Chapter II
Chapter 2

Certain Subclasses of Uniformly $p$–valent Starlike and Convex Functions

An education isn't how much you have committed at memory, or even how much you know, it's being able to differentiate between what you do know and what you don't.

— James Joseph Sylvester

2.1 Introduction

Let $A_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N} = \{1, 2, \ldots\}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in A_p$ is said to be $p$–valent starlike of order $\alpha$ ($0 \leq \alpha < p$), if

$$\Re\left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \mathbb{U}.$$
A function \( f \in A_p \) is said to be \( p \)-valent convex of order \( \alpha \) \((0 \leq \alpha < p)\), if
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \mathbb{U}.
\]
Let \( \mathcal{S}^*_p(\alpha) \) and \( \mathcal{K}_p(\alpha) \) denote, respectively, the classes of \( p \)-valent starlike and convex functions of order \( \alpha \) in \( \mathbb{U} \).

Note that for \( p = 1 \) the classes \( \mathcal{S}^*_1(\alpha) = \mathcal{S}^*(\alpha) \) and \( \mathcal{K}_1(\alpha) = \mathcal{K}(\alpha) \) are the usual classes of univalent starlike and univalent convex functions of order \( \alpha \) \((0 \leq \alpha < 1)\) respectively.

For \( p = 1, \alpha = 0 \) the classes \( \mathcal{S}^*_p(\alpha) \) and \( \mathcal{K}_p(\alpha) \) reduces to \( \mathcal{S}^*(0) = \mathcal{S}^* \) and \( \mathcal{K}(0) = \mathcal{K} \), which are the classes of starlike and convex functions (univalent) with respect to the origin respectively. We know that \( f \in \mathcal{K}_p(\alpha) \) if and only if \( zf'(z) \in \mathcal{S}^*_p(\alpha) \).

For \( p = 1, \alpha = 0 \), Goodman [4, 5] introduced the concepts of uniform starlikeness and uniform convexity, thereby providing proper subclasses of the usual classes of convex and starlike functions, here denoted by \( \mathcal{S}^* \) and \( \mathcal{K} \) respectively. The corresponding “uniform classes” are defined in the following way, by their geometrical mapping properties.

**Definition 2.1.** A function \( f \in A_1 \) is said to be uniformly starlike (convex) in \( \mathbb{U} \) if \( f \in \mathcal{S}^*(\mathcal{K}) \) and has the property that for every circular arc \( \gamma \) contained in \( \mathbb{U} \), with center \( \zeta \), also in \( \mathbb{U} \), the arc \( f(\gamma) \) is starlike (convex w.r.t. \( f(\zeta) \)).

Goodman's denoted the class of uniformly convex functions by UCV and the class of uniformly starlike functions by UST. The proof of the fact that the above definition gives proper subclasses of \( \mathcal{S}^* \) and \( \mathcal{K} \) can be found in [4, 5] through various results. An analytic description of UST
and $UCV$ can be found in [5]. Goodman stated the criterion
\[
\Re \left( 1 + (z - \zeta) \frac{f''(z)}{f'(z)} \right) > 0, \quad \forall z, \zeta \in \mathbb{U} \quad \iff \quad f \in UCV. \tag{2.1}
\]
Following Goodman, Rønning [11, 12] introduced and studied the following subclasses:

1. A function $f(z) \in A_1$ is said to be in the class $S_p(\alpha, \beta)$ of uniformly $\beta$-starlike functions if it satisfies the condition:
\[
\Re \left( \frac{zf'(z)}{f(z)} - \alpha \right) > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \mathbb{U}, \tag{2.2}
\]
where $\beta \geq 0$ and $-1 \leq \alpha < 1$.

2. A function $f(z) \in A_1$ is said to be in the class $UCV(\alpha, \beta)$ of uniformly $\beta$-convex functions if it satisfies the condition:
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) > \beta \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in \mathbb{U}, \tag{2.3}
\]
where $\beta \geq 0$ and $-1 \leq \alpha < 1$.

Rønning obtained a more suitable form of the criterion (2.1), namely
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right|, \quad \forall z \in \mathbb{U} \quad \iff \quad f \in UCV. \tag{2.4}
\]
It follows from (2.2) and (2.3) that
\[
f(z) \in UCV(\alpha, \beta) \iff zf'(z) \in S_p(\alpha, \beta).
\]
For $p \geq 1, p \in \mathbb{N}, \alpha = 0$ and $\beta = 1$, the subclasses of uniformly $p$-valent starlike and convex functions were introduced first by Al-Kharsani and Al-Hajiry [1] and then studied by several authors. Our aim in this chapter is to extend these ideas by considering the classes of uniformly multivalent $\beta$-starlike and uniformly multivalent $\beta$-convex functions and study their properties which will provide a unified study of functions in these classes.
2.2 The Subclasses $SD_p(\alpha, \beta)$ and $KD_p(\alpha, \beta)$

We now introduce two new subclasses denoted by $SD_p(\alpha, \beta)$ and $KD_p(\alpha, \beta)$ of functions $f(z) \in A_p$ as follows:

Definition 2.2. We say that a function $f \in A_p$ is in the class $SD_p(\alpha, \beta)$ if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \beta \left| \frac{zf'(z)}{f(z)} - p \right| + \alpha, \quad z \in \mathbb{U},$$

for some $\beta \geq 0$ and $0 \leq \alpha < p$.

Definition 2.3. We say that a function $f \in A_p$ is in the class $KD_p(\alpha, \beta)$ if

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta \left| \frac{zf''(z)}{f'(z)} - (p - 1) \right| + \alpha, \quad z \in \mathbb{U},$$

for some $\beta \geq 0$ and $0 \leq \alpha < p$.

Note that $f(z) \in KD_p(\alpha, \beta)$ if and only if $zf'(z) \in SD_p(\alpha, \beta)$.

