Chapter-IV
Chapter 4

Univalent Analytic Functions with Negative Coefficients using Ruscheweyh Derivatives

It is mathematics that offers the exact natural sciences a certain measure of security which, without mathematics, they could not attain.

—Albert Einstein

4.1 Introduction

Let $S$ denote the class of functions $f(z)$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, that are analytic and univalent in $U$ and let $S^*(\alpha)$ and $K(\alpha), (0 \leq \alpha < 1)$ denote respectively, the subclasses of $S$ that are starlike of order $\alpha$ and convex of order $\alpha$.

Let $T$ denote the subclass of $S$ consisting of functions whose non-zero coefficients, from the second on, are negative i.e. $T$ consists of functions $f(z)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad (z \in U) \quad (4.1)$$
that are analytic and univalent in $\mathbb{U}$. Let $ST^{*}(\alpha)$ and $\mathcal{K}\mathcal{T}(\alpha), (0 \leq \alpha < 1)$ denote the subclasses of $\mathcal{T}$ that are, respectively, starlike of order $\alpha$ and convex of order $\alpha$. Schild in [5] investigated the subclass of $\mathcal{T}$ consisting of polynomials. In recent years including past three decades, there have been numerous research papers on the study of various subclasses of $\mathcal{T}$. (See for example [1, 2, 6]). On the lines similar to those adopted in earlier chapters, we now propose a unified approach to the study of various subclasses of $\mathcal{T}$. For this purpose, we first introduce the concept of admissible pair of analytic functions.

**Definition 4.1.** Two functions $\psi_{1}(z) = z + \sum_{n=2}^{\infty} \lambda_{n}z^{n}$, $\psi_{2}(z) = z + \sum_{n=2}^{\infty} \mu_{n}z^{n}$, analytic in $\mathbb{U}$, are said to form an admissible pair if $\lambda_{n} \geq 0$, $\mu_{n} \geq 0$ and $\lambda_{n} \geq \mu_{n}$ for all $n \geq 2$.

We denote the admissible pair of functions by $(\psi_{1}, \psi_{2})$. Let $\mathcal{B}_{0}$ denote the class of analytic functions $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| \leq 1$. Functions in $\mathcal{B}_{0}$ are known as Schwartz functions.

**Definition 4.2.** Let $\alpha, \beta$ be real numbers with $-1 \leq \beta \leq 0$, $-1 \leq \alpha \leq 1$, $\beta < \alpha$, and let $(\psi_{1}, \psi_{2})$ be an admissible pair. We say that a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{I}_{\gamma}^{\alpha, \beta}(\psi_{1}, \psi_{2}), \gamma > -1$ if the conditions

$$D^{\gamma}f(z) * \psi_{2}(z) \neq 0, D^{\gamma}f(z) * (\alpha \psi_{2}(z) - \beta \psi_{1}(z)) \neq 0 \quad (4.2)$$

in $0 \leq |z| < 1$ and

$$\frac{D^{\gamma}f(z) * \psi_{2}(z)}{D^{\gamma}f(z) * \psi_{1}(z)} = \frac{1 + \alpha w(z)}{1 + \beta w(z)}, \quad (z \in \mathbb{U}). \quad (4.3)$$

are satisfied for some function $w(z) \in \mathcal{B}_{0}$.

Here $*$ denotes the Hadamard product or convolution of two analytic functions and $D^{\gamma}f(z)$ denotes the Ruscheweyh derivative of $f(z)$ intro-
duced by St. Ruscheweyh [4] as defined in the previous Chapter and is given by

\[ D^\gamma f(z) = \frac{z}{(1 - z)^{\gamma+1}} \ast f(z) = z + \sum_{n=2}^{\infty} a_n A_n(\gamma) z^n, \]

where

\[ A_n(\gamma) = \frac{(\gamma + 1)(\gamma + 2) \cdots (\gamma + n - 1)}{(n - 1)!}. \]

Note that,

\[ A_n(0) = 1, \quad A_n(1) = n \quad \text{and} \quad A_{n+1}(\gamma) = \frac{1}{\gamma + 1} A_n(\gamma + 1). \]

By convention we take \( A_1(\gamma) = 1. \)

On suitably choosing \( \psi_1, \psi_2, \alpha, \beta \) and \( \gamma \) in \( \mathcal{I}^\alpha_\gamma(\psi_1, \psi_2) \) we obtain important subclasses of \( \mathcal{A} \) and also that of \( \mathcal{S} \). For example, if \( 0 \leq \alpha < 1 \) then \( \mathcal{I}^{1-2\alpha,-1}_0 \left( \frac{z}{(1-z)^2}, \frac{z}{1-z} \right) \) and \( \mathcal{I}^{1-2\alpha,-1}_0 \left( \frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2} \right) \) are the well known classes \( \mathcal{S}^*(\alpha) \) and \( \mathcal{K}(\alpha) \) respectively. The class \( \mathcal{I}^{1,-2\theta,-1}_0 \left( \frac{z}{(1-z)^3}, z \right) \) is the class \( \mathcal{P}'(\alpha) \) consisting of functions in \( \mathcal{A} \), whose derivative has real part greater than \( \theta \) in \( \mathbb{U} \) and \( \mathcal{I}^{1-2\alpha,-1}_0 \left( \frac{z}{(1-z)^{N}}, \frac{z}{(1-z)^N} \right) \), \( N \geq 1 \) is the class \( \mathcal{S}^*(\alpha) \) studied by P. Singh [10] while \( \mathcal{I}^{0,-1}_0 \left( \frac{z}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}} \right) \equiv \mathcal{R}_\alpha \) is the class of prestarlike functions of order \( \alpha \). In fact many new subclasses of functions in \( \mathcal{S} \) or in \( \mathcal{A} \) can be defined by suitable selection of \( \psi_1, \psi_2, \alpha, \beta \) and \( \gamma \). It may be observed that in all the above mentioned classes, \( \psi_1 \) and \( \psi_2 \) form an admissible pair of functions.

