Chapter-VI
Chapter 6

Certain Subclasses of Harmonic Univalent Functions with Negative Coefficients

A mathematician is a device for turning coffee into theorems.

- Paul Erdos

6.1 Introduction

In this Chapter, first we shall introduce and study a new subclass of harmonic univalent functions having negative coefficients which is defined using Ruscheweyh derivative operator and then we shall consider another subclass of Sakaguchi type harmonic univalent functions with negative coefficients. The harmonic functions are well known for their natural role in parameterizing minimal surfaces and have been studied by many differential geometers such as Choquet [4], Kneser [12].

Let $S_H$ denote the class of functions $f = h + \bar{g}$ which are harmonic, orientation preserving and univalent in the unit disk $\mathbb{U} = \{z : |z| < 1\}$
normalized by \( f(0) = f_z(0) - 1 = 0 \), where \( h \) and \( g \) are given by

\[
h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1.
\]

In 1984, Cluine and Shiel-Small [5] investigated the class \( S_H \) and studied its geometric subclasses and obtained some coefficient bounds. Since then many authors have worked on harmonic univalent functions and several papers on functions belonging to the class \( S_H \) and its important subclasses has been written. We refer to [1, 2, 6, 8, 9, 13, 14, 17–20] and references therein for a complete description of various subclasses of \( S_H \).

Let \( S_{H_0} \) be the subfamily of \( S_H \) consisting of those functions \( f \in S_H \) whose co-analytic part is zero. We now introduce a subclass of \( S_{H_0} \) denoted by \( T S_H(\lambda, \rho, \gamma) \) consisting of functions \( f \in S_{H_0} \) which satisfy the condition

\[
\Re \left\{ (1 + \rho e^{i\gamma}) z^m \frac{d^m}{dz^m} (D^\lambda f(z)) + (2 + \rho e^{i\gamma}) \sum_{j=1}^{m-1} z^j \frac{d^j}{dz^j} (D^\lambda f(z)) \right\} \geq 0,
\]

where, \( z = r e^{i\theta} \), \( \gamma \) and \( \theta \) are real such that \( 0 \leq \gamma < 1 \), \( 0 \leq \rho < 1 \), \( 0 \leq r < 1 \) and \( D^\lambda(f(z)) \) is the Ruscheweyh derivative [15] of \( f \) and is defined by

\[
D^\lambda(f(z)) = \sum_{n=1}^{\infty} B_n(\lambda) a_n z^n, \quad \lambda > -1,
\]

\[
B_n(\lambda) = \frac{(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1)}{(n-1)!} = \frac{1}{(n-1)!} \prod_{i=1}^{n-1}(\lambda + i)
\]

\[
= \binom{n + \lambda - 1}{n - 1} = B(n, \lambda).
\]

Further, let us denote by \( \overline{S}_H \) the subfamily of \( S_H \) consisting of functions
6.2 Coefficient Bounds

\[ f = h + \bar{g}, \text{ where} \]

\[ h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = -\sum_{k=1}^{\infty} b_k z^k, \quad a_k \geq 0, b_k \geq 0. \quad (6.3) \]

For \( g = 0 \), we denote by \( \overline{S}_{H_0} \), the subfamily of \( S_H \). Finally, let \( \overline{T S}_{H_0}(\lambda, \rho, \gamma) \) be the subclass of functions \( f \in \overline{S}_{H_0} \) and for which (6.1) holds true.

We refer to [10, 11] for many other subclasses of related functions studied by Kanas and Srivastava and Kanas and Wisnioska respectively.

We now briefly describe contents of this chapter. In section 6.2, sufficient condition for a function to be in \( TS_{H_0}(\lambda, \rho, \gamma) \) is given. Coefficients bounds for a function to be in \( \overline{T S}_{H_0}(\lambda, \rho, \gamma) \) are provided. Section 6.3 deals with extreme points and distortion bounds for function belonging to theore classes. Section 6.4, concentrates on convolution and other allied properties.

In section 6.5, we introduce a new subclass denoted by \( \overline{SH}(\alpha) \) of Sakaguchi type univalent harmonic functions with negative coefficient. Coefficient bounds for functions in this case are given in section 6.5. A necessary and sufficient condition for a function to be in \( \overline{SH}(\alpha) \) is given. Growth bounds are also discussed.

### 6.2 Coefficient Bounds

We state the following Theorem due to Jahangiri [8].