For $p = 1, 0 \leq \alpha < 1$ the subclasses $SD_1(\alpha, \beta) = SD(\beta, \alpha)$ and $KD_1(\alpha, \beta) = KD(\beta, \alpha)$ were introduced and studied by Shams, Kulkarni and Jahangiri in [14]. They obtained sufficient coefficient conditions for functions in the classes $SD(\beta, \alpha)$ and $KD(\beta, \alpha)$ along with geometric properties of functions in these classes.

For $p = 1, \alpha = 0$ and $\beta = 1$, we obtain the class $KD_1(0, 1) = UCV$ of uniformly convex functions, defined by Goodman [76]. For $p = 1$ and $\beta = 1$ the class $KD_1(\alpha, 1) = UCV(\alpha)$ of uniformly convex functions of order $\alpha$ was investigated by Rønning [11].

In this Chapter we shall study the geometric properties, coefficient bounds and convolution properties for functions in the classes $SD_p(\alpha, \beta)$ and $KD_p(\alpha, \beta)$. We will show that these classes are closed under certain integral operators. We will also prove certain inequalities which give sufficient conditions for functions in $A_p$ to be in the classes $SD_p(\alpha, \beta)$ and $KD_p(\alpha, \beta)$ using a parabolic domain.
2.3 Geometric Properties

Set \( w(z) = \frac{zf'(z)}{f(z)} \) and \( \Omega_{\alpha,\beta} = \{ w : \Re(w) > \beta|w - p| + \alpha \} \). If \( f(z) \in SD_p(\alpha, \beta) \) then \( w(z) \) belongs to the region \( \Omega_{\alpha,\beta} \).

If \( \beta = 1 \) then \( \Omega_{\alpha,\beta} \) is the interior of a parabola in the right half-plane which is symmetric about the real axis and has vertex at \((p/2, 0)\). Therefore, \( \frac{zf'(z)}{f(z)} \) lies in the region \( \Omega_{\alpha,1} \) which contains \( w = p \) and is bounded by the parabola

\[
v^2 = 2(p - \alpha) \left(u - \frac{p + \alpha}{2}\right).
\]

Figure 1 shows the region \( \Omega_{\alpha,1} \) for \( \alpha = 0 \).

![Figure 2.1: Parabolic domain \( \Omega_{\alpha,1} \) for \( \alpha = 0 \).](image)

If \( \beta > 1 \) then \( \frac{zf'(z)}{f(z)} \) lies in the region \( \Omega_{\alpha,\beta} \) which contains \( w = p \) and is bounded by the ellipse

\[
\left(u - \frac{(p\beta^2 - \alpha)}{(\beta^2 - 1)}\right)^2 + \frac{\beta^2}{(\beta^2 - 1)}v^2 = \frac{\beta^2(p - \alpha)^2}{(\beta^2 - 1)^2}
\]
with vertices at the points \[
\left(\frac{p\beta - \alpha}{\beta - 1}, 0\right), \left(\frac{p\beta + \alpha}{\beta - 1}, 0\right), \left(\frac{(p\beta^2 - \alpha)}{(\beta^2 - 1)}, \frac{(p - \alpha)}{\sqrt{(\beta^2 - 1)}}\right), \]
and \[
\left(\frac{(p\beta^2 - \alpha)}{(\beta^2 - 1)}, \frac{(\alpha - p)}{\sqrt{(\beta^2 - 1)}}\right).
\]

Since \[
\alpha < \frac{p\beta + \alpha}{\beta + 1} < p < \frac{p\beta - \alpha}{\beta - 1},
\]
therefore, we obtain \(\Omega_{\alpha, \beta} \subset \{w : \Re(w) > \alpha\}\). Hence, \(SD_p(\alpha, \beta) \subset S^*(\alpha)\).

We now give sufficient coefficient bounds for functions belonging to the subclasses \(SD_p(\alpha, \beta)\) and \(KSD_p(\alpha, \beta)\). Our first result is contained in

**Theorem 2.1.** If \(f(z) \in A_p\) satisfies
\[
\sum_{n=p+1}^{\infty} \left\{n(1 + \beta) - (\alpha + p\beta)\right\} |a_n| \leq (1 - \alpha),
\]
then \(f(z) \in SD_p(\alpha, \beta)\).

**Proof.** We know that \(f(z) \in SD_p(\beta, \alpha)\) if
\[
\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \left|\frac{zf'(z)}{f(z)} - p\right| + \alpha,
\]
or equivalently,
\[
\left|w - (1 + \alpha) - \beta|w - p|\right| < \left|w + (1 - \alpha) - \beta|w - p|\right|,
\]
where \(w(z) = \frac{zf'(z)}{f(z)}\).

It is sufficient to show that \(R - L > 0\), where
\[
R = \left|w + (1 - \alpha) - \beta|w - p|\right| \quad \text{and} \quad L = \left|w - (1 + \alpha) - \beta|w - p|\right|.
\]
Now,

\[
R = \left| \frac{zf'(z)}{f(z)} + (1 - \alpha) - \beta \left| \frac{zf'(z)}{f(z)} - p \right| \right| \\
= \frac{1}{|f(z)|} \left| zf'(z) + (1 - \alpha) f(z) - \beta \frac{f(z)}{|f(z)|} |zf'(z) - pf(z)| \right| \\
= \frac{1}{|f(z)|} |(p + 1 - \alpha)z^p + \sum_{n=p+1}^{\infty} (n + 1 - \alpha) a_n z^n - \beta e^{i\theta} | \\
\times \left| \sum_{n=p+1}^{\infty} (n - p) a_n z^n \right| \\
\geq \frac{1}{|f(z)|} \left[ (p + 1 - \alpha)|z|^p - \sum_{n=p+1}^{\infty} (n + 1 - \alpha)|a_n||z|^n - \beta \right. \\
\left. \times \sum_{n=p+1}^{\infty} (n - p)|a_n||z|^n \right] \\
> \frac{|z|^p}{|f(z)|} \left[ (p + 1 - \alpha) - \sum_{n=p+1}^{\infty} (n + 1 - \alpha + n \beta - p \beta)|a_n| \right], \quad (2.8)
\]