Let \( \mathcal{I}^{\alpha,\beta}_\gamma,\mathcal{T}(\psi_1, \psi_2) = \mathcal{I}^{\alpha,\beta}_\gamma(\psi_1, \psi_2) \cap \mathcal{T} \) be the class consisting of functions of the form (4.1) which are analytic and univalent in \( \mathbb{U} \) and which belong to \( \mathcal{I}^{\alpha,\beta}_\gamma(\psi_1, \psi_2) \). As pointed out earlier, for \( 0 \leq \theta < 1 \) and \( 0 \leq \beta < 1 \), the classes \( \mathcal{I}^{1-2\alpha,-1}_{0,T} \left( \frac{z}{(1-z)^2}, \frac{z}{1-z} \right) = \mathcal{T}^*(\alpha), \mathcal{I}^{1-2\alpha,-1}_{0,T} \left( \frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2} \right) = \mathcal{K}\mathcal{T}(\alpha) \)
are extensively studied by Silverman [6] and $I_{0,T}^{0,-1} \left( \frac{z}{(1-z)^{2-2\alpha}} \right)$, $R[\alpha]$ is the class of prestarlike functions with negative coefficients studied by Silverman and Silvia [8], while $I_{0,T}^{(1-2\alpha)\beta,-\beta} \left( \frac{z+z^2}{(1-z)^{2}}, \frac{z}{(1-z)^2} \right)$ and $I_{0,T}^{(1-2\alpha)\beta,-\beta} \left( \frac{z}{(1-z)^2}, \frac{z}{(1-z)^2} \right)$ are respectively, the classes $K^*(\alpha, \beta)$ and $S^*(\alpha, \beta)$ considered by Gupta and Jain [1, 2]. One may obtain many other subclasses of $T$ by a suitable selection of functions $\psi_1(z)$ and $\psi_2(z)$ and the parameters in the class $I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)$.

In Section 4.2, we will find a sufficient condition on the Taylor coefficients for a function to be in $I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)$. For functions in $I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)$, this condition is also shown to be necessary. In Section 4.3, we find extreme points of the family $I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)$, distortion properties are also obtained here. In Section 4.4, we consider functions in $I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)$ with fixed second coefficient and determine their extreme points. The characterization of extreme points leads to distortion theorems for functions in this class. Section 4.5 deals with certain containment relations between these classes.

Unless otherwise stated explicitly, we shall assume throughout this chapter that $\alpha$ and $\beta$ satisfy

$$-1 \leq \beta \leq 0, -1 \leq \alpha \leq 1, \beta < \alpha, \alpha > -1$$

and that $\psi_1$ and $\psi_2$ form an admissible pair.

### 4.2 Characterization of the Class $I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)$

In this Section we give a characterization of the class $I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)$ by finding a necessary and sufficient condition for a function to be in $I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)$. Using this characterization we shall obtain coefficient estimates for func-
4.2 Characterization of the Class $\mathcal{I}^\alpha_\gamma(\psi_1, \psi_2)$

tions in this class.

**Theorem 4.1.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $\mathcal{A}$. If for some admissible pair $(\psi_1, \psi_2)$ and $\alpha, \beta$ satisfying (4.4) the inequality

$$
\sum_{n=2}^{\infty} \left\{ (1 - \beta) \lambda_n - (1 - \alpha) \mu_n \right\} A_n(\gamma) |a_n| \leq \alpha - \beta \tag{4.5}
$$

holds, then $f(z) \in \mathcal{I}^\alpha_\gamma(\psi_1, \psi_2)$.

**Proof.** Let (4.5) holds, then for $z \in \mathcal{U}$ consider,

$$
|D^\gamma f(z) * \psi_1(z) - D^\gamma f(z) * \psi_2(z)| - |\alpha D^\gamma f(z) * \psi_1(z) - \beta D^\gamma f(z) * \psi_2(z)|
$$

$$
= \left| \sum_{n=2}^{\infty} (\lambda_n - \mu_n) A_n(\gamma) a_n z^n \right| - (\alpha - \beta) z + \sum_{n=2}^{\infty} (\alpha \mu_n - \beta \lambda_n) A_n(\gamma) |a_n| z^n
$$

$$
\leq \left( \sum_{n=2}^{\infty} (\lambda_n - \mu_n) A_n(\gamma) |a_n| - (\alpha - \beta) + \sum_{n=2}^{\infty} (\alpha \mu_n - \beta \lambda_n) A_n(\gamma) |a_n| \right) |z|
$$

$$
= \left( \sum_{n=2}^{\infty} \left( (1 - \beta) \lambda_n - (1 - \alpha) \mu_n \right) A_n(\gamma) |a_n| - (\alpha - \beta) \right) |z|
$$

$$
\leq 0, \tag{4.6}
$$

since $\alpha \mu_n - \beta \lambda_n > 0$ for $-1 \leq \beta \leq 0, -1 \leq \beta < \alpha \leq 1$ and for all $\lambda_n, \mu_n, n \geq 2$ and (4.6) is non-positive because of (4.5).

Since $\lambda_n \geq \mu_n$ for $n \geq 2$, we have

$$
\frac{(1 - \beta) \lambda_n - (1 - \alpha) \mu_n}{\alpha - \beta} > \mu_n. \tag{4.7}
$$

Hence,

$$
1 - \sum_{n=2}^{\infty} \mu_n A_n(\gamma) |a_n| \geq 1 - \sum_{n=2}^{\infty} \frac{(1 - \beta) \lambda_n - (1 - \alpha) \mu_n}{\alpha - \beta} A_n(\gamma) |a_n| \geq 0 \tag{4.8}
$$
and so

\[ D^\gamma f(z) * \psi_2(z) \neq 0 \text{ in } 0 < |z| < 1, \]

from (4.6) we obtain

\[ \left| \frac{D^\gamma f(z) \* \psi_1(z) - 1}{\alpha - \beta \frac{D^\gamma f(z) \* \psi_1(z)}{D^\gamma f(z) \* \psi_2(z)}} \right| < 1 \quad (4.9) \]

which implies that \( f \in \mathcal{I}_\gamma^{\alpha,\beta}(\psi_1, \psi_2) \). \( \square \)

**Theorem 4.2.** Let \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \) be in \( \mathcal{A} \). Then \( f(z) \in \mathcal{I}_\gamma^{\alpha,\beta}(\psi_1, \psi_2) \) if and only if (4.5) is satisfied.

**Proof.** Let \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{I}_\gamma^{\alpha,\beta}(\psi_1, \psi_2) \). This implies that

\[
\left| \frac{D^\gamma f(z) \* \psi_1(z) - 1}{\alpha - \beta \frac{D^\gamma f(z) \* \psi_1(z)}{D^\gamma f(z) \* \psi_2(z)}} \right| = \left| \frac{\sum_{n=2}^{\infty} (\lambda_n - \mu_n) A_n(\gamma) |a_n| z^n}{\alpha \left( z - \sum_{n=2}^{\infty} A_n(\gamma) |a_n| \mu_n z^n \right) - \beta \left( z - \sum_{n=2}^{\infty} A_n(\gamma) |a_n| \lambda_n z^n \right)} \right| < 1, \text{ for } z \in \mathbb{U}. \quad (4.10)
\]
Using the fact that $|\Re(z)| < |z|$ for all $z$, it follows that

$$\Re\left(\frac{\sum_{n=2}^{\infty}(\lambda_n - \mu_n)A_n(\gamma)|a_n|z^n}{(\alpha - \beta)z + \sum_{n=2}^{\infty}(\beta\lambda_n - \alpha\mu_n)A_n(\gamma)|a_n|z^n}\right) < 1, \text{ for } z \in \mathbb{U}. \quad (4.11)$$

Since the functions $f(z), \psi_1(z)$ and $\psi_2(z)$ are analytic in $\mathbb{U}$, the expressions $\sum_{n=2}^{\infty}(\lambda_n - \mu_n)A_n(\gamma)|a_n|z^n$ and $\sum_{n=2}^{\infty}(\beta\lambda_n - \alpha\mu_n)A_n(\gamma)|a_n|z^n$ in (4.11) are convergent for $z \in \mathbb{U}$.

