CHAPTER - 5

BANACH ALGEBRA OF
A CLASS OF
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5.1 Introduction

In this chapter, we study a class $\Omega_u$ of functions 'f' represented by a Dirichlet series $f(s) = \sum_{k=1}^{\infty} a_k e^{\lambda k}$ as a $B^*$-algebra, for its various properties. In this algebra, characterization of regular and singular elements, topological zero divisors, spectral radius have been obtained in section 5.1, 5.2 and 5.3 respectively. In section 5.4, it has been shown that $\Omega_u$ is not a division algebra. In section 5.5, we consider two ideals $I_1$ and $I_2$ of $\Omega_u$, in such a way that $\Omega_u = I_1 \oplus I_2$. An involution (*) on $\Omega_u$ has been defined and star-homomorphism has been obtained in section 5.6 and 5.7. Further some lattice properties of a subset $\Omega_R$ of $\Omega_u$ has also been studied in section 5.8.

5.2 $\Omega_u$ as a Banach algebra

We have already studied, $\Omega_u$ as a Banach space in chapter 2. Now we will study $\Omega_u$ as a Banach algebra.

For that we recall the definition of $\Omega_u$, once again as follows,
\[ \Omega_u = \left\{ f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} : \left\{ \frac{a_k}{\alpha_k} \right\} \text{ is bounded} \right\} \]

where \( u(s) = \sum_{k=1}^{\infty} \alpha_k e^{s\lambda_k} \) is the fixed Dirichlet series with none of the coefficient \( \alpha_k = 0 \) \( \{i.e. \alpha_k \neq 0 \text{ for any } k\} \), as described in Section [2.3] of Chapter 2, and satisfying the conditions,

\begin{enumerate}
  \item \( \frac{\log k}{\lambda_k} \to 0 \text{ as } k \to \infty \)
  \item \(-\limsup_{k \to \infty} \frac{\log|\alpha_k|}{\lambda_k} = \alpha,\)
\end{enumerate}

so that abscissa of ordinary convergence of \( u \) i.e. \( \sigma_c^u \) coincides with abscissa of the absolute convergence \( \sigma_a^u \) of \( u \) see (7), and it is given by the formula,

\[ \sigma_c^u = \sigma_a^u = -\limsup_{k \to \infty} \frac{\log|\alpha_k|}{\lambda_k} = \alpha. \]

The half plane \( \sigma < \alpha \) is the region of convergence of \( \Omega \) denoted by \( R_u \). It is known that each element \( f \in \Omega_u \) has its region of convergence \( \geq \alpha \) and it is an analytic function in its region of convergence.

Define, in \( \Omega_u \) the pointwise linear operations viz, addition \( (+) \) and scalar multiplication \( (\cdot) \) and a norm \( \| \cdot \| \) as follows;

let \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}, g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \) belong to \( \Omega_u \) and \( c \) is a complex scalar.
(i) \((f + g)(s) = \sum_{k=1}^{\infty} (a_k + b_k) e^{s\lambda_k}\),

(ii) \((c \cdot f)(s) = \sum_{k=1}^{\infty} c \cdot a_k e^{s\lambda_k}\) and

(iii) \(\|f\| = \sup_{k \geq 1} \left| \frac{a_k}{\alpha_k} \right|\).

Then, we have seen in chapter 2 section 2.3 that \(\Omega_u\) is a normed linear space which is complete also, therefore \(\Omega_u\) is a Banach space. In fact, we have,

\[\Omega_u = \left\{ f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} : \left| \frac{a_k}{\alpha_k} \right| \leq M \ \forall k \right\}\]
where \(M > 0\) is an arbitrary constant with \(\|f\| = \sup_{k \geq 1} \left| \frac{a_k}{\alpha_k} \right|\), as a Banach space see [127].