and

\[
L = \left| \frac{zf'(z)}{f(z)} - (1 + \alpha) - \beta \left| \frac{zf'(z)}{f(z)} - p \right| \right| \\
= \frac{1}{|f(z)|} \left| zf'(z) - (1 + \alpha) f(z) - \beta \frac{f(z)}{|f(z)|} |zf'(z) - pf(z)| \right| \\
\leq \frac{1}{|f(z)|} \left[ (p - 1 + \alpha)|z|^p + \sum_{n=p+1}^{\infty} (n - 1 - \alpha)|a_n||z|^n + \beta \\
\times \sum_{n=p+1}^{\infty} (n - p)|a_n||z|^n \right] \\
< \frac{|z|^p}{|f(z)|} \left[ (p - 1 + \alpha) + \sum_{n=p+1}^{\infty} (n - 1 - \alpha + n \beta - p \beta)|a_n| \right]. \quad (2.9)
\]
From (2.8) and (2.9), we have
\[
R - L > \frac{|z|^p}{|f(z)|} \left[ 2(1 - \alpha) - 2 \sum_{n=p+1}^{\infty} 2 \{n(1 + \beta) - (\alpha + p\beta)\} |a_n| \right]
\]
using (2.7), we get
\[
R - L > 0.
\]

For \( p = 1 \), as immediate consequence of Theorem 2.1 we obtain the following corollary due to Shams et al. [14]:

**Corollary 2.1.** If
\[
\sum_{n=p+1}^{\infty} \{ n(1 + \beta) - (\alpha + \beta) \} |a_n| \leq (1 - \alpha),
\]
then \( f(z) \in SD(\alpha, \beta) \).

For \( \beta = 0 \) and \( p = 1 \), we have

**Corollary 2.2.** If \( \sum_{n=2}^{\infty} (n - \alpha)|a_n| \leq (1 - \alpha) \) then \( f(z) \in S^*(\alpha) \).

Here \( S^*(\alpha) \) is the usual class of starlike functions of order \( \alpha \). Next, we state corresponding result for functions belonging to the subclass \( KD_p(\alpha, \beta) \).

**Theorem 2.2.** If \( f(z) \in A_p \) satisfies
\[
\sum_{n=p+1}^{\infty} n\{n(1 + \beta) - (\alpha + p\beta)\} |a_n| \leq (1 - \alpha), \tag{2.10}
\]
then \( f(z) \in KD_p(\alpha, \beta) \).

**Proof.** Proof follows from the proof of previous theorem and the fact that \( f(z) \in KD_p(\alpha, \beta) \) if and only if \( z f'(z) \in SD_p(\alpha, \beta) \). □
Taking \( p = 1 \) in Theorem 2.2, we obtain

**Corollary 2.3.** If

\[
\sum_{n=p+1}^{\infty} n\{n(1 + \beta) - (\alpha + \beta)\}|a_n| \leq (1 - \alpha),
\]

then \( f(z) \in KD(\alpha, \beta) \).

This was proved by Shams et al. [14]. For \( \beta = 0 \) and \( p = 1 \), we have

**Corollary 2.4.** If \( \sum_{n=2}^{\infty} n(n - \alpha)|a_n| \leq (1 - \alpha) \) then \( f(z) \in K(\alpha) \).

It is easy to see that for \( \alpha = 0, \beta = 0 \) and \( p = 1 \) in Theorem 2.1 and 2.2 we obtain well known coefficient estimates for functions in the classes of starlike and convex functions denoted by \( S^* \) and \( K \) respectively [7, 8].

### 2.4 Coefficient Inequalities

We now give coefficient inequalities for functions belonging to the subclasses \( SD_p(\alpha, \beta) \) and \( KD_p(\alpha, \beta) \). Our first result is contained in

**Theorem 2.3.** If \( f(z) \in SD_p(\beta, \alpha) \) with \( 0 \leq \beta \leq \alpha < p \) or \( \beta > \frac{p+\alpha}{2p} \) then \( f(z) \in S_p^* \left( \frac{\alpha - p\beta}{1 - \beta} \right) \).

**Proof.** We know that \( \Re(z) \leq |z| \) for any complex number \( z \). Therefore \( f(z) \in SD_p(\alpha, \beta) \) gives us

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \beta \Re \left( \frac{zf'(z)}{f(z)} - 1 \right) + \alpha. \tag{2.11}
\]

From this we get

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \frac{\alpha - p\beta}{1 - \beta} \quad (z \in U). \tag{2.12}
\]
Now, if $0 \leq \beta \leq \alpha < p$, then it follows that

$$0 \leq \frac{\alpha - \beta}{1 - \beta} < p,$$

and if $\beta > \frac{p+\alpha}{2p}$, then we have

$$-p < \frac{p\beta - \alpha}{\beta - 1} \leq 0,$$

The conclusion of the theorem now follows.

For $p = 1$, we obtain the following corollary due to Owa, Polatoglu, and Yuvaz [9].

**Corollary 2.5.** If $f(z) \in SD(\alpha, \beta)$ with $0 < \beta < \alpha$ or $\beta > \frac{1+\alpha}{2}$ then $f(z) \in S^* \left( \frac{\alpha-\beta}{1-\beta} \right)$.

Next, we state the corresponding result for functions belonging to the subclass $KD_p(\alpha, \beta)$.

**Theorem 2.4.** If $f(z) \in KD_p(\alpha, \beta)$ with $0 \leq \beta \leq \alpha < p$ or $\beta > \frac{p+\alpha}{2p}$ then $f(z) \in K_p \left( \frac{\alpha-\beta}{1-\beta} \right)$.