**Theorem 6.1.** Let \( f = h + \bar{g} \), where

\[ h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = -\sum_{k=1}^{\infty} b_k z^k. \]

Furthermore let

\[ \sum_{k=1}^{\infty} \left( \frac{k(k - \alpha)}{1 - \alpha} |a_k| + \frac{k(k + \alpha)}{1 - \alpha} |b_k| \right) \leq 2, \]

where \( a_1 = 1 \) and \( 0 \leq \alpha < 1 \). Then \( f \) is harmonic univalent in \( \mathbb{U} = \{ z : |z| < 1 \} \) and \( f \in \overline{TS_H}(\alpha) \). \( \overline{TS_H}(\alpha) \) is the subclass of \( TS_H(\alpha) \) consisting of harmonic univalent functions of order \( \alpha \).

Using the above theorem we now give sufficient condition for a function \( f \) to be in the class \( TS_{H_0}(\lambda, \rho, \gamma) \).

**Theorem 6.2.** Let \( f \in \mathcal{S}_{H_0} \), with

\[ \sum_{k=2}^{\infty} \left( \frac{k(1 + \rho)(k^m - 1)}{2(k - 1)} \right) |a_k| B_k(\lambda) \leq 2, \tag{6.4} \]

where \( a_1 = 1, \lambda > -1 \) and \( 0 \leq \rho < 1 \). Then \( f \) is harmonic univalent in \( \mathbb{U} \) and \( f \in TS_{H_0}(\lambda, \rho, \gamma) \).

**Proof.** Using Theorem 6.1, we obtain \( f \) is harmonic, since

\[ \sum_{k=2}^{\infty} k^2 |a_k| \leq \sum_{k=2}^{\infty} \left( \frac{k(1 + \rho)(k^m - 1)}{2(k - 1)} |a_k| \right) B_k(\lambda), \]

where \( 0 \leq \rho < 1, \lambda > -1 \).

To prove that \( f \in TS_{H_0}(\lambda, \rho, \gamma) \) it is sufficient to show that (6.2) holds by making use of the fact that

\[ \Re z \geq \alpha \iff |z + 1 - \alpha| \geq |1 - z + \alpha|, (z \in \mathbb{C}, \alpha \in \mathbb{R}). \]

For \( \alpha = 0 \), above condition takes the form

\[ \Re z \geq 0 \iff |z + 1| \geq |1 - z|, (z \in \mathbb{C}). \]
Let

\[ \phi(z, \lambda) = (1 + \rho e^{i\gamma}) z^m \frac{d^m}{dz^m} (D^\lambda f(z)) \]
\[ + (2 + \rho e^{i\gamma}) \sum_{j=1}^{m-1} z^j \frac{d^j}{dz^j} (D^\lambda f(z)) \]  

(6.5)

and

\[ \psi(z, \lambda) = \sum_{j=1}^{m-1} z^j \frac{d^j}{dz^j} (D^\lambda f(z)). \]  

(6.6)

Thus we have to show that

\[ |\phi(z, \lambda) + \psi(z, \lambda)| - |\phi(z, \lambda) - \psi(z, \lambda)| \geq 0. \]  

(6.7)

Replacing values of \( \phi \) and \( \psi \) from (6.5) and (6.6) respectively in (6.7) we
obtain