Now for \(f, g \in \Omega_u\) with \(f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}\), \(g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k}\),

let us define vector multiplication \((. )\) in \(\Omega_u\) as follows;

\[(f \cdot g)(s) = \sum_{k=1}^{\infty} \left( \frac{a_k b_k}{\alpha_k} \right) e^{s\lambda_k}.\]  \hspace{1cm} (5.2.1)

Now, we can easily see that \(f \cdot g \in \Omega_u\), since

\[\left| \frac{a_k b_k}{\alpha_k^2} \right| = \left| \frac{a_k b_k}{\alpha_k \alpha_k} \right| = \left| \frac{a_k}{\alpha_k} \right| \left| \frac{b_k}{\alpha_k} \right| \leq M^2 < \infty.\]

**Lemma:** \(5.2.1:-\) \(\Omega_u\) is a commutative Banach algebra with identity 'u'.

**Proof:** For all \(f, g, h \in \Omega_u\), it can easily be verified that,

(i) \(f(gh) = (fg)h\)
(ii) \( f \cdot (g + h) = fg + fh, \quad (f + g) \cdot h = fh + gh \)

(iii) \( \mu(f \cdot g) = (\mu f)g = f(\mu g) \) for all scalar \( \mu \in \mathbb{C} \) (set of the complex number)

(iv) \( f \cdot g = g \cdot f \)

(v) \( u \cdot f = f \cdot u \)

(vi) \( \|f\| \leq \|f\| \|g\| \) hold in \( \Omega_u \).

Therefore \( \Omega_u \) is a Banach algebra with identity 'u' see [def. 1.23, 1.24]

Further, we also see that

\[
\|f\|^2 = \sup_k \left| \frac{a_k}{a_k} \right| = \sup_k \left| \frac{a_k}{a_k} \right| = \sup_k \left| \frac{a_k}{a_k} \right| = \|f\| \|f\| = \|f\|^2.
\]

This shows that \( \Omega_u \) is a Regular Banach algebra [28, 131].

5.3 Topological zero divisors (TZD) and spectral radius

In this section, we obtained Invertible elements, Topological zero divisors, Spectral radius of an element \( f \) in \( \Omega_u \) and few other important properties of the Banach algebra \( \Omega_u \).

**Theorem - 5.3.1:** \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) belong to \( \Omega_u \) is invertible in \( \Omega_u \) iff

(i) \( a_k \neq 0 \) for any suffix \( k \) and

(ii) \( \left\{ \frac{a_k}{a_k} \right\} \) should be bounded.
Proof: Let \( g \) be the inverse of \( f \in \Omega_u \). Then by definition of inverse [def. 1.25], we must have,

\[
f \cdot g = u
\]

\[
(f \cdot g)(s) = u(s) \quad \forall s \in R_u
\]

Or

\[
\sum_{k=1}^{\infty} \frac{a_kb_k}{a_k} e^{s\lambda_k} = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}
\]

\[
\Leftrightarrow \quad \frac{a_kb_k}{a_k} = a_k, \quad \text{for each } k = 1, 2, 3, \ldots...
\]

\[
\Leftrightarrow \quad a_kb_k = a_k^2, \quad \text{for each } k = 1, 2, 3, \ldots...
\]

\[
\Leftrightarrow \quad b_k = \frac{a_k^2}{a_k}, \quad \text{for each } k = 1, 2, 3, \ldots...
\]

Now, \( b_k \) will exist if \( a_k \neq 0 \) for any \( k \). So \( f \in \Omega_u \) will be invertible if \( a_k \neq 0 \) for any \( k \) and in that case,

\[
f^{-1}(s) = g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} = \sum_{k=1}^{\infty} \frac{a_k^2}{a_k} e^{s\lambda_k}.
\]

Which is the requirement (i).

Further in order that \( g \in \Omega_u \), we must also have,

\[
\left| \frac{a_k^2}{a_k} \right| = \left| \frac{a_k}{a_k} \right| < \infty
\]
i.e. \( \{ \frac{a_k}{a_k} \} \) should be bounded. Which is the requirement (ii) of the theorem.

An element in a Banach algebra which is invertible is called regular otherwise it is called singular see [def. 1.25].

**Corollary - 5.3.1:** \( f \in \Omega_u \) is singular element of \( \Omega_u \) iff either of the following two conditions holds,

(i) \( a_k = 0 \) for some k. if \( a_k \neq 0 \) for any k then

(ii) \( \left\{ \frac{a_k}{a_k} \right\} \) is unbounded.

**Theorem - 5.3.2:** An element \( f \in \Omega_u \) is a Topological Zero Divisor (TZD) in \( \Omega_u \) iff it is singular.

