x, z) + (y, z)
\end{align*}
(iii) \((ax, y) = \alpha(x, y)\)

(iv) \((x, x) \geq 0 \text{ and } (x, x) = 0 \text{ if and only if } x = 0\)

If we define \((x, x) = \|x\|^2\) then \(X\) becomes a Normed linear space (NLS).

Further if, \(X\) is complete under the norm obtained from the inner-product, then \(X\) is called a Hilbert Space.

In other word a complex Banach space is a Hilbert space in which an inner product is defined.

**Theorem: 1.2 [126, p-348]** - (Cauchy-Schwarz inequality). Let \(X\) be a Hilbert space then for every \(x, y \in X\),

\[ |(x, y)| \leq \|x\| \|y\| \]

**Remark: 1.4 [89, p-176]** - The angle between any two non-zero vectors \(x, y\) is given by the formula

\[ \cos \theta = \frac{|(x, y)|}{\|x\| \|y\|}, \quad 0 \leq \theta \leq \pi \]

In particular, orthogonality \((\theta = \frac{\pi}{2})\) of \(x\) and \(y\) is characterized by \((x, y) = 0\)

**Theorem: 1.3 [103, p-218]** - The Pythagorean theorem in a Hilbert space \(H\) can be stated as follows, \(x \perp y\) implies \(\|x\|^2 + \|y\|^2 = \|x + y\|^2\)
Theorem: 1.4 [40, p-124] – Let $X$ be a Hilbert space, then for every $x, y \in X$ we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

This is known as parallelogram law.

Theorem: 1.5 [125, p-88] – A Banach space is a Hilbert space if and only if the parallelogram law holds in it.

1.15 [52, p-220] – Let $X, Y$ be normed linear spaces. A mapping $T : X \to Y$ which assigns to each element $x$ of $X$ a unique element $y \in Y$ is called an operator (or transformation). The operator $T$ is called linear if

(a) $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha T(x_2)$ for all scalars $\alpha_1, \alpha_2$ and all elements $x_1, x_2 \in X$

(b) Range space of $T$ $R(T) = \{y \in Y : Tx = y \text{ for some } x \in X\}$.

(c) Null space of $T$ $N(T) = \{x \in X : Tx = 0\}$.

Remark: 1.5 [96, p-91] – An operator $T$ on a NLS $X$ is called bounded if there is a constant $M$ such that

$$\|Tx\| \leq M\|x\|, \quad x \in X$$

An operator $T$ is called continuous at a point $x_0 \in X$ if $x_n \to x_0$ implies $Tx_n \to Tx_0$ in $Y$. 

[28]
Theorem: 1.6 [96, p-96] – If a linear operator $T$ is continuous at one point $x_0 \in X$ then it is bounded, and hence continuous at every point of $X$.

Definition: 1.14 (Compact Linear Operator) [96, p-405] – Let $X$, $Y$ be two Normed linear spaces. An operator $T : X \to Y$ is called compact linear operator (or completely continuous linear operator) if for every bounded subset $M$ of $X$, $\overline{T(M)}$ is compact.

1.16 [96, p-407] – Let $X$ and $Y$ be normed linear spaces and $T : X \to Y$ a linear operator. Then, if $T$ is bounded and $\dim(T(x)) < \infty$, the operator $T$ is compact.

Theorem: 1.7 [96, p-408] – Let $\{T_n\}$ be a sequence of compact linear operators from a NLS $X$ into a Banach space $Y$ such that $\|T_n - T\| \to 0$ as $n \to \infty$ i.e. $\{T_n\}$ is uniformly operator convergent to $T$, then the limit operator $T$ is compact.

1.17 [96, p-201] – An operator $T$ on a Hilbert space $H$ is called to be self adjoint if $T^* = T$. Equivalently, $T$ is self adjoint if $(Tx, x)$ is real for all $x \in H$.

1.18 [96, p-470] – A self adjoint operator on a Hilbert space $H$ is said to be positive operator i.e $T \geq 0$ if $(Tx, x) \geq 0 \forall x \in H$.

1.19 [96, p-201] – An operator $T$ on a Hilbert space $H$ is said to be normal if it commutes with its adjoint i.e. if $TT^* = T^*T$. 
1.20 [96, p-201] – An operator $T$ on a Hilbert space $H$ is said to be unitary if $TT^* = T^*T = I$.