\[
|\phi(z, \lambda) + \psi(z, \lambda)| - |\phi(z, \lambda) - \psi(z, \lambda)| = \\
(1 + \rho e^{i\gamma}) z^m \frac{d^m}{dz^m}(D^\lambda f(z)) + (3 + \rho e^{i\gamma}) \sum_{j=1}^{m-1} z^j \frac{d^j}{dz^j}(D^\lambda f(z)) \\
- (1 + \rho e^{i\gamma}) \sum_{j=1}^{m} z^j \frac{d^j}{dz^j}(D^\lambda f(z)) \\
= (1 + \rho e^{i\gamma}) z^m \frac{d^m}{dz^m} \left( \sum_{k=1}^{\infty} B_k(\lambda) a_k z^k \right) + (3 + \rho e^{i\gamma}) \sum_{j=1}^{m-1} z^j \frac{d^j}{dz^j} \left( \sum_{k=1}^{\infty} B_k(\lambda) a_k z^k \right) \\
\times \left( \sum_{k=1}^{\infty} B_k(\lambda) a_k z^k \right) - (1 + \rho e^{i\gamma}) \sum_{j=1}^{m} z^j \frac{d^j}{dz^j} \left( \sum_{k=1}^{\infty} B_k(\lambda) a_k z^k \right) \\
= (1 + \rho e^{i\gamma}) \sum_{k=m}^{\infty} k(k-1)(k-2) \cdots (k-m+1) B_k(\lambda) a_k z^k \\
+ (3 + \rho e^{i\gamma}) \sum_{k=1}^{\infty} \{ k + k(k-1) + k(k-1)(k-2) + \cdots \\
+ k(k-1)(k-2) \cdots (k-m+1) \} B_k(\lambda) a_k z^k \\
\times \{ k + k(k-1) + k(k-1)(k-2) + \cdots \\
+ k(k-1)(k-2) \cdots (k-m) \} B_k(\lambda) a_k z^k \]
\[ \begin{align*} 
&= \left( 3 + \rho e^{i\gamma} \right) z + (3 + \rho e^{i\gamma}) \left\{ \sum_{k=2}^{\infty} k^2 + \sum_{k=3}^{\infty} k(k-1)(k-2) + \cdots \right\} \\
&+ \sum_{k=m-1}^{\infty} k(k-1)(k-2) \cdots (k-m+2) \right\} B_k(\lambda) a_k z^k \\
&+ 2(2 + \rho e^{i\gamma}) \sum_{k=m-1}^{\infty} k(k-1)(k-2) \cdots (k-m+1) B_k(\lambda) a_k z^k \\
&- \left( 1 + \rho e^{i\gamma} \right) z + (1 + \rho e^{i\gamma}) \left\{ \sum_{k=2}^{\infty} k^2 + \sum_{k=3}^{\infty} k(k-1)(k-2) + \cdots \right\} B_k(\lambda) a_k z^k \\
&+ \sum_{k=m+1}^{\infty} k(k-1)(k-2) \cdots (k-m) \right\} B_k(\lambda) a_k z^k \\
&\geq (3 + \rho) |z| - (1 + \rho) \left\{ \sum_{k=2}^{\infty} k^2 + \sum_{k=3}^{\infty} k(k-1)(k-2) + \cdots \right\} B_k(\lambda) |a_k||z|^k \\
&+ \sum_{k=m-1}^{\infty} k(k-1)(k-2) \cdots (k-m+2) \right\} B_k(\lambda) |a_k||z|^k \\
&- 2(2 + \rho) \sum_{k=m}^{\infty} k(k-1)(k-2) \cdots (k-m+1) B_k(\lambda) |a_k||z|^k \\
&- (1 + \rho) |z| - (1 + \rho) \left\{ \sum_{k=2}^{\infty} k^2 + \sum_{k=3}^{\infty} k(k-1)(k-2) + \cdots \right\} B_k(\lambda) |a_k||z|^k \\
&+ \sum_{k=m+1}^{\infty} k(k-1)(k-2) \cdots (k-m) \right\} B_k(\lambda) |a_k||z|^k \\
&\geq 2|z| - (1 + \rho) \sum_{k=2}^{\infty} (k + k^2 + k^3 + \cdots + k^{m-2}) B_k(\lambda) |a_k||z|^k \\
&- 2(2 + \rho) \sum_{k=2}^{m-1} B_k(\lambda) |a_k||z|^k - (1 + \rho) \sum_{k=2}^{m} B_k(\lambda) |a_k||z|^k \\
&\geq 2|z| \left\{ 1 - \frac{(1 + \rho)}{2} \sum_{k=2}^{m-1} \frac{k(k^m-1)}{(k-1)} B_k(\lambda) |a_k||z|^{k-1} \right\} \\
&\geq 0. 
\end{align*} \]
Thus, \( f(z) \in TSH_{0}(\lambda, \rho, \gamma) \).

\textbf{Remark 6.1.} Note that the above result is sharp. To see this, consider a harmonic function of the form

\[
f(z) = z + \sum_{k=2}^{\infty} \frac{2(k-1)}{(1 + \rho)k(k^m - 1)} A_k z^k,
\]

with \( \sum_{k=2}^{\infty} A_k = 1 \).

\textbf{Theorem 6.3.} Let \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ a_k \geq 0 \). Then \( f(z) \) is in \( TSH_{0}(\lambda, \rho, \gamma) \) if and only if

\[
\sum_{k=2}^{\infty} \left( \frac{k(1 + \rho)(k^m - 1)}{2(k-1)} \right) |a_k| B_k(\lambda) \leq 2,
\]

where \( a_1 = 1, \lambda > -1, 0 \leq \gamma < 1, 0 \leq \rho < 1 \) and \( B_k(\lambda) = \frac{1}{(k-1)!} \prod_{j=1}^{k-1} (\lambda + j) \).

\textbf{Proof.} Note that, \( TSH_{0}(\lambda, \rho, \gamma) \subset TSH_{0}(\lambda, \rho, \gamma) \). Thus the ‘necessary’ part follows from Theorem 6.2.