**Proof:** Necessary part is obvious, as we know that every Topological Zero Divisor (TZD) [def. 1.28] is singular see [68]. For the sufficient part, let us suppose that \( f \) is singular then to prove that it is a TZD. Let \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \), then as \( f \) is singular so by coro [5.3.1] either

(i) \( a_k = 0 \) for some k or

(ii) If \( a_k \neq 0 \) for any k then \( \left\{ \frac{a_k}{a_k} \right\} \) should be unbounded.

In the first case if \( a_k = 0 \) for some k say for \( k = k_o \) i.e. \( a_{k_o} = 0 \), then define \( \{g_k\} \), as follows;
$g_k(s) = \alpha_{k_0} e^{s\lambda_{k_0}}$ for each $k = 1, 2, 3, \ldots$.

Obviously $g_k \in \Omega_u$ and $\|g_k\| = 1$ for each $k$ and

$$\|f \cdot g_k\| = \sup \left| \frac{a_{k_0} \lambda_{k_0}}{\lambda_{k_0}} \right| = a_{k_0} = 0$$

i.e.

$$\|f \cdot g_k\| = 0 \quad \text{for each } k$$

so that

$$\lim_{k \to \infty} \|f \cdot g_k\| = 0.$$ 

Hence $f$ is a TZD in $\Omega_u$ by definition 1.28 or see [102]. In the (ii) case let $a_k \neq 0$ for any $k$, then $\left\{ \frac{a_k}{a_k} \right\}$ should be unbounded, so there must exist sub-sequence say $\{k_n\}$ of $\{k\}$ such that

$$\lim_{n \to \infty} \left| \frac{a_{k_n}}{a_{k_n}} \right| = 0.$$ 

(5.3.1)

Define a sequence $\{g_n\}$ as follows;

$$g_n(s) = \alpha_{k_n} e^{s\lambda_{k_n}} \quad \text{for } n = 1, 2, 3, \ldots$$

Clearly $g_n \in \Omega_u$ and $\|g_n\| = 1 \quad \forall n$ and

$$\|f \cdot g_n\| = \left| \frac{a_{k_n}}{a_{k_n}} \right| \to 0 \quad \text{as } n \to \infty \quad \text{by (5.3.1)},$$

so $f$ is a TZD in $\Omega_u$. Which proves the theorem.
Theorem - 5.3.3:- Let \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) belong to \( \Omega_u \) and let \( E = \left\{ \frac{a_k}{a_k} \right\}_{k=1}^{\infty} \),

then spectrum of \( f \) i.e. \( \sigma(f) = \bar{E} \) where \( \bar{E} \) is the closure of \( E \).

Proof:- Let \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) belong to \( \Omega_u \). Since by the definition \( \sigma(f) \) is the set of all complex number \( \lambda \neq 0 \) s.t. \((f - \lambda u)\) is singular [102]. Now it follows by coro [5.3.1] that either

(i) \((a_k - \lambda a_k) = 0\) for some suffix k which implies \( \lambda = \frac{a_k}{a_k} \in E \) or

(ii) \( \left\{ \frac{a_k}{a_k - \lambda a_k} \right\} \) is unbounded. In this case there must exist a sub-sequence \( \{k_n\} \) of positive integers such that

\[
\lim_{n \to \infty} \left| \frac{a_{k_n} - \lambda a_{k_n}}{a_{k_n}} \right| = 0
\]

or

\[
\lim_{n \to \infty} \left| \frac{a_{k_n} - \lambda}{a_{k_n}} \right| = 0,
\]

showing that \( \lambda \) is a limit point of set \( E = \left\{ \frac{a_k}{a_k} \right\} \), hence \( \lambda \in \bar{E} \).

Combining these two case we see that \( \sigma(f) = \bar{E} \).

Corollary - 5.3.2:- Resolvent set of \( f \) is therefore given by \( \rho(f) = C - \bar{E} \) where \( C \) is the set of complex numbers.