**Proof.** Proof is similar to the proof of last theorem.

The following corollary is due to Owa, Polatoglu, and Yuvaz [9] for $p = 1$.

**Corollary 2.6.** If $f(z) \in KD(\alpha, \beta)$ with $0 \leq \beta \leq \alpha$ or $\beta > \frac{1+\alpha}{2}$ then $f(z) \in K \left( \frac{\alpha-\beta}{1-\beta} \right)$.

We now state the main theorem of this chapter which gives a necessary coefficient condition for functions $f(z) \in SD_p(\alpha, \beta)$. 
Theorem 2.5. If \( f(z) \in \mathcal{SD}_p(\alpha, \beta) \) then

\[
|a_{p+1}| \leq \frac{2(p - \alpha)}{|1 - \beta|} \tag{2.13}
\]

and

\[
|a_{p+n}| \leq \frac{2(p - \alpha)}{n|1 - \beta|} \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \quad (n \geq 2). \tag{2.14}
\]

Proof. We know that if \( f(z) \in \mathcal{SD}_p(\alpha, \beta) \) then

\[
\Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{\alpha - p\beta}{1 - \beta} \quad (z \in \mathbb{U}).
\]

Let us define a function \( p(z) \) by

\[
q(z) = \frac{(1 - \beta) \left( \frac{zf'(z)}{f(z)} \right) - (\alpha - p\beta)}{(p - \alpha)} \quad (z \in \mathbb{U}). \tag{2.15}
\]

Note that \( q(z) \) is analytic in \( \mathbb{U} \) with \( q(0) = 1 \) and \( \Re(q(z)) > 0 \). Let

\[
q(z) = 1 + q_1 z + q_2 z^2 + \ldots,
\]

then we can write

\[
\frac{zf'(z)}{f(z)} = \frac{\alpha - p\beta}{1 - \beta} + \frac{p - \alpha}{1 - \beta} \sum_{n=0}^{\infty} q_n z^n.
\]

Or

\[
zf'(z) = f(z) \left( p + \left( \frac{p - \alpha}{1 - \beta} \right) \sum_{n=1}^{\infty} q_n z^n \right). \tag{2.16}
\]

From this, we obtain

\[
n a_{p+n} = \left( \frac{p - \alpha}{1 - \beta} \right) (q_n + a_{p+1} q_{n-1} + a_{p+2} q_{n-2} + \ldots + a_{p+n-1} q_1). \tag{2.17}
\]
From the co-efficient estimates of [3] for Carathéodory functions, we know that $q_n \leq 2$ for all $n \geq 1$. Making use of it in (2.17) we see that

$$|a_{p+n}| \leq \frac{2(p-\alpha)}{n|1-\beta|}(1 + |a_{p+1}| + |a_{p+2}| + \ldots + |a_{p+n-1}|). \quad (2.18)$$

Therefore, for $n = 1$, we have

$$|a_{p+1}| \leq \frac{2(p-\alpha)}{|1-\beta|}, \quad (2.19)$$

this proves (2.13). Now for $n = 2$, we obtain

$$|a_{p+2}| \leq \frac{2(p-\alpha)}{2|1-\beta|}(1 + |a_{p+1}|)$$

$$\leq \frac{2(p-\alpha)}{2|1-\beta|} \left(1 + \frac{2(p-\alpha)}{|1-\beta|}\right).$$

This shows that (2.14) is true for $n = 2$. For $n = 3$, we see that

$$|a_{p+3}| \leq \frac{2(p-\alpha)}{3|1-\beta|}(1 + |a_{p+1}| + |a_{p+2}|)$$

$$\leq \frac{2(p-\alpha)}{3|1-\beta|} \left(1 + \frac{2(p-\alpha)}{|1-\beta|} + \frac{2(p-\alpha)}{2|1-\beta|} + \frac{2^2(p-\alpha)^2}{2|1-\beta|^2}\right).$$

Thus, (2.14) holds for $n = 3$. Next, we assume that (2.14) is true for $n = k$ and therefore

$$|a_{p+k+1}| \leq \frac{2(p-\alpha)}{(k+1)|1-\beta|} \left(1 + \frac{2(p-\alpha)}{|1-\beta|} + \frac{2(p-\alpha)}{|1-\beta|}\left(1 + \frac{2(p-\alpha)}{|1-\beta|}\right) + \ldots + \frac{2(p-\alpha)}{k|1-\beta|} \prod_{j=1}^{k-2} \left(1 + \frac{2(p-\alpha)}{j|1-\beta|}\right)\right)$$

$$\leq \frac{2(p-\alpha)}{(k+1)|1-\beta|} \prod_{j=1}^{k-1} \left(1 + \frac{2(p-\alpha)}{j|1-\beta|}\right).$$

This shows that (2.14) is true for $n = k + 1$. Hence, using principle of mathematical induction, (2.14) holds for all $n \geq 2$. \hfill \square
Remark 2.1. Taking $p = 1$ in Theorem 2.5 we obtain

$$|a_{n+1}| \leq \frac{2(1 - \alpha)}{n|1 - \beta|} \prod_{j=1}^{n-1} \left( 1 + \frac{2(1 - \alpha)}{j|1 - \beta|} \right) \quad (n \geq 2) \quad (2.20)$$

which was given by Owa, Polatoğlu and Yavuz [9].

Remark 2.2. Taking $p = 1$ and $\beta = 0$ in Theorem 2.5, we have

$$|a_{n+1}| \leq \frac{1}{n!} \prod_{j=2}^{n+1} (j - 2\alpha) \quad (n \geq 1), \quad (2.21)$$

this was proved by Robertson [167].

We know that $f(z) \in K\mathcal{D}_p(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{S}\mathcal{D}_p(\alpha, \beta)$.