Now choose the values of $z$ on the real axis in $\mathbb{U}$, so that $\overline{D^\gamma f(z)*\psi_1(z)} = \overline{D^\gamma f(z)*\psi_2(z)}$ is real. Upon clearing the denominator in (4.11) and making use of the fact that $\overline{D^\gamma f(z)* (\alpha\psi_2 - \beta\psi_1)} \neq 0$ in $0 < |z| < 1$, we have

$$\sum_{n=2}^{\infty}(\lambda_n - \mu_n)A_n(\gamma)|a_n|z^n < \left((\alpha - \beta)z + \sum_{n=2}^{\infty}(\beta\lambda_n - \alpha\mu_n)A_n(\gamma)|a_n|z^n\right),$$

for all $z \in (-1, 1)$. Letting $z \to 1$ through real values we get

$$\sum_{n=2}^{\infty}((1 - \beta)\lambda_n - (1 - \alpha)\mu_n) A_n(\gamma)|a_n| \leq \alpha - \beta,$$

which is the required condition. Now other way around, condition (4.5) implies $f(z) \in \mathcal{I}_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)$, is an immediate consequence of the Theorem ???. This completes the proof. \[\square\]

**Remark 4.1.** For suitable choices of the functions $\psi_1, \psi_2$ and suitable values of the parameters $\alpha, \beta$ and $\gamma$ the above theorem gives necessary and sufficient condition for functions of the form $f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n$ to belong to a particular class. For instance, for the class $\mathcal{S}T(\alpha)$ we get Theorem 2 of Silverman [6] and for the class $\mathcal{R}[\alpha]$ we obtain...
4.3 Extreme Points and Distortion Properties

2 of Silverman and Silvia [8]. Similarly, the above theorem reduces to
Theorem 1 of [82] obtained earlier by Gupta and Jain [1] for the classes
$S^*(\alpha, \beta)$ and $P^*(\alpha, \beta)$, $(0 < \beta \leq 1)$ respectively. In addition one may
define new subclasses and get necessary and sufficient conditions for them.

**Corollary 4.1.** If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in $I_{\gamma_1,\gamma_2}(\psi_1, \psi_2)$ then

$$|a_n| \leq \frac{\alpha - \beta}{A_n(\gamma) ((1 - \beta)\lambda_n - (1 - \alpha)\mu_n)} \quad (4.12)$$

for all admissible values of $\alpha, \beta, \gamma, \lambda_n$ and $\mu_n$, $n \geq 2$.

The above result is sharp, for each $n$, for functions of the form

$$f_n(z) = z - \frac{\alpha - \beta}{A_n(\gamma) ((1 - \beta)\lambda_n - (1 - \alpha)\mu_n)} z^n, n \geq 2. \quad (4.13)$$

### 4.3 Extreme Points and Distortion Properties

It can be easily seen that for each admissible pair $(\psi_1, \psi_2)$, $-1 \leq \beta \leq 0$
and $-1 \leq \beta < \alpha \leq 1$ the class $I_{\gamma_1,\gamma_2}(\psi_1, \psi_2)$ is a compact subset of $\mathcal{A}$.
Also this class is a convex set. It is known that an extreme value of
a real valued continuous linear functional on a compact convex subset
of a locally convex space is attained at an extreme point only. Also it
has been observed that the maximum value of the $n$th coefficient $|a_n|$ is
attained for the function $f_n$ given by (4.13) for $n = 2, 3, \ldots$. Hence these
functions are extreme points. In view of this observation in the following
theorem we show that together with the identity functions these are
the only extreme points for the class $I_{\gamma_1,\gamma_2}(\psi_1, \psi_2)$. For properly chosen
functions $\psi_1, \psi_2$ and parameters $\alpha, \beta$ this theorem yields corresponding
results of Silverman [6], Silverman and Silvia [8], Mishra [3] (for the case
4.3 Extreme Points and Distortion Properties

$p = 1$, Gupta and Jain [1, 2] etc. For the sake of notational simplicity let us denote henceforth by

\[ \delta_n = ((1 - \beta)\lambda_n - (1 - \alpha)\mu_n). \quad (4.14) \]

**Theorem 4.3.** Let

\[ f_1(z) = z \] and \[ f_n(z) = z - \frac{\alpha - \beta}{A_n(\gamma)\delta_n}z^n. \]

Then \( f \in \mathcal{I}_{\gamma, T}^{\alpha, \beta}(\psi_1, \psi_2) \) if and only if it can be expressed in the form

\[ f(z) = \sum_{n=1}^{\infty} \chi_n f_n(z), \quad \text{where } \chi_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \chi_n = 1. \quad (4.16) \]

**Proof.** Let \( f(z) = \sum_{n=1}^{\infty} \chi_n f_n(z), \) where \( \chi_n \geq 0, \sum_{n=1}^{\infty} \chi_n = 1, f_1(z) \) and \( f_n(z) \) are given by (4.15). Then

\[
\begin{align*}
  f(z) &= \chi_1 f_1(z) + \sum_{n=2}^{\infty} \chi_n f_n(z) \\
  &= \left(1 - \sum_{n=2}^{\infty} \chi_n\right) z + \sum_{n=2}^{\infty} \chi_n \left(z - \frac{\alpha - \beta}{A_n(\gamma)\delta_n}z^n\right) \\
  &= z - \sum_{n=2}^{\infty} \frac{\alpha - \beta}{A_n(\gamma)\delta_n} \chi_n z^n.
\end{align*}
\]

Since

\[
\sum_{n=2}^{\infty} \frac{A_n(\gamma)\delta_n \chi_n(\alpha - \beta)}{(\alpha - \beta) A_n(\gamma)\delta_n} = \sum_{n=2}^{\infty} \chi_n = 1 - \chi_1 \leq 1,
\]

it follows from (4.5) that \( f \in \mathcal{I}_{\gamma, T}^{\alpha, \beta}(\psi_1, \psi_2) \).