Theorem - 5.3.4:- Let \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) belong to \( \Omega_u \). Then spectral radius of \( f \) i.e. \( r(f) = \|f\| \).
Proof:- By the definition of spectral Radius of an element in a Banach algebra, we know that spectral radius \( r(f) \) of \( f \) is given by,

\[
r(f) = \sup \{|\lambda| : \lambda \in \sigma(f)\}
\]

see [131] or [def. 1.30 & theorem 1.9].

Here in our case \( \sigma(f) = \bar{E} \) where \( E = \{\alpha_k\} \), therefore

\[
r(f) = \sup_k \left\{ \left| \frac{\alpha_k}{a_k} \right| \right\} = \|f\|.
\]

5.4:- Division Algebra

In this section, we show that \( \Omega_u \) is not a division algebra.

A division-algebra is an algebra with identity such that every non-zero element in it, is invertible, see [96, p-403].

Theorem - 5.4.1:- \( \Omega_u \) is not a division-algebra.

Proof:- Let \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k} \) belong to \( \Omega_u \) where \( a_k = \frac{\alpha_k}{k}, \ k \geq 1 \).

Clearly \( f \in \Omega_u \). We claim that \( f \) does not possess a multiplicative inverse in \( \Omega_u \).

Let, if possible, \( g, g(s) = \sum_{k=1}^{\infty} b_k e^{s \lambda_k} \) in \( \Omega_u \) be the multiplicative inverse of \( f \), then

\[
f \cdot g = u
\]

or \( (f \cdot g)(s) = u(s) \) for all \( s \in R_u \).
\[ \sum_{k=1}^{\infty} \frac{a_k b_k}{\alpha_k} e^{s \lambda_k} = \sum_{k=1}^{\infty} a_k e^{s \lambda_k} \]

\[ \Rightarrow \quad \frac{a_k b_k}{\alpha_k} = a_k \quad \text{or} \quad b_k = \frac{a_k^2}{\alpha_k} \quad \text{for each} \quad k = 1, 2, 3, \ldots \]

But here \[ a_k = \frac{a_k}{k} \quad \text{for each} \quad k \]

\[ \Rightarrow \quad b_k = a_k \cdot k \]

\[ \Rightarrow \quad \frac{b_k}{\alpha_k} = k \to \infty . \]

This shows that \( g \notin \Omega_u \).

### 5.5 ideals of \( \Omega_u \)

In this section, we consider two ideals \( I_1 \) and \( I_2 \) of \( \Omega_u \), in such a way that \( \Omega_u \) is the direct sum of these two ideals.

**Theorem - 5.5.1:** There exist two ideals \( I_1 \) and \( I_2 \) in \( \Omega_u \) such that \( \Omega_u = I_1 \oplus I_2 \).

**Proof:** Let \( d_1(s) = \sum_{k=1}^{n} \alpha_k e^{s \lambda_k} \) and \( d_2(s) = \sum_{k=n+1}^{\infty} \alpha_k e^{s \lambda_k} \).

Then clearly \( u(s) = d_1(s) + d_2(s) \) i.e. \( u = d_1 + d_2 \) and we have also

\[ d_1^2(s) = (d_1 \cdot d_1)(s) = \sum_{k=1}^{n} \frac{\alpha_k \alpha_k}{\alpha_k} e^{s \lambda_k} \]
\[
= \sum_{k=1}^{n} \alpha_k e^{s\lambda_k} = d_1(s)
\]

so \(d_1^2 = d_1\) and \((d_1 f)(s) = \sum_{k=1}^{n} \alpha_k e^{s\lambda_k} = f_1(s)\) (say).

Similarly \(d_2^2 = d_2\) and \((d_2 f)(s) = \sum_{k=n+1}^{\infty} \alpha_k e^{s\lambda_k} = f_2(s)\) (say).

Further \((d_1, d_2)(s) = \sum 0. e^{s\lambda_k} = 0\), so that \(d_1 d_2 = 0\).

Let \(I_1 = \{d_1 f : f \in \Omega_u\}\) and \(I_2 = \{d_2 f : f \in \Omega_u\}\),

then it can easily be seen, as follows, that \(I_1\) and \(I_2\) are ideals of \(\Omega_u\).