Then, we have

Theorem 2.6. If $f(z) \in K\mathcal{D}_p(\alpha, \beta)$, then

$$|a_{p+1}| \leq \frac{(p - \alpha)}{|1 - \beta|} \quad (2.22)$$

and

$$|a_{p+n}| \leq \frac{2(p - \alpha)}{n(n + 1)|1 - \beta|} \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \quad (n \geq 2). \quad (2.23)$$

Remark 2.3. If we take $p = 1$ in Theorem 2.6, then

$$|a_{n+1}| \leq \frac{2(1 - \alpha)}{(n + 1)n|1 - \beta|} \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \quad (n \geq 2)$$

proved by Owa, Polatoğlu and Yavuz [9].

Remark 2.4. Taking $p = 1$ and $\beta = 0$ in Theorem 2.6, we get

$$|a_{n+1}| \leq \frac{1}{(n + 1)!} \prod_{j=2}^{n+1} (j - 2\beta) \quad (n \geq 1),$$

which was proved by Robertson [167].
Theorem 2.7. If $f(z) \in \mathcal{SD}_p(\alpha, \beta)$, then

$$
\max \left\{ 0, |z|^p - \frac{2(p - \alpha)}{|1 - \beta|} |z|^{p+1} - \sum_{n=2}^{\infty} \frac{2(p - \alpha)}{n|1 - \beta|} \left( \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \right) |z|^{p+n} \right\}
\leq |f(z)|
$$

$$
\leq |z|^p + \frac{2(p - \alpha)}{|1 - \beta|} |z|^{p+1} + \sum_{n=2}^{\infty} \frac{2(p - \alpha)}{n|1 - \beta|} \left( \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \right) |z|^{p+n}
$$

and

$$
\max \left\{ 0, p|z|^{p-1} - \frac{2(p + 1)(p - \alpha)}{|1 - \beta|} |z|^p - \sum_{n=2}^{\infty} \frac{2(p + n)(p - \alpha)}{n|1 - \beta|} \left( \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \right) |z|^{p+n-1} \right\}
\leq |f'(z)| \leq p|z|^{p-1} + \frac{2(p + 1)(p - \alpha)}{|1 - \beta|} |z|^p
$$

$$
+ \sum_{n=2}^{\infty} \frac{2(p + n)(p - \alpha)}{n|1 - \beta|} \left( \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \right) |z|^{p+n-1}.
$$

Proof. Proof follows from the fact that

$$
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ p = 1, 2, \ldots
$$

and using Theorem 2.7.
Corollary 2.7. If \( f(z) \in \mathcal{KD}_p(\alpha, \beta) \), then

\[
\max \left\{ 0, \left| z \right|^p - \frac{(p - \alpha)}{|1 - \beta|} \left| z \right|^{p+1} - \sum_{n=2}^{\infty} \frac{2(p - \alpha)}{n(n+1)|1 - \beta|} \left( \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \right) \times \left| z \right|^{p+n} \right\} 
\leq |f(z)|
\leq \left| z \right|^p + \frac{(p - \alpha)}{|1 - \beta|} \left| z \right|^{p+1} + \sum_{n=2}^{\infty} \frac{2(p - \alpha)}{n(n+1)|1 - \beta|} \left( \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \right) \times \left| z \right|^{p+n}
\]

and

\[
\max \left\{ 0, \left| z \right|^{p-1} - \frac{(p + 1)(p - \alpha)}{|1 - \beta|} \left| z \right|^p - \sum_{n=2}^{\infty} \frac{2(p - \alpha)}{n|1 - \beta|} \left( \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \right) \times \left| z \right|^{p+n-1} \right\} 
\leq |f'(z)|
\leq \left| z \right|^{p-1} + \frac{(p + 1)(p - \alpha)}{|1 - \beta|} \left| z \right|^p + \sum_{n=2}^{\infty} \frac{2(p - \alpha)}{n|1 - \beta|} \left( \prod_{j=1}^{n-1} \left( 1 + \frac{2(p - \alpha)}{j|1 - \beta|} \right) \right) \times \left| z \right|^{p+n-1}.
\]

2.5 Hadamard Product

Let \( f(z), g(z) \in \mathcal{A}_p \) be given by

\[
f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n
\]
then the Hadamard product (convolution) of \( f(z) \) and \( g(z) \) is defined by

\[
(f \ast g)(z) = z^p + \sum_{n=p+1}^{\infty} b_n a_n z^n
\]

We now state a theorem, the proof of which follows using a convolution result of Shams et al. [14] and Theorem 2.1.

**Theorem 2.8.** The classes \( SD_p(\alpha, \beta) \) and \( KD_p(\alpha, \beta) \) are closed under Hadamard product with convex functions in \( \mathbb{U} \).

As a consequence of the above theorem, we have

**Corollary 2.8.** If \( f(z) \) is in \( SD_p(\alpha, \beta) \) (or \( KD_p(\alpha, \beta) \)) then the function \( g(z) \) defined by

\[
g(z) = \frac{1 + \mu}{z^\mu} \int_0^z t^{\mu-p} f(t) dt, \quad \Re(\mu) \geq 0
\]

is also in \( SD_p(\alpha, \beta) \) (or \( KD_p(\alpha, \beta) \)).