Conversely suppose that \( f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n \in \mathcal{I}_{\gamma, T}^{\alpha, \beta}(\psi_1, \psi_2) \). Since from (4.12), we have \( |a_n| \leq \frac{\alpha - \beta}{A_n(\gamma)\delta_n}, n = 2, 3, \ldots \), hence putting

\[
\chi_n = \frac{A_n(\gamma)\delta_n}{\alpha - \beta} |a_n|,
\]

we have
\[
f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n = z - \sum_{n=2}^{\infty} \frac{x_n(\alpha - \beta)}{A_n(\gamma)\delta_n} z^n
\]
\[
= \left(1 - \sum_{n=2}^{\infty} \chi_n\right) z + \sum_{n=2}^{\infty} \chi_n \left(z - \frac{\alpha - \beta}{A_n(\gamma)\delta_n} z^n\right)
\]
\[
= \left(1 - \sum_{n=2}^{\infty} \chi_n\right) f_1(z) + \sum_{n=2}^{\infty} \chi_n f_n(z) = \sum_{n=1}^{\infty} \chi_n f_n(z).
\]
This completes the proof of the theorem. □

From above theorem we conclude that the extreme points of \(\mathcal{I}_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)\) are the functions defined by (4.15). Since \(\max |f(z)|, \min |f(z)|\) on \(|z| = r\) for \(f \in \mathcal{I}_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)\) occurs at one of its extreme points, therefore it is sufficient to examine the maximum and minimum on \(|z| = r\) of the functions \(f_n\) and their derivative \(f_n', n = 1, 2, \ldots\) as defined in (4.15). Hence we have the following distortion properties for functions \(f(z)\) in \(\mathcal{I}_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)\).

**Theorem 4.4.** Let \(f \in \mathcal{I}_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2)\) and \(\gamma_n\) are as defined in (4.14), \(n = 2, 3, \ldots\). Then
\[
r - \max_n \frac{(\alpha - \beta)}{A_n(\gamma)\delta_n} r^n \leq |f(re^{i\theta})| \leq r + \max_n \frac{(\alpha - \beta)}{A_n(\gamma)\delta_n} r^n \tag{4.17}
\]
and
\[
r - \max_n \frac{n(\alpha - \beta)}{A_n(\gamma)\delta_n} r^n \leq |f'(re^{i\theta})| \leq r + \max_n \frac{n(\alpha - \beta)}{A_n(\gamma)\delta_n} r^n. \tag{4.18}
\]
Further if \(\{\delta_n\}\) is a non-decreasing function of \(n\), then
\[
r - \frac{(\alpha - \beta)}{(\gamma + 1)\delta_2} r^2 \leq |f(re^{i\theta})| \leq r + \frac{(\alpha - \beta)}{(\gamma + 1)\delta_2} r^2 \tag{4.19}
\]
and if \( \left\{ \frac{n}{\delta_n} \right\} \) is a non decreasing function of \( n \), then

\[
1 - \frac{2(\alpha - \beta)}{(\gamma + 1)\delta_2} r^2 \leq |f'(re^{i\theta})| \leq r + \frac{(\alpha - \beta)}{(\gamma + 1)\delta_2} r^2. \tag{4.20}
\]

Equality holds in (4.19) and (4.20) are obtained for each \( n \geq 2 \), for the function

\[
f_2(z) = z - \frac{(\alpha - \beta)}{(\gamma + 1)\delta_2} z^2. \tag{4.21}
\]

Proof. Let \( f \in T_{\gamma, T}(\psi_1, \psi_2) \). Then for \( |z| = r \)

\[
|f_2(z)| \leq \max_n \max_{|z|=r} \left| z - \frac{(\alpha - \beta)}{A_n(\gamma)\delta_n} z^n \right|
\leq \max_n \left( r + \frac{(\alpha - \beta)}{A_n(\gamma)\delta_n} r^n \right) = r + \max_n \frac{(\alpha - \beta)}{A_n(\gamma)\delta_n} r^n \tag{4.22}
\]

which is right hand side of (4.17). Also since

\[
|f_2(z)| \geq \min_n \min_{|z|=r} \left| z - \frac{(\alpha - \beta)}{A_n(\gamma)\delta_n} z^n \right|
\geq \min_n \left( r - \frac{(\alpha - \beta)}{A_n(\gamma)\delta_n} r^n \right)
\geq r - \max_n \frac{(\alpha - \beta)}{A_n(\gamma)\delta_n} r^n \tag{4.23}
\]

which is the left side of (4.17).

For the derivative a similar argument gives (4.18).

If \( \{\delta_n\} \) is a non-decreasing function of \( n \), then for \( |z| = r \),

\[
\max_n \frac{(\alpha - \beta)}{A_n(\gamma)\delta_n} r^{n-1} = \frac{(\alpha - \beta)}{(\gamma + 1)\delta_2} r
\]

and hence from (4.22) and (4.23) we obtain (4.19). Further if \( \left\{ \frac{n}{\delta_n} \right\} \) is a non decreasing function of \( n \), then for \( |z| = r \),

\[
\max_n \frac{n(\alpha - \beta)}{A_n(\gamma)\delta_n} r^{n-1} = \frac{2(\alpha - \beta)}{(\gamma + 1)\delta_2} r
\]
and in this case (4.20) follows from (4.18). Equalities in (4.19) and (4.20) are attained for the function \( f_2(z) \) defined in (4.21). This completes the proof. □

**Corollary 4.2.** Let \( f \in I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2) \) and \( \delta_n \) be as defined in (4.14). If \( \{\delta_n\}_{n=2}^{\infty} \) is a non decreasing function of \( n \), then \( f(U) \) is contained in a disk with center at the origin and of radius \( 1 - \frac{(\alpha-\beta)}{(\gamma+1)\delta_n} \). This result is sharp.

*Proof.* The proof follows by taking \( r \to 1 \) in (4.19). □

### 4.4 The Class \( T_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2, b) \) and its Properties

Since for \( f \in I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2) \), we have seen from (4.12) that

\[
|a_n| \leq \frac{(\alpha - \beta)}{(\gamma + 1)\delta_n},
\]

we may write \( |a_2| = \frac{(\alpha - \beta)b}{(\gamma + 1)\delta_2} \), where \( 0 \leq b \leq 1 \).

Let us denote by \( T_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2, b) \), functions in \( I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2) \) which are of the form

\[
f^b(z) = z - \frac{(\alpha - \beta)b}{\gamma_2}z^2 - \sum_{n=3}^{\infty} |a_n| z^n,
\]

where \( 0 \leq b \leq 1 \) is kept fixed.