1.21 [96, p-148] – A linear operator $T$ on a Hilbert space $H$ is perpendicular projection $\iff T = T^2 = T^*$.

1.22 [114, p-192] – If $T_1$ and $T_2$ are perpendicular projection on closed subspaces $M_1$ and $M_2$ respectively on a Hilbert space $H$ then $M_1$ and $M_2$ are orthogonal if and only if $T_1T_2 = 0 \iff T_2T_1 = 0$.

1.23 [68, p-3] – A vector space $X$ over the field of complex number $\mathbb{C}$ is said to be an algebra if it equipped with a binary operation, referred to as multiplication and denoted by Juxtaposition, from $X \times X \to X$ such that for $x, y, z \in X$ and $a \in \mathbb{C}$

(i) $x(yz) = (xy)z$

(ii) $x(y + z) = xy + xz$; $(y + z)x = yx + zx$

(iii) $a(xy) = (ax)y = x(ay)$,

$X$ is said to be commutative algebra if $X$ is an algebra and

(iv) $xy = yx \quad x, y \in X$, where $X$ is said to an algebra with identity if $X$ is an algebra and there exist some element $e \in X$ such that

(v) $ex = xe = x$ for every $x \in X$,
It is evident that, if $X$ is an algebra with identity, then the identify element i.e. 'e' unique.

1.24 [68, p-4] – A normed space $(X, \| \cdot \|)$ over $\mathbb{C}$ is said to be a normed algebra if $X$ is an algebra and satisfies the inequality

$$\|xy\| \leq \|x\| \|y\| \text{ for } x, y \in X$$

A normed algebra $X$ is said to be a Banach algebra if the normed space $(X, \| \cdot \|)$ is a Banach space. Let $X$ be an algebra. Then $X_1 \subseteq X$ is said to be subalgebra if $X_1$ is a subspace of $X$ such that

$$x, y \in X_1 \text{ implies } xy \in X_1$$

**Theorem: 1.8 [68, p-4]** – A closed subalgebra of a Banach algebra is a Banach algebra.

1.25 [68, p-4] – Let $X$ be an algebra with identity $e$. An element $x \in X$ is said to have a left (right) inverse if there exist same $y \in X$ such that $yx = e$ ($xy = e$), where $x$ is said to have an inverse if there exist same $y \in X$, such that $xy = yx = e$. If $x \in X$ has an inverse, then $X$ is said to be regular or invertible.

$x \in X$ is said to be singular if it is not regular. It is easy to verify that, if $x \in X$ has a left inverse $y$ and a right inverse $z$, then $y = z$ is an inverse and that inverse is unique.
1.26 [68, p-4] – Let $X$ be an algebra. An element $x \in X$ is said to have a right (left) \textit{quasi-inverse} if there exists some $y \in X$ such that

$$x \circ y = x + y - xy = 0 \quad (y \circ x = y + x - yx = 0),$$

and $x$ is said to have a quasi-inverse if there exists some $y \in X$ such that $y \circ x = x \circ y = 0$. If $x \in X$ has a quasi-inverse, then $x$ is said to be quasi-regular or quasi-invertible. $x \in X$ is said to be quasi-singular if it is not quasi-regular.

1.27 [68, p-33] – Let $(X, || \cdot ||)$ be a normed algebra. Then $x \in X$ is said to be \textit{nilpotent} if there exists some non-negative integer $n$ such that $x^n = 0$, and $x \in X$ is said to be \textit{topological nilpotent} if

$$\lim_{n \to \infty} ||x^n||^{\frac{1}{n}} = 0$$

1.28 [68, p-40] – Let $(X, || \cdot ||)$ be a normed algebra. Then $x \in X$ is said to be a left (right) \textit{topological zero divisor} (TZD) if there exists a sequence $\{y_k\}$ in $X$ such that $||y_k|| = 1, k = 1, 2, 3, \ldots \ldots$ and $\lim||xy_k|| = 0$ ($\lim_{k \to \infty} ||y_kx|| = 0$), and $x$ is said to be a \textit{two sided topological zero divisor} if there exist a sequence $\{y_k\} \subset X$ for which $||y_k|| = 1, k = 1, 2, 3, \ldots \ldots$, and

$$\lim_{k \to \infty} ||xy_k|| = \lim_{k \to \infty} ||y_kx|| = 0$$

If $X$ is commutative, then we speak of only topological zero divisors.
1.29 [68, P.54] – Let X be a Banach algebra and let \( x \in X \). If X has an identity e, then the spectrum of x, denoted by \( \sigma(x) \), is the set of all \( \zeta \in \mathbb{C} \). Such that \( (x - \zeta e) \) is singular, if X is without identify, then \( \sigma(x) \) is the set of all \( \zeta \in \mathbb{C}, \zeta \neq 0 \), such that \( \left( \frac{x}{\zeta} \right) \) is quasi-singular, together with \( \zeta = 0 \).