Let \(x = d_1 f\) and \(y = d_1 f'\), belong to \(I_1\), where \(f, f' \in \Omega_u\) then

\[x - y = d_1(f - f') \in I_1.\]

Next, let \(g \in \Omega_u\)

\[g(d_1 f) = d_1(g. f) = d g'\] where \(g' = g. f \in \Omega_u\),

so that \(I_1\) is an ideal. Similarly \(I_2\) is an ideal and it is such that \(I_1 \cap I_2 = \{0\}\).

Next, let \(f \in \Omega_u\) then we have,

\[f = u. f = (d_1 + d_2). f = d_1 f + d_2 f = f_1 + f_2,\]

where \(d_1 f = f_1 \in I_1\) and \(d_2 f = f_2 \in I_2\).

Hence \(\Omega_u = I_1 \oplus I_2\).
5.6 \( \Omega_u \) as B*-algebra

In this section, we defined \( \Omega_u \) as a B*-algebra with an involution defined on it. We study some of the important properties of this B*-algebra [102] or see [def. 1.32].

We know that \( \Omega_u \) is a Banach algebra. Now, \( \Omega_u \) is a Banach algebra with involution \((\cdot^*)\), if there exist a mapping \(*: \Omega_u \to \Omega_u\) such that for any \(f, g \in \Omega_u\) and \(t \in \mathbb{C}\) we have,

\begin{align*}
(i) \quad (f + g)^* &= f^* + g^* \\
(ii) \quad (tf)^* &= \overline{t}f^* \quad (t \text{ is a complex scalar}) \\
(iii) \quad (fg)^* &= g^*f^* \\
(iv) \quad (f^*)^* &= f^{**} = f \\
(v) \quad (f^{-1})^* &= (f^*)^{-1} \quad \text{provided } f \text{ is invertible.}
\end{align*}

The Banach algebra \( \Omega_u \) with involution is said to be a B*-algebra [def. 1.32] if

\[ ||f^*f|| = ||f||^2, \quad f \in \Omega_u. \]

**Theorem – 5.6.1:-** \((\Omega_u, \cdot^*)\) is a B*-algebra.

**Proof:** Let \( *: \Omega_u \to \Omega_u \) defined \( * (f) = f^* \) where \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) belongs to \( \Omega_u \) and
Then clearly \( f^* \in \Omega_u \).

Further, it can easily be seen as given below that the above properties of involution \((\ast)\) hold in \(\Omega_u\). That is

(i) \( (f + g)^* = f^* + g^* \).

We have,

\[
(f + g)^*(s) = \sum_{k=1}^{\infty} \left( \frac{a_k + b_k}{a_k} \right) \alpha_k e^{s\lambda_k} \quad \forall s \in R_u
\]

\[
= f^*(s) + g^*(s)
\]

\( \Rightarrow \) \( (f + g)^* = f^* + g^* \);

(ii) \( (tf)^*(s) = \sum_{k=1}^{\infty} \left( \frac{\alpha_k}{\alpha_k} \right) t \alpha_k e^{s\lambda_k} \)

or \( (tf)^*(s) = \bar{t} \sum_{k=1}^{\infty} \frac{\alpha_k}{\alpha_k} \alpha_k e^{s\lambda_k} \)

or \( (tf)^*(s) = \bar{t} f^*(s) \)

\( \Rightarrow \) \( (tf)^* = \bar{t} f^* \);
(iii) we have, \((fg)^*(s) = \sum_{k=1}^{\infty} \frac{a_k b_k}{a_k \bar{a}_k} \alpha_k e^{s\lambda_k}\)

or \((fg)^*(s) = \sum_{k=1}^{\infty} \frac{a_k b_k}{a_k \bar{a}_k} \alpha_k e^{s\lambda_k}\)

or \((fg)^*(s) = \sum_{k=1}^{\infty} \frac{(a_k \alpha_k)(b_k \alpha_k)}{a_k} e^{s\lambda_k}\)

\[= f^*(s) \cdot g^*(s) = g^*(s) \cdot f^*(s)\]

\[\Rightarrow (fg)^* = f^* \cdot g^* = g^* \cdot f^* .\]