In this Section we determine extreme points and distortion properties of functions in \( T_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2, b) \). From Theorem 4.2, we conclude that the family \( T_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2, b) \) is closed under convex linear combinations and consequently the closure, \( \overline{CLX} \left( I_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2) \right) \) is simply \( T_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2, b) \). The following theorem, which can be proved analogous to the Theorem 4.3, gives characterization of extreme points for the class \( T_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2, b) \).
4.4 The Class $I_{\gamma,T}^{\alpha,\beta}(\psi_1,\psi_2,b)$ and its Properties

**Theorem 4.5.** Let $\gamma_n$ be defined as in (4.14),

$$f_2^b(z) = z - \frac{(\alpha - \beta)b}{(\gamma + 1)\delta_2}z^2,$$

and

$$f_n^b(z) = z - \frac{(\alpha - \beta)b}{(\gamma + 1)\delta_2}z^2 - \frac{(1 - b)(\alpha - \beta)b}{A_n(\gamma)\delta_n}z^n. \quad (4.26)$$

Then $f \in I_{\gamma,T}^{\alpha,\beta}(\psi_1,\psi_2,b)$ if and only if it can be expressed as

$$f(z) = \sum_{n=2}^{\infty} \delta_n f_n^b(z),$$

where $\delta_n \geq 0$ and $\sum_{n=2}^{\infty} \delta_n = 1. \quad (4.27)$

Extreme points of $I_{\gamma,T}^{\alpha,\beta}(\psi_1,\psi_2,b)$ are functions of the form (4.25) and (4.26).

**Remark 4.2.** It may be noted that the function $f_1(z) = z$ is an extreme point of $I_{\gamma,T}^{\alpha,\beta}(\psi_1,\psi_2)$ whereas it is not even a member of $I_{\gamma,T}^{\alpha,\beta}(\psi_1,\psi_2,b)$ unless $b = 0.\)

**Remark 4.3.** For suitable functions $\psi_1$, $\psi_2$ and parameters $\alpha$, $\beta$ in Theorem 4.5 we get results of Silverman and Silvia [9] and new results in terms of fixed second coefficient of corresponding classes studied by Silverman and Silvia [8], Gupta and Jain [1, 2] etc.

Since the family $I_{\gamma,T}^{\alpha,\beta}(\psi_1,\psi_2,b)$ is compact in view of the observation made at the beginning of Section 4.3, we conclude that the sharp upper and lower bounds for $|f|$ and $|f'|$ for functions $f(z)$ belonging to $I_{\gamma,T}^{\alpha,\beta}(\psi_1,\psi_2,b)$ occur at one of the extreme points given by (4.25) and (4.26). However the bounds are not straight forward as noted by Silverman and Silvia [9] even in the particular case of the classes $ST^*(\alpha)$ and $ST(\alpha), \, 0 \leq \alpha < 1$. For this purpose we shall need the following lemmas which are analogous to Lemmas 1, 2 and 3 of Silverman and Silvia [9] and
which generalize those Lemmas. At the same time for suitable choices
of \( \psi_1, \psi_2, \alpha \) and \( \beta \) as mentioned earlier, we obtain corresponding results
for various subclasses of univalent functions with negative coefficients.

**Lemma 4.1.** Let \( \delta_n, n \geq 2 \) be as defined in (4.14) and \( 0 < b < 1 \),

\[
b_0 = -((\gamma + 1)(\gamma + 2)\delta_3 + 8(\gamma + 1)\delta_2 - 2(\alpha - \beta)) \]
\[
+ \sqrt{((\gamma + 1)(\gamma + 2)\delta_3 + 8(\gamma + 1)\delta_2 - 2(\alpha - \beta))^2 + 64(\gamma + 1)(\alpha - \beta)\delta_2}
\]
\[
4(\alpha - \beta),
\]

\[
r_{0,b} = -4(1 - b)(\gamma + 1)\delta_2
\]
\[
+ \sqrt{16(1 - b)^2(\gamma + 1)^2\delta_3^2 + 2b^2(1 - b)(\gamma + 1)(\gamma + 2)(\alpha - \beta)\delta_3}
\]
\[
2b(1 - b)(\alpha - \beta)
\]

and

\[
f^b_3(z) = z - b(\alpha - \beta)z^2 - \frac{2(1 - b)(\alpha - \beta)}{(\gamma + 1)(\gamma + 2)\delta_3}z^3.
\]

Then for \( 0 \leq r \leq 1 \), \( 0 < b < 1 \),

\[
|f^b_3(re^{i\theta})| \geq r - \frac{b(\alpha - \beta)}{(\gamma + 1)\delta_2}r^2 - \frac{2(1 - b)(\alpha - \beta)}{(\gamma + 1)(\gamma + 2)\delta_3}r^3
\]

with equality for \( \theta = 0 \). For either \( 0 < b \leq b_0 \) and \( 0 < r \leq r_{0,b} \) or \( b_0 \leq b < 1 \),

\[
|f^b_3(re^{i\theta})| \leq r + \frac{b(\alpha - \beta)}{(\gamma + 1)\delta_2}r^2 - \frac{2(1 - b)(\alpha - \beta)}{(\gamma + 1)(\gamma + 2)\delta_3}r^3
\]

with equality for \( \theta = \pi \). For either \( 0 < b < b_0 \) and \( r_{0,b} < r < 1 \),

\[
|f^b_3(re^{i\theta})| \leq r \left( 1 + \frac{b^2(\gamma + 2)(\alpha - \beta)(1 - (\gamma + 2)\delta_3)\delta_3}{4(1 - b)(\gamma + 1)\delta_2^2}
\]
\[
+ \frac{b^2(\alpha - \beta)^2(1 + 16(\gamma + 1)^2(\gamma + 2)\delta_2^2)}{2(1 - b)(\gamma + 1)^2\delta_2^2}r^2
\]
\[
+ \frac{4(1 - b)^2(\alpha - \beta)^2}{(\gamma + 1)^2(\gamma + 2)^2\delta_2^3}r^4 \right)^{1/2}
\]

with equality for \( \theta = \cos^{-1} \left( \frac{2b(1 - b)(\alpha - \beta)r^2 - b(\gamma + 1)(\gamma + 2)\delta_2}{5(1 - b)(\gamma + 1)\delta_2 r} \right) \).
Proof. We employ the same technique as used by Silverman and Silveria \[9\]. Let \( f^b_3(z) \) be as defined in the hypothesis. Then

\[
|f^b_3(re^{i\theta})|^2 = r^2 + \frac{b^2(\alpha - \beta)^2}{(\gamma + 1)^2 \delta_2^2} r^4 + \frac{4(1 - b)^2(\alpha - \beta)^2}{(\gamma + 1)^2(\gamma + 2)^2 \delta_3^2} r^6
- \frac{2b(\alpha - \beta) \cos \theta}{(\gamma + 1) \delta_2} r^3
- \frac{4(1 - b)(\alpha - \beta) \cos 2\theta}{(\gamma + 1)(\gamma + 2) \delta_3} r^4
+ \frac{4b(1 - b)(\alpha - \beta)^2 \cos \theta}{(\gamma + 1)^2(\gamma + 2) \delta_2 \delta_3} r^5.
\]