1.30 [68, p-58] – Let X be a Banach algebra. If \( x \in X \), then we set \( v(x) = \sup_{\zeta \in \sigma(x)} |\zeta| \), then \( v(x) \) is called the spectral radius of x.

Theorem: 1.9 [68, p-58] (Spectral radius Formula) – Let X be a Banach algebra. If \( x \in X \), then spectral radius \( v(x) \) of x is given by

\[
v(x) = \sup\{ |\lambda| : \lambda \in \sigma(f) \} \quad \text{or equivalently by the formula}
\]

\[
v(x) = \lim_{n \to \infty} \|x^n\|^{1/n}
\]

1.31 [68, p-109] – Let X be a commutative Banach algebra. A net \( \{u_{\alpha}\} \subset X \) is said to be an approximate identity if

(i) \( \sup_{\alpha} \|u_{\alpha}\| < \infty \)

(ii) \( \lim_{\alpha} \|u_{\alpha}x - x\| = 0 \) for every \( x \in X \).

1.32 [68, p-273] – Let X be an algebra. Then X is said to be a algebra with involution \( (*) \) if there exists a mapping \( * : x \to x \) such that for any \( x, y \in X \) and \( a \in \mathbb{C} \), we have,

(i) \( (x + y)^* = x^* + y^* \)
(ii) \((ax)^* = \overline{ax}\)

(iii) \((xy)^* = y^*x^*

(iv) \((x^*)^* = x^{**} = x\)

A Banach algebra \(X\) with involution \(*\) is said to be B*-algebra if \(\|x^*x\| = \|x\|^2\), \(x \in X\) and \((*)\) is called isometry if \(\|x^*\| = \|x\|\).

1.33 [68, p-275] – Let \(X\) be a Banach algebra with involution \(*\). A subset \(X_1\) of \(X\) is said to be symmetric if \(x \in X_1\), implies \(x^* \in X_1\). An element \(x \in X\) is said to be self adjoint if \(x^* = x\), unitary if \(xx^* = x^*x = 1\) and normal if \(xx^* = x^{**}\).

1.34 [29, p-357] – A mapping \(\Phi\) of commutative algebra \(X_1\) into another commutative algebra \(X_2\) is said to be a homomorphism if it is linear and preserves the multiplication; i.e.

\[\varphi(ax + by) = a \varphi(x) + b \varphi(y)\] and \(\varphi(xy) = \varphi(x)\varphi(y)\) for all \(x, y \in X_1\) and \(a, b \in C\).

1.35 [68, p-277] – Let \(X_1\) and \(X_2\) be Banach algebra with involution \(*\) and \(\sim\), respectively. A homomorphism \(\varphi: X_1 \to X_2\) is said to be a \(\ast\)-homomorphism if \(\varphi(x^*) = (\overline{\varphi(x)})\) for every \(x \in X_1\).

1.36 [28, p-212] – Let \(X\) be an algebra with involution \(*\). A linear functional \(\psi\) on \(X\) is said to be positive if \(\psi(x^*x) \geq 0\) for all \(x \in X\).
Theorem: 1.10 [28, p-246] – Let \( \psi \) be a positive functional on a B*-algebra \( X \). Then \( \psi \) is bounded and
\[
|\psi(x)|^2 \leq \|\psi\|\psi(x^*x), \ x \in X.
\]

Theorem: 1.11 [28, p-247] – If a B*-algebra \( X \) has an identity element \( e \), then a linear functional \( \psi \) on \( X \) is positive if and only if \( \|\psi\| = \psi(e) \).