**Proof.** Using definition of convolution of functions in \( SD_p(\alpha, \beta) \) (or \( KD_p(\alpha, \beta) \)),

\( g(z) \) can be written as

\[
g(z) = f(z) \ast \sum_{n=p+1}^{\infty} \left( \frac{1 + \mu}{n + \mu} \right) z^n,
\]

where \( \sum_{n=p+1}^{\infty} \left( \frac{1 + \mu}{n + \mu} \right) z^n \) is convex in \( \mathbb{U} \). The conclusion now follows from Theorem 2.8. \( \square \)

Similarly, we have

**Corollary 2.9.** If \( f(z) \) is in \( SD_p(\alpha, \beta) \) (or \( KD_p(\alpha, \beta) \)) then the function \( h(z) \) defined by

\[
h(z) = \int_0^z \frac{f(t) - f(\lambda t)}{t^p(1 - \lambda)} dt, \quad |\lambda| \leq 1, \lambda \neq 1
\]

is also in \( SD_p(\alpha, \beta) \)(or \( KD_p(\alpha, \beta) \)).
2.5 Hadamard Product

Proof. As earlier, we can write \( h(z) \) as

\[
h(z) = f(z) \ast \left( z^p + \sum_{n=p+1}^{\infty} \frac{1-\lambda^n}{n(1-\lambda)} z^n \right).
\]

Note that the function \( z^p + \sum_{n=p+1}^{\infty} \frac{1-\lambda^n}{n(1-\lambda)} z^n \) is convex in \( U \). Therefore, using Theorem 2.8 the conclusion follows. □

We know that if \( f(z) \in SD_p(\alpha, \beta) \) and \( \beta > 1 \) then \( \frac{zf'(z)}{f(z)} \subset \Omega_{\alpha,\beta} \), where \( \Omega_{\alpha,\beta} \) is the region bounded by the ellipse

\[
\left( u - \frac{(p\beta^2 - \alpha)}{(\beta^2 - 1)} \right)^2 + \frac{\beta^2}{(\beta^2 - 1)} v^2 = \frac{\beta^2(p - \alpha)^2}{(\beta^2 - 1)^2}.
\]

In parametric form the equation of the ellipse becomes

\[
w(t) = \left( \frac{(p\beta^2 - \alpha)}{(\beta^2 - 1)} + \frac{\beta(p - \alpha)}{(\beta^2 - 1)} \cos t, \frac{(p - \alpha)}{\sqrt{(\beta^2 - 1)}} \sin t \right), \quad 0 \leq t \leq 2\pi.
\]

Hence, for \( \beta > 1 \), \( z \in U \setminus \{0\} \), we have \( f(z) \in SD_p(\alpha, \beta) \) if and only if \( \frac{zf'(z)}{f(z)} \neq w(t) \) or \( zf'(z) - f(z)w(t) \neq 0 \). Define \( F(z) \) by

\[
F(z) = \frac{1}{1 - w(t)} \left\{ \frac{z}{(1-z)^2} - \frac{z}{(1-z)}w(t) \right\}
\]

Then using results from [176], we obtain \( f(z) \in SD_p(\alpha, \beta) \) if and only if \( f(z) \neq \frac{F(z)}{z} \).

Conversely, if \( f(z) \neq \frac{F(z)}{z} \), then \( \frac{zf'(z)}{f(z)} \neq w(t), \ 0 \leq t \leq 2\pi \). Therefore, \( \frac{zf'(z)}{f(z)} \) lies inside \( \Omega_{\alpha,\beta} \) or outside \( \Omega_{\alpha,\beta} \) for \( z \in U \). But \( \left. \frac{zf'(z)}{f(z)} \right|_{z=0} = p \in \Omega_{\alpha,\beta} \), thus \( \frac{zf'(z)}{f(z)} \mid_{z \in U} \subset \Omega_{\alpha,\beta} \). Hence \( f(z) \in SD_p(\alpha, \beta) \) and we have

Theorem 2.9. A function \( f(z) \in SD_p(\alpha, \beta) \), \( \beta > 1 \) if and only if \( f(z) \neq \frac{F(z)}{z} \neq 0 \) where \( F(z) \) is given by (2.25).
2.6 Certain Sufficient Estimates

In order to prove our results in this section we need the following lemma due to Jack [6].

**Lemma 2.1.** (Jack's Lemma) Let \( w(z) \) be (non-constant) analytic function in \( \mathbb{U} \) with \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value on the circle \( |z| = r < 1 \) at a point \( z_0 \), then

\[
z_0 w'(z_0) = cw(z_0)
\]

where \( c \) is real and \( c \geq 1 \).

Making use of Lemma 2.1 our first result in this section is the following:

**Theorem 2.10.** Let \( f(z) \in A_p \), then \( f(z) \) is uniformly \( p \)-valent starlike of order \( \alpha \) in \( \mathbb{U} \) if the inequality

\[
\Re \left( \frac{1 + \frac{zf''(z)}{f'(z)} - p}{\frac{zf'(z)}{f(z)} - p} \right) < 1 + \frac{2\beta}{p(1 + 2\beta)},
\]

holds.

**Proof.** Define \( w(z) \) by

\[
w(z) = \frac{2\beta}{p} \left( \frac{zf'(z)}{f(z)} - p \right), \quad p \in \mathbb{N}, z \in \mathbb{U}.
\]

Note that \( w(z) \) is analytic in \( \mathbb{U} \) and \( w(0) = 0 \). Differentiating (2.27) logarithmically we get

\[
\left( \frac{zf'(z)}{f(z)} - p \right) \frac{zw'(z)}{w(z)} = \frac{zf'(z)}{f(z)} \left( 1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \right),
\]
which leads to
\[
\left( 1 + \frac{zf'''(z)}{f''(z)} - p \right) = \frac{p}{2\beta} w(z) + \frac{zw'(z)}{2\beta + w(z)}.
\] (2.28)

Combining (2.27) and (2.28) we see that
\[
\left( 1 + \frac{zf''(z)}{f'(z)} - p \right) = 1 + \frac{2\beta}{p} \frac{zw'(z)}{w(z)(2\beta + w(z))}, \quad p \in \mathbb{N}, \; z \in \mathbb{U}. \] (2.29)

Now assume that there exists a point $z_0 \in \mathbb{U}$ with $|z_0| = 1$ such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1, \quad (w(z_0) \neq 1),
\]
and let $w(z_0) = e^{i\theta}, \theta \neq -\pi$. Now applying Jack's Lemma 2.1, we get
\[
\frac{z_0w'(z_0)}{w(z_0)} = c, \quad c \geq 1. \] (2.30)