Hence

\[
\frac{\partial |f^b_3(re^{i\theta})|^2}{\partial \theta} = \frac{2b(\alpha - \beta) \sin \theta}{(\gamma + 1) \delta_2} r^3
+ \frac{16(1 - b)(\alpha - \beta) \sin \theta \cos \theta}{(\gamma + 1)(\gamma + 2) \delta_3} r^4
- \frac{4b(1 - b)(\alpha - \beta)^2 \sin \theta}{(\gamma + 1)^2(\gamma + 2) \delta_2 \delta_3} r^5.
\]

Note that,

\[
\frac{\partial |f^b_3(re^{i\theta})|^2}{\partial \theta} = 0 \text{ for } \theta_1 = 0, \theta_2 = \pi \text{ and } \theta_3 = \cos^{-1}\left(\frac{2b(1 - b)(\alpha - \beta)r^2 - b(\gamma + 1)(\gamma + 2)\delta_3}{8(1 - b)(\gamma + 1)\delta_2 r}\right), \tag{4.28}
\]

\(\theta_3\) is a valid root only when

\[-1 \leq \frac{2b(1 - b)(\alpha - \beta)r^2 - b(\gamma + 1)(\gamma + 2)\delta_3}{8(1 - b)(\gamma + 1)\delta_2 r} \leq 1 \tag{4.29}\]

or \(\theta_3\) is a valid third root only when both inequalities

\[2b(1 - b)(\alpha - \beta)r^2 - 8(1 - b)(\gamma + 1)\delta_2 r - b(\gamma + 1)(\gamma + 2)\delta_3 \leq 0 \tag{4.30}\]

and

\[2b(1 - b)(\alpha - \beta)r^2 + 8(1 - b)(\gamma + 1)\delta_2 r - b(\gamma + 1)(\gamma + 2)\delta_3 \geq 0 \tag{4.31}\]

are satisfied for some range of \(r\) and \(b\).

To find the range of \(r\) and \(b\) for which both inequalities (4.30) and (4.31)
are to be satisfied we solve the equation

\[ 2b(1 - b)(\alpha - \beta)r^2 - 8(1 - b)(\gamma + 1)\delta_2 r - b(\gamma + 1)(\gamma + 2)\delta_3 = 0 \] (4.32)

in \( r \) and obtain

\[ r_1 = \frac{k_1 + k_2}{k_3} \quad \text{and} \quad r_2 = \frac{k_1 - k_2}{k_3}, \]

where

\[ k_1 = 4(1 - b)(\gamma + 1)\delta_2, \]

\[ k_2 = \sqrt{16(1 - b)^2(\gamma + 1)^2\delta_2^2 + 2b^2(1 - b)(\gamma + 1)(\gamma + 2)(\alpha - \beta)\delta_3} \]

and \( k_3 = 2b(1 - b)(\alpha - \beta) \).

Similarly solving the equation

\[ 2b(1 - b)(\alpha - \beta)r^2 + 8(1 - b)(\gamma + 1)\delta_2 r - b(\gamma + 1)(\gamma + 2)\delta_3 = 0 \] (4.33)

in \( r \), we get

\[ r_3 = \frac{-k_1 + k_2}{k_3} \quad \text{and} \quad r_4 = -\frac{k_1 - k_2}{k_3}. \]

It can be easily seen that only root \( r_3 \) lies between \((0, 1)\) which is equal to \( r_{0,b} \) as in the hypothesis. By taking \( r = -1 \) in (4.32) and (4.33) and solving in terms of \( b \) we get four roots out of which the only root which lies in \((0, 1)\) is \( b_0 \). Now it can be seen that for \( r_{0,b} \leq r < 1 \) and \( 0 < b < b_0 \), both inequalities (4.30) and (4.31) are satisfied simultaneously. Since,

\[ |f^b_3(re^{i\theta})| = \left| re^{i\theta} - \frac{b(\alpha - \beta)}{(\gamma + 1)\delta_2} r^2 e^{2i\theta} - \frac{2(1 - b)(\alpha - \beta)}{(\gamma + 1)(\gamma + 2)\delta_3} r^3 e^{3i\theta} \right| \]

\[ \geq r - \frac{b(\alpha - \beta)}{(\gamma + 1)\delta_2} r^2 - \frac{2(1 - b)(\alpha - \beta)}{(\gamma + 1)(\gamma + 2)\delta_3} r^3 \]

for \( 0 \leq r < 1 \) and \( 0 < b < 1 \).

To obtain the upper bound for \( |f^b_3(re^{i\theta})| \) one can compare extremal values of \( |f^b_3(-r)| \) and \( |f^b_3(re^{i\theta_3})| \) where \( \theta_3 \) is as defined in (4.28) on the appropriate intervals.
Lemma 4.2. Let $\delta_n, n \geq 2$ defined by (4.14) is a non-decreasing function of $n$, then for $n \geq 4$
\[ |f_n^b(re^{i\theta})| \leq |f_4^b(-r)|.\]

Proof. The assumption made in the hypothesis implies that $\{r_n\}$ is a non-decreasing function of $n$. Thus we have
\[
|f_n^b(re^{i\theta})| \leq r + \frac{b(\alpha - \beta)}{(\gamma + 1)\delta_2} r^2 + \frac{2(1 - b)(\alpha - \beta)}{A_n(\gamma)\delta_n} r^n \\
\leq r + \frac{b(\alpha - \beta)}{(\gamma + 1)\delta_2} r^2 + \frac{2(1 - b)(\alpha - \beta)}{A_n(4)\delta_4} r^4 \\
= -f_4^b(-r).
\]
Hence the lemma.

Combining Lemmas 4.1 and 4.2 we obtain

Theorem 4.6. If $f \in \mathcal{I}_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2, b)$ and $\delta_n, n \geq 2$ as defined in (4.14), is a non-decreasing function of $n$, then
\[ |f(re^{i\theta})| \geq r - \frac{b(\alpha - \beta)}{(\gamma + 1)\delta_2} r^2 - \frac{2(1 - b)(\alpha - \beta)}{(\gamma + 1)(\gamma + 2)\delta_3} r^3 \]
with equality for $f_3^b(z)$ at $z = r$ and
\[ |f(re^{i\theta})| \leq \max \left\{ \max_{\theta} |f_3^b(re^{i\theta})|, -f_4^b(-r) \right\}, \]
where $\max_{\theta} |f_3^b(re^{i\theta})|$ is given by Lemma 4.1.