To prove the ‘only if’ part we will show that \( f \notin TSH_{0}(\lambda, \rho, \gamma) \) if the inequality (6.10) does not hold good.

A function \( f \in TSH_{0}(\lambda, \rho, \gamma) \) if and only if

\[
\Re \left\{ (1 + \rho e^{i\gamma}) z^m \frac{d^m}{dz^m} (D^\lambda f(z)) + (2 + \rho e^{i\gamma}) \sum_{j=1}^{m-1} z^j \frac{d^j}{dz^j} (D^\lambda f(z)) \right\} \geq 0,
\]

with \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k \).
Therefore,
\[
\mathcal{R}e \left\{ 2|z| - \sum_{k=2}^{\infty} \frac{k(1+\rho)(k^m - 1)}{(k-1)} |a_k|B_k(\lambda)|z|^k \right\} \geq 0.
\]

The last inequality must hold for all \(z, |z| = r < 1\). Choosing the values of \(z\) on the positive real axis, where \(0 < |z| = r < 1\), we have
\[
\left\{ 2 - \sum_{k=2}^{\infty} \frac{k(1+\rho)(k^m - 1)}{(k-1)} |a_k|B_k(\lambda)r^{k-1} \right\} \geq 0. 
\tag{6.11}
\]

Note that if the condition (6.10) does not holds then the numerator in (6.11) when \(r \to 1\) is negative, which contradicts the fact that \(f \in \mathcal{T}S_{H_0}(\lambda, \rho, \gamma)\). This completes the proof of the theorem. \(\Box\)

### 6.3 Extreme Points and Distortion Bounds

**Definition 6.1.** Let \(E \subseteq X\) be a subset of a topological vector space \(X\) over the field of complex numbers. The convex hull of \(E\) is the smallest convex set containing \(E\). It is denoted by \(\text{CLX}(E)\).

**Theorem 6.4.** \(f \in \text{CLX}(\mathcal{T}S_{H_0}(\lambda, \rho, \gamma))\) if and only if
\[
f(z) = \sum_{k=1}^{\infty} A_k h_k(z), \tag{6.12}
\]
where,
\[
h_1(z) = z, \quad h_k(z) = z - \frac{2(k-1)}{k(1+\rho)(k^m - 1)B_k(\lambda)} z^k, \quad k = 2, 3, \ldots ,
\]
and \(\sum_{k=1}^{\infty} A_k = 1, A_k \geq 0\).

In particular, the extreme points of \(\mathcal{T}S_{H_0}(\lambda, \rho, \gamma)\) are \(\{h_k\}\).
Proof. Let \( f \) be given by (6.12), then

\[
f(z) = \sum_{k=1}^{\infty} A_k z - \sum_{k=2}^{\infty} \frac{2(k-1)}{k(1+\rho)(k^m-1)B_k(\lambda)} A_k z^k
\]

Therefore,

\[
\sum_{k=2}^{\infty} \frac{k(1+\rho)(k^m-1)B_k(\lambda)}{2(k-1)} |a_k| = \sum_{k=2}^{\infty} A_k
\]

\[
= 1 - A_2 \leq 1.
\]

Showing that \( f \in CLX(\overline{TS}_{H_0}(\lambda, \rho, \gamma)) \).

Conversely, assume that \( f \in CLX(\overline{TS}_{H_0}(\lambda, \rho, \gamma)) \). Putting

\[
A_k = \frac{k(1+\rho)(k^m-1)B_k(\lambda)}{2(k-1)} |a_k|, \quad k = 2, 3, \ldots,
\]

where, \( \sum_{k=1}^{\infty} A_k = 1 \), we obtain by simple calculations

\[
f(z) = \sum_{k=1}^{\infty} A_k h_k(z)
\]

and the is proof complete.

\[\Box\]

**Theorem 6.5.** Let \( f \in \overline{TS}_{H_0}(\lambda, \rho, \gamma) \), then

\[
|f(z)| \leq r + B_2(\lambda)(1+\rho)(2^m-1)r^2, \quad |z| = r < 1, \quad (6.13)
\]

and

\[
|f(z)| \geq r - B_2(\lambda)(1+\rho)(2^m-1)r^2, \quad |z| = r < 1. \quad (6.14)
\]
6.4 Hadamard Product and Other Properties

Proof.