(iv) We have, \(f^*(s) = \sum_{k=1}^{\infty} \frac{a_k}{a_k} \alpha_k e^{s\lambda_k}\)

\[\Rightarrow (f^*)^*(s) = f^{**}(s) = \sum_{k=1}^{\infty} \frac{(a_k \alpha_k)}{a_k} \alpha_k e^{s\lambda_k}\]

\[\Rightarrow (f^*)^*(s) = f^{**}(s) = \sum_{k=1}^{\infty} \frac{(a_k \alpha_k)}{a_k} \alpha_k e^{s\lambda_k}\]

\[\Rightarrow (f^*)^*(s) = f^{**}(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} = f(s)\]

\[\Rightarrow f^{**}(s) = f(s)\]

\[\Rightarrow (f^*)^* = f^{**} = f .\]

(v) To Prove, \((f^{-1})^* = (f^*)^{-1} ,\)
we have \[ f^{-1}(s) = \sum_{k=1}^{\infty} \frac{a_k^2}{a_k} e^{s\lambda_k} \] and
\[
(f^{-1})^*(s) = \sum_{k=1}^{\infty} \frac{\overline{a_k}}{a_k} \alpha_k e^{s\lambda_k}
\]
or
\[
(f^{-1})^*(s) = \sum_{k=1}^{\infty} \frac{\overline{a_k}}{a_k} \alpha_k e^{s\lambda_k}
\]
so that,
\[
(f^{-1})^* = (f^*)^{-1}.
\]

(vi) Next, we have,
\[
f^*(s) = \sum_{k=1}^{\infty} \frac{\overline{a_k}}{a_k} \alpha_k e^{s\lambda_k}
\]
where \( f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) in \( \Omega_u \),
so that,
\[
(f^* f)(s) = \sum_{k=1}^{\infty} \left( \frac{\overline{a_k}}{a_k} \right) \left( \frac{a_k a_k}{\alpha_k} \right) e^{s\lambda_k},
\]
therefore,
\[
\|f^* f\| = \sup_k \left| \frac{\overline{a_k} a_k a_k}{\alpha_k a_k} \right|
\]
\[
\|f^* f\| = \sup_k \left| \frac{\overline{a_k}}{a_k} \frac{a_k}{a_k} \right| \text{ as } |\bar{z}| = z
\]
\[
= \sup_k \left| \frac{a_k}{a_k} \right| \cdot \sup_k \left| \frac{a_k}{a_k} \right|
\]
\[
\|f^* f\| = \|f\| \cdot \|f\|
\]
\[
\|f^* f\| = \|f\|^2.
\]
Thus, we see that, $\Omega_u$ is a $B^*$-algebra.

In the next theorem, we find self-adjoint, unitary and normal elements in the Banach algebra $\Omega_u$. For the definitions of these, see [def. 1.33].

**Theorem - 5.6.2:** Let $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k}$ belong to $\Omega_u$ then $f$ is

(i) Self-adjoint, if $\left\{ \frac{a_k}{\alpha_k} \right\}$ is real.

(ii) Unitary, iff $\left\{ \frac{a_k}{\alpha_k} \right\}$ is invertible sequence of complex number.

(iii) Normal, iff $a_k = \alpha_k$ for each $k$ i.e when $f = u$.

**Proof (i):** We know that $f \in \Omega_u$ is self-adjoint iff,

$$f^* = f$$

or

$$f^*(s) = f(s) \quad \forall s \in R_u$$

or

$$\sum_{k=1}^{\infty} \frac{a_k}{\alpha_k} \alpha_k e^{s \lambda_k} = \sum_{k=1}^{\infty} a_k e^{s \lambda_k}$$

or

$$\frac{a_k}{\alpha_k} \alpha_k = a_k \quad \forall k$$

or

$$\frac{\alpha_k}{a_k} = \frac{a_k}{\alpha_k} \quad \forall k$$

$\Leftrightarrow$ $\left\{ \frac{a_k}{\alpha_k} \right\}$ should be real for each $k$.