Proof. In view of Lemma 4.2 the sharp upper bound of $|f|$ for $f \in \mathcal{I}_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2, b)$ occurs for $f_3^b$ or $f_4^b$. Since the sharp lower bound for $|f|$ clearly occurs at $f_3^b(r)$, the bounds on $|f|$ are obtained on comparing Lemmas 4.1 and 4.2. \qed
4.4 The Class $I^\alpha_\gamma T(\psi_1, \psi_2, b)$ and its Properties

Making use of the fact that $f(z) \in I^\alpha_\gamma T(\psi_1, \psi_2, b)$ if and only if

$$zf'(z) \in I^\alpha_\gamma T\left(\int_0^z \frac{\psi_1(t)}{t}, \int_0^z \frac{\psi_2(t)}{t}, b\right),$$

the following Lemmas and Theorem follow from Lemmas 4.1 and 4.2 and Theorem 4.6 respectively.

**Lemma 4.3.** Let $\delta_n$ be as defined in (4.14) and let

\[
b_0 = -((\gamma + 1)(\gamma + 2)\delta_3 + 8(\gamma + 1)\delta_2 - 6(\alpha - \beta))
+ \sqrt{((\gamma + 1)(\gamma + 2)\delta_3 + 8(\gamma + 1)\delta_2 - 6(\alpha - \beta))^2 + 192(\gamma + 1)(\alpha - \beta)\delta_2} \over 12(\alpha - \beta),
\]

\[
r_1 = -8(1 - b)(\gamma + 1)\delta_2
+ \sqrt{64(1 - b)^2(\gamma + 1)\delta_2^2 + 24b^2(1 - b)(\gamma + 1)(\gamma + 2)(\alpha - \beta)\delta_3} \over 12b(1 - b)(\alpha - \beta)
\]

and

\[
f_3^b(z) = z - b(\alpha - \beta)z^2 - 2(1 - b)(\alpha - \beta)z^3.
\]

Then for $0 \leq r \leq 1$, $0 < b < 1$,

\[
|f_3^b(re^{i\theta})| \geq 1 - \frac{2b(\alpha - \beta)}{(\gamma + 1)\delta_2}r - \frac{6(1 - b)(\alpha - \beta)}{(\gamma + 1)(\gamma + 2)\delta_3}r^2,
\]

with equality for $\theta = 0$. For either $0 \leq r < r_1$ and $0 < b < b_1$ or $b_1 \leq b < 1$

\[
|f_3^b(re^{i\theta})| \leq 1 + \frac{2b(\alpha - \beta)}{(\gamma + 1)\delta_2}r - \frac{6(1 - b)(\alpha - \beta)}{(\gamma + 1)(\gamma + 2)\delta_3}r^2,
\]

with equality for $\theta = \pi$. For either $0 < b < b_1$ and $r_1 \leq r < 1$,

\[
|J_3^b(re^{i\theta})| \leq r \left(1 + \frac{b^2(\gamma + 2)(\alpha - \beta)(1 - (\gamma + 2)\delta_3)\delta_3}{4(1 - b)(\gamma + 1)\delta_2^2}
+ b^2(\alpha - \beta)^21 + 16(\gamma + 1)^2(\gamma + 2)\delta_2r^2
+ \frac{4(1 - b)^2(\alpha - \beta)^2}{(\gamma + 1)^2(\gamma + 2)^2\delta_2\delta_3}r^4\right)^{1/2},
\]
with equality for \( \theta = \cos^{-1}\left(\frac{6b(1-b)(\alpha-\beta)r^2-b(\gamma+1)(\gamma+2)\delta_3}{8(1-b)(\gamma+1)\delta_2r}\right) \).

**Lemma 4.4.** Let \( \delta_n \) defined in (4.14), is a non-decreasing function of \( n \), then for \( n \geq 4 \),

\[
|f_n^{rb}(re^{i\theta})| \leq |f_4^{rb}(-r)|
\]

**Theorem 4.7.** Suppose that \( f \in \mathcal{I}_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2; b) \) and \( \delta_n \) as defined in (4.14), is a non-decreasing function of \( n \), then

\[
|f'(re^{i\theta})| \geq 1 - \frac{2b(\alpha-\beta)}{(\gamma+1)\delta_2}r - \frac{6(1-b)(\alpha-\beta)}{(\gamma+1)(\gamma+2)\delta_3}r^2, (0 \leq r < 1)
\]

with equality for \( f_3^{rb} \) at \( z = r \) and

\[
|f'(re^{i\theta})| \leq \max_{\theta}\left\{ \max_{\theta}|f_3^{rb}(re^{i\theta})|, f_4^{rb}(-r) \right\},
\]

where \( \max_{\theta}|f_3^{rb}(re^{i\theta})| \) is given by Lemma 4.3.

**Corollary 4.3.** If \( f \in \mathcal{I}_{\gamma,T}^{\alpha,\beta}(\psi_1, \psi_2, b) \), then \( f(U) \) contains a disk with center at the origin and radius

\[
1 - \frac{(\alpha-\beta)}{(\gamma+1)\delta_2} - \frac{2(1-b)(\alpha-\beta)}{(\gamma+1)(\gamma+2)\delta_3}.
\]

**Remark 4.4.** As noted earlier for the selection of the functions \( \psi_1(z) = \frac{z}{(1-z)^2}, \psi_2(z) = \frac{z}{1-z}, \alpha = 1 - 2\alpha_1 (0 \leq \alpha_1 < 1) \) and \( \beta = -1 \), and \( \psi_1(z + z^2) = \frac{z}{(1-z^2)^3}, \psi_2(z) = \frac{z}{(1-z)^3}, \alpha = 1 - 2\alpha_1 (0 \leq \alpha_1 < 1) \) and \( \beta = -1 \), we get the subclasses \( T_b^*(\alpha) \) and \( K_b^*(\alpha) \) respectively. Silverman and Silvia [9] obtained earlier the corresponding results mentioned in this Section for the subclasses \( T_b^*(\alpha) \) and \( K_b^*(\alpha) \) separately, where they proved corresponding Lemmas 4.1 to 4.4 and Theorems 4.6 and 4.7 separately for these classes.
Remark 4.5. Once more we remark here that properly chosen functions $\psi_1, \psi_2$ and the parameters $\alpha, \beta$ and $\gamma$ we can obtain several new results corresponding to the results of this Section for the classes $\mathcal{R}[\alpha], \mathcal{S}^*(\alpha, \beta), \mathcal{S}(\alpha, \beta)$ etc. Thus our approach unifies the study of these classes.