\[ |f(z)| \leq r + \sum_{k=2}^{\infty} |a_k|r^k \]
\[ \leq r + \sum_{n=2}^{\infty} |a_k|r^n \]
\[ = r + \frac{1}{B_2(\lambda)(1 + \rho)(2^m - 1)} \sum_{k=2}^{\infty} (1 + \rho)(2^m - 1)|a_k|B_k(\lambda)r^2 \]
\[ \leq r + B_2(\lambda)(1 + \rho)(2^m - 1)r^2 \]

which is (6.13). To prove the second inequality, let us note that

\[ |f(z)| \geq r - \sum_{k=2}^{\infty} |a_k|r^k \]
\[ \geq r - \sum_{n=2}^{\infty} |a_k|r^n \]
\[ = r - \frac{1}{B_2(\lambda)(1 + \rho)(2^m - 1)} \sum_{k=2}^{\infty} (1 + \rho)(2^m - 1)|a_k|B_k(\lambda)r^2 \]
\[ \geq r + B_2(\lambda)(1 + \rho)(2^m - 1)r^2 \]

Hence, the Theorem. \(\square\)

6.4 Hadamard Product and Other Properties

Definition 6.2. Let

\[ f(z) = z - \sum_{k=2}^{\infty} |a_k|z^k \quad \text{and} \quad g(z) = z - \sum_{k=2}^{\infty} |b_k|z^k. \]

We define the convolution of \(f(z)\) and \(g(z)\) as

\[ (f \ast g)(z) = f(z) \ast g(z) = z - \sum_{k=2}^{\infty} |a_k||b_k|z^k. \]
Theorem 6.6. Let \( f, g \in TS_{H_0}(\lambda, \rho, \gamma) \), then we have

\[
(f * g)(z) \in TS_{H_0}(\lambda, \rho, \gamma).
\]

Proof. Since \( f(z) \) and \( g(z) \in TS_{H_0}(\lambda, \rho, \gamma) \) therefore both of them satisfy (6.10) and since \( |b_k| \leq 1 \), we may write

\[
\sum_{k=1}^{\infty} \left( \frac{k(1 + \rho)(k^m - 1)}{2(k - 1)} |a_kb_k| \right) B_k(\lambda) \leq \sum_{k=1}^{\infty} \left( \frac{k(1 + \rho)(k^m - 1)}{2(k - 1)} |a_k| \right) B_k(\lambda) \leq 2.
\]

Hence \( f * g \in TS_{H_0}(\lambda, \rho, \gamma) \).

Theorem 6.7. The class \( TS_{H_0}(\lambda, \rho, \gamma) \) is closed under convex combinations.

Proof. Assume that

\[
f_i \in TS_{H_0}(\lambda, \rho, \gamma), i = 1, 2, \ldots
\]

are defined by

\[
f_i(z) = z - \sum_{k=2}^{\infty} |a_{i,k}| z^k.
\]

Using (6.10), we get

\[
\sum_{k=1}^{\infty} \left( \frac{k(1 + \rho)(k^m - 1)}{2(k - 1)} |a_{i,k}| \right) B_k(\lambda) \leq 2. \ \forall \ i = 1, 2, \ldots
\]

For \( \sum_{i=1}^{\infty} \alpha_i, 0 \leq \alpha_i \leq 1 \), we may write convex combination of \( f_i \) as

\[
\sum_{i=1}^{\infty} \alpha_i f_i = z - \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} \alpha_i |a_{i,k}| \right) z^k.
\]
Thus,
\[
\sum_{k=1}^{\infty} \left\{ \frac{k(1 + \rho)(k^m - 1)}{2(k - 1)} \left( \sum_{i=1}^{\infty} |a_{i,k}| \right) \right\} B_k(\lambda) \\
\sum_{i=1}^{\infty} \left\{ \sum_{k=1}^{\infty} \left( \frac{k(1 + \rho)(k^m - 1)}{2(k - 1)} |a_{i,k}| \right) B_k(\lambda) \right\} \alpha_i \\
\leq 2 \sum_{i=1}^{\infty} \alpha_i = 2.
\]

Hence, \(\sum_{i=1}^{\infty} \alpha_i f_i \in \overline{TSH_0}(\lambda, \rho, \gamma).\) \qed

6.5 Sakaguchi Type Harmonic Univalent Functions with Negative Coefficients

In this section we introduce a new subclass \(\overline{SH}(\alpha)\) of Sakaguchi type harmonic univalent functions. This kind of class of functions was first introduced and studied by Ahuja and Jahangiri [3] with positive coefficients. We extend the results of [6] to the functions belonging to the subclass \(\overline{SH}(\alpha)\) and obtain a necessary and sufficient condition for the functions belonging to the class \(\overline{SH}(\alpha)\) in the unit disk and obtain coefficient condition and distortion bounds.