(ii): Next $f \in \Omega_u$ will be unitary iff,
\[ f^*f = u \]

or

\[(f^*f)(s) = u(s), \quad \forall s \in R_u \]

\[ \iff \]

\[ \left( \frac{a_k}{\alpha_k} \right) \alpha_k a_k = a_k, \quad \forall k \]

\[ \iff \]

\[ \left( \frac{a_k}{\alpha_k} \right) \left( \frac{a_k}{\alpha_k} \right) = 1, \quad \forall k \]

or

\[ \left| \frac{a_k}{\alpha_k} \right|^2 = 1, \quad \forall k \]

or

\[ \left| \frac{a_k}{\alpha_k} \right| = 1, \quad \forall k \]

i.e. all the points \( \left\{ \frac{a_k}{\alpha_k} \right\} \) should lie on the unit circle \(|z| = 1\) of the complex plane.

(iii): \( f \in \Omega_u \) will be normal iff,

\[(ff^*)(s) = f^{**}(s) \quad \forall s \in R_u \]

or

\[(ff^*)(s) = f(s) \quad \forall s \in R_u \]

or

\[ \alpha_k \frac{\overline{a_k} a_k}{\overline{\alpha_k} \alpha_k} = a_k, \quad \forall k \]

or

\[ \overline{a_k} = \overline{\alpha_k}, \quad \forall k \]

or

\[ a_k = \alpha_k, \quad \forall k \]

\[ \iff \]

\[ f = u. \]
So the only normal element in $\Omega_u$ is $u$.

5.7: * - Homomorphism on $\Omega_u$

In this section, we study an * - Homomorphism on $\Omega_u$ and obtained some results related to it. Before to this, we introduce another class of functions $\Omega_v$ as described below.

Let $v$, $v(s) = \sum_{k=1}^{\infty} \beta_k e^{s\lambda_k}$ be another fixed Dirichlet series with $\beta_k \neq 0$ for any $k$, with same exponents $\{\lambda_k\}$ as that of $u(s)$ and satisfying exactly those conditions, as stated about $u(s)$, in chapter 2. We also assume

$$\lim_{k \to \infty} \sup \frac{\log|\beta_k|}{\lambda_k} = -\beta, \quad (\beta \leq \alpha).$$

Then $v(s)$ is an holomorphic function in its region of convergence $R_v$ given by $\sigma < \beta$, like that of $u$, which has its region of convergence $R_u$ given by $\sigma < \alpha$.

Let us define a similar class $\Omega_v$ like $\Omega_u$ as follows,

$$\Omega_v = \left\{ f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} : \left| \frac{a_k}{\lambda_k} \right| < \infty \right\}.$$

Then like $\Omega_u$, $\Omega_v$ is also a commutative B*-algebra with product defined as $(f, g)(s) = \sum_{k=1}^{\infty} \left( \frac{a_k b_k}{\beta_k} \right) e^{s\lambda_k}$ for $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}, \ g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k}$ belonging to $\Omega_v$ and the involution '$\sim$' defined on $\Omega_v$ as follows.
Let \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) belong to \( \Omega_v \). Then

\[
\tilde{f}(s) = \tilde{f} \text{ where } \tilde{f}(s) = \sum_{k=1}^{\infty} \left( \frac{a_k}{\alpha_k} \right) \beta_k e^{s\lambda_k} .
\]

........................ (5.7.1)

Now, we come to the following theorem.

**Theorem - 5.7.1:** The mapping \( \psi : \Omega_u \rightarrow \Omega_v \) defined as

\[
\psi(f)(s) = \sum_{k=1}^{\infty} a_k \left( \frac{\beta_k}{\alpha_k} \right) e^{s\lambda_k}
\]

is an \( \ast \)-homomorphism.

where \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) belong to \( \Omega_u \).

**Proof:** For the definition of \( \ast \)-homomorphism see [def. 1.35]. Clearly \( \psi(f) \in \Omega_v \) whenever \( f \in \Omega_u \). Next let \( t_1, t_2 \) belong to \( \mathbb{C} \) and \( f, g \in \Omega_u \)

where \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) and \( g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \).

Then for \( s \in \mathbb{R}_v \), we can easily check that

(i) \( \psi(t_1 f + t_2 g) = t_1 \psi(f) + t_2 \psi(g) \)

(ii) \( \psi(fg) = \psi(f) . \psi(g) \) hold.