1.37 [46, p-1] – A partially ordered set (poset) \( P \) is a set in which a binary relation \( x \leq y \) is defined which satisfies for all \( x, y, z \in P \) the following conditions,

(i) \( x \leq x \)

(ii) If \( x \leq y \) and \( y \leq x \) then \( x = y \) (Anti-symmetry)

(iii) If \( x \leq y \) and \( y \leq z \) then \( x \leq z \) (transitivity)

1.38 [46, p-6] – An upper bound (lower bound) of a subset \( P_1 \) of a poset \( P \) is an element \( a \in P \) such that \( x \leq a \) \( (a \leq x) \) for every \( x \in P_1 \). An element \( z \in P \) is a least upper bound (l.u.b) of \( P_1 \) if \( z \) is an upper bound of \( P_1 \) and if \( z_1 \) is any upper bound of \( P_1 \) then \( z \leq z_1 \). Similarly, greatest-lower bound (g.l.b) can also be defined.

1.39 [46, p-6] – A lattice is poset \( P \), any two of whose elements have a g.l.b. denoted by \( (x \land y) \), and a l.u.b. denoted by \( (x \lor y) \).
1.40 [46, p-7] - A sublattice of a lattice $L$ is a subset $L_1$ of $L$ such that $a \in L_1, b \in L_1 \Rightarrow a \land b \in L_1$ and $a \lor b \in L_1$.

1.41 [71, p-49] - A vector space $X$ over the real numbers field, endowed with an order relation $\leq$; is called a ordered vector-space if following axioms are satisfied:

(i) $x \leq y \Rightarrow x + z \leq y + z$ for all $x, y, z \in X$

(ii) $x \leq y \Rightarrow tx \leq ty$ for all $x, y \in X$ and $t$ is a positive real number.

A vector-lattice is an ordered vector space such that $x \lor y \land x \land y$ exists for all $x, y \in X$.

1.42 [46, p-2] - A function $\theta : P \rightarrow Q$ from a poset $P$ to another poset $Q$ is called order-preserving or isotone if it satisfies

$$x \leq y \Rightarrow \theta(x) \leq \theta(y) \text{ for } x, y \in P.$$

1.43 [46, p-13] - A lattice $L$ is said to be modular if it satisfies the modular identity viz;

If $x \leq z$, then $x \lor (y \land z) = (x \lor y) \land z$ holds for every $y \in L$.

1.44 [46, p-366] - A Banach lattice $L$ is real Banach space $(L, \| \cdot \|)$ which is also a lattice under $\leq$ satisfying the following condition

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|, x, y \in L \text{ where } |x| = x \lor (-x).$$
1.45 [46, p-287] – A pogroup \((G_1, +)\) is a group which is also a poset and in which every group translation is isotone, i.e. if \(x \leq y; \ x, y \in G_1\) then \(a + x + b \leq a + y + b\) for all \(a, b \in G_1\). If the Pogroup is also a lattice, it is called a lattice-ordered group or l-group. Further a Pogroup \(G_1\) is called Archimedean when \(x \ll y\) implies \(x = 0\). Where \(a \ll b\) means \(na \leq b\) for every integer \(n\). (known as incomparably smaller).

1.46 [46, p-397] – A p-ring is a ring \(R\) which is also a poset under a relation \(\leq\), and in which

(i) \(y \leq x \Rightarrow a + y \leq a + x\) for all \(a \in R\)

(ii) \(0 \leq x\) and \(0 \leq y\) imply \(0 \leq xy\) in \(R\).

An l-ring is a po-ring \(R\) which is also a lattice under \(\leq\).

A function ring or f-ring is a l-ring in which \(a \wedge b = 0\) and \(0 \leq c\) imply \(ca \wedge b = ac \wedge b = 0\). Here \(0\) denoted the group identity for addition (+).

1.47 [71, p-101] – A lattice norm \(\|x\|\) on a vector lattice \(L\) is called an M-norm if it satisfies the axiom-

\[
\|x \vee y\| = \max\{\|x\|, \|y\|\}, 0 \leq x, 0 \leq y \& x, y \in L.
\]
is called an $M$-normed space and an $M$-normed Banach lattice is called an abstract $M$-space (briefly, $AM$-space)

1.48 [71, p-112] – A lattice norm $\| \|$ on a vector-lattice $L$ is called $L$-norm if it satisfies the axiom-

$$\|x + y\| = \|x\| + \|y\|, \quad 0 \leq x, 0 \leq y \quad \text{and} \quad x, y \in L$$

$(L, \| . \|)$ is called an $L$-normed space and an $L$-normed Banach space is called an Abstract $L$-space ($AL$-space or Lebesgue-space).