Let \(S_{\mathcal{H}}\) denote the class of all complex valued, harmonic and sense preserving univalent functions normalized by \(f(0) = 0 = f_z(0) - 1\) with \(f_z(0)\) denotes partial derivative of \(f(z)\) at \(z = 0\). We call \(h\) and \(g\) analytic part and co-analytic part of \(f\) respectively. The harmonic function \(f = h + \bar{g}\) for \(g \equiv 0\) reduces to an analytic function \(f = h\).

Sakaguchi [16] introduced the class \(S_S\) of analytic functions in \(U\) which are star like with respect to symmetrical points. An analytic function
6.6 Coefficient Condition

$f(z)$ is said to be starlike with respect to symmetrical points if and only if

$$\Re \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > 0 \quad (6.15)$$

Extending the above definition to include harmonic functions, Ahuja and Jahangiri in [3] denoted the class of complex valued, sense preserving, harmonic starlike functions $f$ of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad b_n \geq 0. \quad (6.16)$$

by $S'H(\alpha)$ which satisfy the condition

$$\Re \left\{ \frac{2\frac{\partial}{\partial \theta} f(re^{i\theta})}{f(re^{i\theta}) - f(-re^{i\theta})} \right\} \geq \alpha \quad (6.17)$$

We now denote by $\overline{S'H}(\alpha)$ the subclass of $S'H(\alpha)$ consisting of functions of the form $f = h + \overline{g}$, where

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \quad \text{and} \quad g(z) = -\sum_{n=1}^{\infty} b_n z^n, \quad b_n \geq 0. \quad (6.18)$$

In the next section we show that the coefficient bounds which had proved sufficient in [3], is necessary too for functions in the class $\overline{S'H}(\alpha)$. Further we give distortion bounds.

6.6 Coefficient Condition

We state a lemma from [3] which will we needed in the sequel. It gives a sufficient condition for functions in $S'H(\alpha)$.
Lemma 6.1. For $h$ and $g$ as in (6.16), let the harmonic function $f = h + \bar{g}$ satisfy

$$\sum_{n=1}^{\infty} \left\{ \frac{2(n-1)}{1-\alpha} (|a_{2n-2}| + |b_{2n-2}|) + \frac{2n-1-\alpha}{1-\alpha} |a_{2n-1}| \\ + \frac{2n-1+\alpha}{1-\alpha} |b_{2n-1}| \right\} \leq 2$$ (6.19)

where $a_1 = 1$ and $0 \leq \alpha < 1$. Then $f$ is sense-preserving harmonic univalent function in $U$ and $f \in \mathcal{SH}(\alpha)$.

We now state main theorem of this section.

**Theorem 6.8.** Let $f = h + \bar{g}$ be as given in (6.16). Then $f \in \overline{\mathcal{SH}}(\alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left\{ \frac{2(n-1)}{1-\alpha} (|a_{2n-2}| + |b_{2n-2}|) + \frac{2n-1-\alpha}{1-\alpha} |a_{2n-1}| \\ + \frac{2n-1+\alpha}{1-\alpha} |b_{2n-1}| \right\} \leq 2$$ (6.20)

where $a_1 = 1$ and $0 \leq \alpha < 1$.

**Proof.** Assume that (6.20) holds then the if part follows from Lemma 6.1 by observing that if the analytic and co-analytic part of $f = h + \bar{g} \in \mathcal{SH}(\alpha)$ are of the form (6.18) then $f \in \overline{\mathcal{SH}}(\alpha)$.

To prove the converse, we will show that $f = h + \bar{g} \notin \overline{\mathcal{SH}}(\alpha)$ if the condition (6.20) does not hold. Notice that a necessary and sufficient condition for $f = h + \bar{g}$ given by (6.18) to be starlike of order $\alpha$, $0 \leq \alpha < 1$, is that $\Re \left\{ \frac{2g(ze^{i\theta})}{f(ze^{i\theta})-f(-ze^{i\theta})} \right\} > \alpha$. This is equivalent to

$$\Re \left\{ \frac{2(\bar{z}h'(z) - zg'(z))}{(h(z) - h(-\bar{z})) + (g(z) - g(-\bar{z}))} \right\} - \alpha \geq 0.$$ (6.21)
Now we may write,

\[
h(z) - h(-z) = 2z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} (-1)^n a_n z^n \\
= 2z - \sum_{n=2}^{\infty} 2a_{2n-1} z^{2n-1}.
\] (6.22)