From (2.29) and (2.30) we obtain
\[
\Re \left( 1 + \frac{z_0f''(z_0)}{f'(z_0)} - p \right) = \Re \left( 1 + \frac{2\beta}{p} \frac{z_0w'(z_0)}{w(z_0)(2\beta + w(z_0))} \right)
\]
\[
= \Re \left( 1 + \frac{2\beta c}{p} \frac{1}{2\beta + w(z_0)} \right)
\]
\[
= 1 + \frac{2\beta c}{p} \Re \left( \frac{1}{2\beta + e^{i\theta}} \right)
\]
\[
\geq 1 + \frac{2\beta c}{p} \frac{1}{2\beta + e^{i\theta}}
\]
\[
\geq 1 + \frac{c}{p} \left( \frac{2\beta}{1 + 2\beta} \right)
\]
\[
\geq 1 + \frac{2\beta}{p(1 + 2\beta)},
\]
which contradicts the hypothesis (2.26). Therefore, we conclude that \(|w(z)| < 1\) for all \(z \in \mathbb{U}\) and (2.27) yields the inequality
\[
\left| \frac{zf'(z)}{f(z)} - p \right| < \frac{p}{2\beta}, \quad (p \in \mathbb{N}, z \in \mathbb{U})
\]
which implies that \(\frac{zf'(z)}{f(z)}\) lies in a circle centered at \(p\) and with radius \(\frac{p}{2\beta}\).

This amounts to say that \(\frac{zf'(z)}{f(z)} \in \Omega_{\alpha, \beta}\), and so
\[
\Re \left( \frac{zf'(z)}{f(z)} - \alpha \right) > \beta \left| \frac{zf'(z)}{f(z)} - p \right|
\]
(2.31)
i.e. \(f(z)\) is uniformly \(p\)-valent starlike of order \(\alpha\) in \(\mathbb{U}\).

Next, we determine the sufficient coefficient bound for uniformly \(p\)-valent convex functions of order \(\alpha\).

**Theorem 2.11.** Let \(f(z) \in \mathcal{A}_p\), then \(f(z)\) is uniformly \(p\)-valent convex of order \(\alpha\) in \(\mathbb{U}\) if the inequality
\[
\Re \left( \frac{1 + \frac{zf'''(z)}{f''(z)} - p}{1 + \frac{zf''(z)}{f'(z)} - p} \right) < 1 + \frac{2\beta}{p(1 + 2\beta) - 2}, \quad (2.32)
\]
holds.

**Proof.** Define \(w(z)\) by
\[
w(z) = \frac{2\beta}{p} \left( 1 + \frac{zf'''(z)}{f''(z)} - p \right), \quad p \in \mathbb{N}, z \in \mathbb{U}. \quad (2.33)
\]
Note that \(w(z)\) is analytic in \(\mathbb{U}\) and \(w(0) = 0\). Differentiating (2.33) logarithmically we get
\[
\left( 1 + \frac{zf'''(z)}{f''(z)} - p \right) = p - 1 + \frac{p}{2\beta}w(z) + \frac{pzw'(z)}{2\beta} \left( p - 1 + \frac{p}{2\beta}w(z) \right). \quad (2.34)
\]
Combining (2.33) and (2.34) we get
\[
\left(1 + \frac{zf'''(z)}{f''(z)} - p \right) = 1 + \frac{p - 1}{p} \frac{z \cdot f'(z)}{w(z)} + \frac{z \cdot w'(z)}{w(z) \left(p - 1 + \frac{p}{2\beta} w(z)\right)}, \quad (2.35)
\]
where \(p \in \mathbb{N}, z \in \mathbb{U} \).

Suppose now that there exists a point \(z_0 \in \mathbb{U}\) with \(|z_0| = 1\) such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1, \quad (w(z_0) \neq 1),
\]
and let \(w(z_0) = e^{i\theta}, \theta \neq -\pi\). Applying Jack's Lemma 2.1, we obtain
\[
\frac{z_0 w'(z_0)}{w(z_0)} = c, \quad c \geq 1. \quad (2.36)
\]
Now (2.35) and (2.36) yield
\[
\Re \left(1 + \frac{zf'''(z_0)}{f''(z_0)} - p \right) = \Re \left(1 + \frac{2\beta(p - 1)}{p} \frac{z \cdot w'(z_0)}{w(z_0)(2\beta(p - 1) + pw(z_0))} \right)
\]
\[
= 1 + \frac{2\beta}{p} (p - 1) \Re \left(\frac{1}{w(z_0)}\right) + \frac{2\beta c}{p} \Re \left(\frac{1}{2\beta(p - 1) + pw(z_0)}\right)
\]
\[
\geq 1 + \frac{2\beta}{p} (p - 1) \left|\frac{1}{e^{i\theta}}\right| + \frac{2\beta c}{p} \left|\frac{1}{2\beta(p - 1) + p e^{i\theta}}\right|
\]
\[
\geq 1 + \frac{2\beta}{p} (p - 1) + \frac{2\beta c}{p} \left(\frac{1}{p(1 + 2\beta) - 2\beta}\right)
\]
\[
\geq 1 + \frac{2\beta}{p} (p - 1) + \frac{2\beta}{p} \left(\frac{1}{p(1 + 2\beta) - 2\beta}\right)
\]
\[
\geq 1 + \frac{2\beta}{p(1 + 2\beta) - 2\beta}
\]
which contradicts the hypothesis (2.32). Therefore, we conclude that 
\[ |w(z)| < 1 \] for all \( z \in \mathbb{U} \) and as in the proof of the previous theorem, (2.33) yields the inequality
\[
\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| < \frac{p}{2\beta}, \quad (p \in \mathbb{N}, z \in \mathbb{U})
\]
which implies that \( 1 + \frac{zf''(z)}{f'(z)} \) lies in a circle centered at \( p \) and with radius \( \frac{p}{2\beta} \). This implies that \( 1 + \frac{zf''(z)}{f'(z)} \in \Omega_{\alpha,\beta} \), and therefore
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) > \beta \left| \frac{zf''(z)}{f'(z)} - (p - 1) \right| \tag{2.37}
\]
i.e. \( f(z) \) is uniformly \( p \)-valent convex of order \( \alpha \) in \( \mathbb{U} \).