4.5 Containment Relations

In this Section we will find some containment relations between the classes $\mathcal{I}_{\gamma, T}^{\alpha, \beta}(\phi_1, \phi_2)$ and $\mathcal{I}_{\gamma, T}^{\alpha, \beta}(\psi_1, \psi_2)$, where $(\phi_1, \phi_2)$ and $(\psi_1, \psi_2)$ are two admissible pairs of functions with corresponding coefficients $\lambda_n, \mu_n$ and $\lambda'_n, \mu'_n$ respectively for $n \geq 2$. It can be easily seen that if $f \in \mathcal{I}_{\gamma, T}^{\alpha, \beta}(\phi_1, \phi_2)$ then $f \in \mathcal{I}_{\gamma, T}^{\alpha, \beta}(\psi_1, \psi_2)$, provided $\lambda'_n - \lambda_n \leq \frac{1-\alpha}{1-\beta}(\mu'_n - \mu_n)$ holds for all $n \geq 2$. In the following theorem we obtain several partial containment relation between the classes $\mathcal{I}_{\gamma, T}^{\alpha, \beta}(\phi_1, \phi_2)$ and $\mathcal{I}_{\gamma, T}^{\alpha, \beta}(\psi_1, \psi_2)$.

Theorem 4.8. Let $(\phi_1, \phi_2)$ and $(\psi_1, \psi_2)$ be two admissible pairs of functions, with corresponding coefficients $\lambda_n, \mu_n$ and $\lambda'_n, \mu'_n$ respectively and let $f \in \mathcal{I}_{\gamma, T}^{\alpha, \beta}(\phi_1, \phi_2)$. If $\delta_n$ defined by (4.14) is such that

$$\delta_2(\lambda'_n - \mu'_n) + (\alpha - \beta)(\lambda'_2 \mu'_n - \lambda'_n \mu'_2) \leq (\lambda'_2 - \mu'_2)\delta_n,$$

for all $n \geq 2$, (4.34) then $f \in \mathcal{I}_{\gamma, T}^{1-2\rho, -1}(\psi_1, \psi_2)$, where

$$\rho = \frac{\delta_2 - (\alpha - \beta)\lambda'_2}{\delta_2 - (\alpha - \beta)\mu'_2}.\quad (4.35)$$

The result is sharp in the sense that if $\rho_1 > \rho$ then $f \notin \mathcal{I}_{\gamma, T}^{1-2\rho_1, -1}(\psi_1, \psi_2)$.

Proof. In view of Theorem 4.2 it is sufficient to show that

$$\sum_{n=2}^{\infty} \frac{A_n(\gamma)\delta_n}{\alpha - \beta} |a_n| \leq 1,$$

(4.36)
implies that
\[ \sum_{n=2}^{\infty} \frac{\lambda_n' - \rho \mu_n'}{1 - \rho} A_n(\gamma) |a_n| \leq 1. \] (4.37)

It can be easily verified that the value \( \rho \) given by (4.35) lies in \((0, 1)\). Thus (4.37) would follow if
\[ \frac{\delta_n}{\alpha - \beta} \geq \frac{\lambda_n' - \rho \mu_n'}{1 - \rho} \quad \text{for all } n \geq 2. \] (4.38)

Substituting the value of \( \rho \) from (4.35) in (4.38) we see that (4.37) would follow if
\[ \frac{\delta_n}{\alpha - \beta} \geq \frac{\delta_2(\lambda_n' - \mu_n') + (\alpha - \beta)(\lambda_2 \mu_n' - \lambda_n \mu_2')}{(\alpha - \beta)(\lambda_2' - \mu_2')} \]
which is equivalent to
\[ (\lambda_n' - \mu_n') + (\alpha - \beta)(\lambda_2 \mu_n' - \lambda_n \mu_2') \leq \gamma_n(\lambda_2' - \mu_2'). \]

However this is true by assumption. This completes the proof.

To verify the sharpness of this result we note that if \( \rho_1 > \rho \), then for this \( \rho_1 \) the inequality (4.38) is not satisfied.

As remarked earlier selecting \( \psi_1, \psi_2, \alpha, \beta \) and \( \gamma \) suitably in \( T_{\gamma, \alpha}^{\psi_1, \alpha}(\psi_1, \psi_2) \) we immediately have the following corollaries to the above theorem in which case condition (4.34) for the corresponding classes is satisfied. The first corollary is Theorem 7 due to Silverman [6].

**Corollary 4.4.** If \( f \in K^*(\alpha) \), then \( f \in T^* \left( \frac{2}{3 - \alpha} \right) \) \((0 \leq \alpha < 1)\).

Next Corollary is due to Silverman and Silvia [8].

**Corollary 4.5.** If \( f \in R[\alpha], (0 \leq \alpha < \frac{1}{2}) \), then \( f \in T^* \left( \frac{2(1 - \alpha)}{3 - 2 \alpha} \right) \). The result is sharp with extremal function
\[ f_2(z) = z - \frac{1}{2(2 - \alpha)} z^2. \] (4.39)
Gupta and Jain [2] proved the following.

**Corollary 4.6.** If \( f \in \mathcal{K}^*(\alpha, \beta) \), then \( f \in \mathcal{P}^* \left( \frac{1}{3-2\alpha}, \frac{1}{3-2\beta} \right), (0 \leq \alpha < 1, 0 < \beta \leq 1) \).

### 4.6 Concluding Remarks

For any point \( z_0 \) with \(-1 < z_0 < 1\) and \( z_0 \neq 0 \) one can consider on the lines of Silverman [7] another subclass of functions of the form \( f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \) with \( a_n \geq 0 \), \( a_1 > 0 \) and satisfying the conditions (4.2) and (4.3) for an admissible pair \( \psi_1, \psi_2 \) and for some \( \alpha, \beta \) satisfying (4.4). Denoting this class of functions by \( T_{\gamma, T(z_0)}^{\alpha, \beta}(\psi_1, \psi_2) \), results analogous to those of Sections 4.2, 4.3 and 4.4 can be obtained for functions in the class \( T_{\gamma, T(z_0)}^{\alpha, \beta}(\psi_1, \psi_2) \), which would include the results of Silverman [7] and others as a particular case. However, we shall not pursue these results in this work. As stated earlier the study of these classes unifies the study of various subclasses of univalent functions with negative coefficients. For properly selected values of the functions \( \psi_1, \psi_2 \) and the parameters \( \alpha, \beta, \gamma \) our results reduces to those of Silverman [6], Silverman and Silvia [8, 9], Gupta and Jain [1, 2] and Mishra [3].
References