Similarly,

\[
g(z) - g(-z) = -\sum_{n=1}^{\infty} b_n z^n + \sum_{n=1}^{\infty} (-1)^n b_n z^n = -\sum_{n=1}^{\infty} 2b_{2n-1} z^{2n-1}.
\] (6.23)

Therefore,

\[
2(z h'(z) - zg'(z)) - \alpha \left\{ (h(z) - h(-z)) + (g(z) - g(-z)) \right\}
\]

\[
= 2(1 - \alpha) z - \sum_{n=2}^{\infty} (2n - \alpha + (-1)^n \alpha) a_n z^n + \sum_{n=1}^{\infty} (2n - \alpha + (-1)^n \alpha) b_n z^n \\
= 2(1 - \alpha) z - \sum_{n=2}^{\infty} \left\{ 2(2n - 2)a_{2n-1} z^{2n-2} + (2(2n - 1) - 2\alpha)a_{2n-1} z^{2n-1} \right\} \\
+ \sum_{n=1}^{\infty} \left\{ 2(2n - 2)b_{2n-1} z^{2n-2} + (2(2n - 1) + 2\alpha)b_{2n-1} z^{2n-1} \right\},
\] (6.24a)

and

\[
(h(z) - h(-z)) + (g(z) - g(-z)) = 2 \left\{ z - \sum_{n=2}^{\infty} 2a_{2n-1} z^{2n-1} \\
- \sum_{n=1}^{\infty} 2b_{2n-1} z^{2n-1} \right\}.
\] (6.24b)
6.6 Coefficient Condition

Hence,

\[
0 \leq \Re \left\{ \frac{2(zh'(z) - zg'(z))}{(h(z) - h(-z)) + (g(z) - g(-z))} \right\} - \alpha \\
= \Re \left\{ \frac{2(zh'(z) - zg'(z)) - \alpha [(h(z) - h(-z)) + (g(z) - g(-z))]}{(h(z) - h(-z)) + (g(z) - g(-z))} \right\}
\]

\[
(1 - \alpha)z - \sum_{n=2}^{\infty} \left\{ 2(n-1)a_{2n-2}z^{2n-2} + (2n - 1 - \alpha)a_{2n-1}z^{2n-1} \right\} \\
\leq \frac{z - \sum_{n=2}^{\infty} a_{2n-1}z^{2n-1} - \sum_{n=1}^{\infty} b_{2n-1}z^{2n-1}}{z - \sum_{n=2}^{\infty} a_{2n-1}z^{2n-1} - \sum_{n=1}^{\infty} b_{2n-1}z^{2n-1}} \\
(1 - \alpha) - \sum_{n=2}^{\infty} \left\{ 2(n-1)a_{2n-2} + (2n - 1 - \alpha)a_{2n-1} \right\} \\
\leq \frac{1 + \sum_{n=2}^{\infty} a_{2n-1} + \sum_{n=2}^{\infty} b_{2n-1}}{1 + \sum_{n=2}^{\infty} a_{2n-1} + \sum_{n=2}^{\infty} b_{2n-1}} \\
\sum_{n=2}^{\infty} \left\{ 2(n-1)b_{2n-2} + (2n - 1 + \alpha)b_{2n-1} \right\} \\
+ \frac{1 + \sum_{n=2}^{\infty} a_{2n-1} + \sum_{n=2}^{\infty} b_{2n-1}}{1 + \sum_{n=2}^{\infty} a_{2n-1} + \sum_{n=2}^{\infty} b_{2n-1}} \cdot 
\]

If the condition (6.20) does not hold good then the numerator in the above is \( \leq 0 \) i.e.

\[
(1 - \alpha) - \sum_{n=2}^{\infty} \left\{ 2(n-1)(a_{2n-2} + b_{2n-2}) + (2n - 1 - \alpha)a_{2n-1} \\
+ (2n - 1 + \alpha)b_{2n-1} \right\} \leq 0.
\]
Hence \( f \notin \overline{SH}(\alpha) \). This completes the proof. \( \square \)

**Theorem 6.9.** Let \( f \in \overline{SH}(\alpha) \) then

\[
(1 - b_1)r - \left( \frac{1 - \alpha}{2} - \frac{1 + \alpha}{2} |b_1| \right) r^2 \leq |f(z)| \leq (1 + b_1)r + \left( \frac{1 - \alpha}{2} - \frac{1 + \alpha}{2} |b_1| \right) r^2,
\]

with \( |z| = r < 1 \).