Taking \( \beta = 1 \) in Theorems 2.10 and 2.11 we get the following corollaries proved by Al-Kharsani and Al-Hajiry in [2]

**Corollary 2.10.** Let \( f(z) \in A_p \) satisfies the inequality
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} - p \right) < 1 + \frac{2}{3p},
\]
then \( f(z) \) is uniformly \( p \)-valent starlike in \( \mathbb{U} \).

**Corollary 2.11.** Let \( f(z) \in A_p \) satisfies the inequality
\[
\Re \left( \frac{1 + zf'''(z)}{1 + zf''(z)} - p \right) < 1 + \frac{2}{3p - 2},
\]
then \( f(z) \) is uniformly \( p \)-valent convex in \( \mathbb{U} \).

In order to prove our next result we need the following definition:
Definition 2.4. A function \( f(z) \in \mathcal{A}_p \) is said to be uniformly \( p \)-valent close-to-convex (or uniformly close-to-convex when \( p = 1 \)) of order \( \alpha \) in \( \mathbb{U} \) if it satisfies the inequality

\[
\Re \left( \frac{zf'(z)}{g(z)} \right) \geq \beta \left| \frac{zf'(z)}{g(z)} - p \right| + \alpha,
\]

for some \( g(z) \in \mathcal{SD}_p(\alpha, \beta) \).

The following theorem gives the sufficient condition for uniformly \( p \)-valent close-to-convex functions.

**Theorem 2.12.** Let \( f(z) \in \mathcal{A}_p \) satisfies the inequality

\[
\Re \left( \frac{zf''(z)}{f'(z)} \right) < p - \frac{2\beta}{1 + 2\beta}, \tag{2.38}
\]

then \( f(z) \) is uniformly \( p \)-valent close-to-convex of order \( \alpha \) in \( \mathbb{U} \).

**Proof.** Let us define \( w(z) \) by

\[
w(z) = \frac{2\beta}{p} \left( \frac{f'(z)}{z^{p-1}} - p \right), \quad p \in \mathbb{N}, z \in \mathbb{U}. \tag{2.39}
\]

Clearly, \( w(z) \) is analytic in \( \mathbb{U} \) and \( w(0) = 0 \). Moreover, logarithmic differentiation of (2.39) give rise to

\[
\frac{zf''(z)}{f'(z)} = (p - 1) + \frac{zw'(z)}{2\beta + w(z)}. \tag{2.40}
\]
As earlier, using conditions of Jack’s Lemma and (2.38), we get
\[
\Re \left( \frac{z_0 f''(z_0)}{f'(z_0)} \right) = (p - 1) + c \Re \left( \frac{w(z_0)}{2 \beta + w(z_0)} \right)
\]
\[
= (p - 1) + c \Re \left( \frac{e^{i \theta}}{2 \beta + e^{i \theta}} \right)
\]
\[
\geq (p - 1) + c \frac{1}{1 + 2 \beta}
\]
\[
\geq (p - 1) + \frac{1}{1 + 2 \beta}
\]
\[
\geq p - \frac{2 \beta}{1 + 2 \beta}
\]
which contradicts the hypothesis (2.38). Therefore, we conclude that
\[|w(z)| < 1\] for all \(z \in U\) and (2.39) yields the inequality
\[
\left| \frac{f'(z)}{z^{p-1}} \right| < \frac{p}{2 \beta}, \quad (p \in \mathbb{N}, z \in U)
\]
which implies that \(\frac{f'(z)}{z^{p-1}} \in \Omega_{\alpha, \beta}\), and therefore
\[
\Re \left( \frac{f'(z)}{z^{p-1} - \alpha} \right) > \beta \left| \frac{f'(z)}{z^{p-1} - p} \right|
\]
\[
\text{(2.41)}
\]
i.e. \(f(z)\) is uniformly \(p\)-valent close-to-convex of order \(\alpha\) in \(U\).

If \(\beta = 1\), then Theorem 2.12 gives us the corresponding result of Al-Kharsani and Al-Hajiry established in [2].

**Corollary 2.12.** Let \(f(z) \in A_p\) satisfies the inequality
\[
\Re \left( \frac{z f''(z)}{f'(z)} \right) < p - \frac{2}{3},
\]
then \(f(z)\) is uniformly \(p\)-valent close-to-convex of order \(\alpha\) in \(U\).

For parametric values \(p = 1, \beta = 1\) in Theorems 2.10, 2.11 and 2.12 we get
Corollary 2.13. Let $f(z) \in A$ satisfies the inequality
\[
\Re \left( \frac{zf''(z)}{f'(z)} - 1 \right) < \frac{5}{3},
\]
then $f(z)$ is uniformly starlike in $\mathbb{U}$.

Corollary 2.14. Let $f(z) \in A$ satisfies the inequality
\[
\Re \left( \frac{zf'''(z)}{f''(z)} \right) < 3,
\]
then $f(z)$ is uniformly convex in $\mathbb{U}$.

Corollary 2.15. Let $f(z) \in A_p$ satisfies the inequality
\[
\Re \left( \frac{zf''(z)}{f'(z)} \right) < \frac{1}{3},
\]
then $f(z)$ is uniformly $p$-valent close-to-convex in $\mathbb{U}$.

Remark 2.5. For different values of the parameters $p, \alpha$ and $\beta$ in all our results of this chapter one can easily obtain many other interesting results that have been provided in [1, 2, 9, 10, 14] and references therein.
References