**Proof.** Let \( f \in \overline{SH}(\alpha) \). We can now write

\[
|f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n + \sum_{n=1}^{\infty} b_n |z|^n
\]

\[
\leq (1 + b_1)r + \sum_{n=2}^{\infty} (a_{2n-2} + b_{2n-2}) r^{2n-2} + \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1}) r^{2n-1}
\]

\[
\leq (1 + b_1)r + r^2 \sum_{n=2}^{\infty} (a_{2n-2} + b_{2n-2}) + r^2 \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1}) r
\]

\[
= (1 + b_1)r + \frac{(1 - \alpha)}{2} r^2 \sum_{n=2}^{\infty} \frac{2}{(1 - \alpha)} (a_{2n-2} + b_{2n-2}) + r^2 \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1}) r
\]

\[
\leq (1 + b_1)r + \frac{(1 - \alpha)}{2} r^2 \sum_{n=2}^{\infty} \frac{2(n-1)}{(1 - \alpha)} (a_{2n-2} + b_{2n-2}) + r^2 \sum_{n=2}^{\infty} \frac{2n-1 - \alpha}{1 - \alpha} a_{2n-1} + \frac{2n-1 + \alpha}{1 - \alpha} b_{2n-1}
\]

\[
\leq (1 + b_1)r + \frac{(1 - \alpha)}{2} \left( 1 - \frac{1 + \alpha}{1 - \alpha} b_1 \right) r^2 \quad \text{(using 6.20)}
\]

\[
= (1 + b_1)r + \left( \frac{1 - \alpha}{2} - \frac{1 + \alpha}{2} |b_1| \right) r^2.
\]
Similarly,

\[ |f(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^n - \sum_{n=1}^{\infty} b_n |z|^n \]

\[ \geq (1 - b_1)r - \sum_{n=2}^{\infty} (a_{2n-2} + b_{2n-2})r^{2n-2} - \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1})r^{2n-1} \]

\[ \geq (1 - b_1)r - r^2 \sum_{n=2}^{\infty} (a_{2n-2} + b_{2n-2}) - r^2 \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1})r \]

\[ = (1 - b_1)r - \frac{(1 - \alpha)}{2} r^2 \sum_{n=2}^{\infty} \frac{2}{(1 - \alpha)} (a_{2n-2} + b_{2n-2}) \]

\[ - r^2 \sum_{n=2}^{\infty} (a_{2n-1} + b_{2n-1})r \]

\[ \geq (1 - b_1)r - \frac{(1 - \alpha)}{2} r^2 \sum_{n=2}^{\infty} \frac{2(n - 1)}{(1 - \alpha)} (a_{2n-2} + b_{2n-2}) \]

\[ - r^2 \sum_{n=2}^{\infty} \left( \frac{2n - 1 - \alpha}{1 - \alpha} a_{2n-1} + \frac{2n - 1 + \alpha}{1 - \alpha} b_{2n-1} \right) \]

\[ \geq (1 - b_1)r - \frac{(1 - \alpha)}{2} \left( 1 - \frac{1 + \alpha}{1 - \alpha} b_1 \right) r^2 \quad \text{(using 6.20)} \]

\[ = (1 + b_1)r - \left( \frac{1 - \alpha}{2} - \frac{1 + \alpha}{2} |b_1| \right) r^2. \]

Combining the two inequalities we obtain the result. \( \square \)

**Corollary 6.1.** If \( f \in \overline{SH}(\alpha) \) then

\[ \left\{ w : |w| < \frac{1 + \alpha - (3 + \alpha)b_1}{2} \right\} \subset f(\overline{U}). \]

### 6.7 Concluding Remarks

We have already seen in this chapter that for suitably chosen parametric values many of our results reduces to those of [3, 10? , 11] and others.
However, we mention here that for $m \geq 1$, by considering subclasses of harmonic functions $f = h + \bar{g}$ of the form

$$h(z) = z^m - \sum_{k=2}^{\infty} a_{k+m-1} z^{k+m-1},$$

$$g(z) = \sum_{k=1}^{\infty} b_{k+m-1} z^{k+m-1}, \quad |b_m| < 1,$$

and denoting these classes by $TSS_{H_0}^m(\lambda, \rho, \gamma), SH(m, \alpha)$ etc. several extensions to the subclasses $TSS_{H_0}(\lambda, \rho, \gamma)$ and $SH(\alpha)$ are possible which will consists of $m$-valent harmonic functions. Most of our results of this chapter will then follow as particular cases to these new subclasses of harmonic multivalent functions along with many new and interesting results. We plan to consider few such extensions in the future works.
References


REFERENCES