Thus \( \psi \) is a homomorphism. Next, we show that \( \psi \) is \( \ast \)-homomorphism. For this,

we see that,

\[
\psi(f^\ast)(s) = \sum_{k=1}^{\infty} \left( \frac{a_k}{\alpha_k} \right) \alpha_k . \left( \frac{\beta_k}{\alpha_k} \right) e^{s\lambda_k}
\]

\[122\]
\[ \sum_{k=1}^{\infty} \left( \frac{a_k}{\alpha_k} \right) \beta_k e^{s\lambda_k}, \]

and we have, \[ \psi(f)(s) = \sum_{k=1}^{\infty} a_k \left( \frac{\beta_k}{\alpha_k} \right) e^{s\lambda_k} \]

\[ \Rightarrow \quad (\overline{\psi(f)})(s) = \sum_{k=1}^{\infty} \left( \frac{a_k \beta_k}{\alpha_k \beta_k} \right) \beta_k e^{s\lambda_k} \quad \text{by (5.7.1)} \]

\[ \Rightarrow \quad (\overline{\psi(f)})(s) = \sum_{k=1}^{\infty} \left( \frac{a_k}{\alpha_k} \right) \beta_k e^{s\lambda_k} = \psi(f^*)(s) \]

so that \[ \psi(f^*) = \overline{\psi(f)}. \]

Hence \( \psi \) is an *-homomorphism.

**Remark - 5.7.1:-** The above study of \( \Omega_u \) as a B*-algebra generalize and extend the results in [99] for \( \alpha_k = \left( \frac{\lambda_k}{e} \right)^{\lambda_k} \) and corrects the work of [101] if \( \alpha_k \) is taken \( \frac{1}{\omega_k} \) and \( u(s) \) representing an entire function i.e when \( \alpha = \infty \).

### 5.8 Banach Lattice

This section is concerned with the study of a few properties of a Banach lattice \( \Omega_R \), defined as the class of self-adjoint element of \( (\Omega_u, \| \cdot \|) \) with suitable partial order relation defined on it.
As we have seen in section 5.6 that if \((\Omega_u, \| \|, \ast)\), where \(\ast\) is defined by (5.6.1), is a Banch algebra with involution \(\ast\), then an element \(f \in \Omega_u\) is self-adjoint if and only if \([\frac{a_k}{\alpha_k}]\) is real for all \(k\).

Let \(\Omega_R = \{ f \in \Omega_u, \ f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} : \left(\frac{a_k}{\alpha_k}\right) \text{ is real for each } k \}\).

If \(f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}, \ g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k}\) belong to \(\Omega_R\),

then clearly (i) \(f - g \in \Omega_R\) and

(ii) \(f \cdot g \in \Omega_R\),

so that \((\Omega_R, \| \|)\) is a commutative subring of \((\Omega_u, \| \|)\) with the unit element 'u' given by \(u(s) = \sum_{k=1}^{\infty} \alpha_k e^{s\lambda_k}\).

For \(f, g \in \Omega_R\), given by \(f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}\) and \(g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k}\), define

\[
f < g \ \text{iff} \ \frac{a_k}{\alpha_k} \leq \frac{b_k}{\alpha_k} \ \text{for each } k. \ \ \ (5.8.1)
\]

Clearly, the relation < satisfies the following properties:

(i) \(f < f\) for all \(f \in \Omega_R\)

(ii) \(f < g\) and \(g < f \Rightarrow f = g\)

(iii) \(f < g\) and \(g < h \Rightarrow f < h\).
So the relation $<$ defined in (5.8.1) is an ordered relation over $(\Omega_R, \| \|)$. Hence $(\Omega_R, \| \|)$ is poset [def. 1.37].

In the next theorem, we show that $(\Omega_R, \| \|)$ is a Banach-lattice. For the definition of a Banach lattice see [def. 1.44].

**Theorem - 5.8.1:** $(\Omega_R, \| \|)$ is a Banach lattice.

**Proof:** For $f, g \in \Omega_R$, given by $f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$, $g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k}$, define;

$$h(s) = \sum_{k=1}^{\infty} c_k e^{s\lambda_k} \quad \text{and} \quad m(s) = \sum_{k=1}^{\infty} d_k e^{s\lambda_k}$$

with

$$\frac{c_k}{a_k} = \min \left\{ \frac{a_k}{a_k}, \frac{b_k}{a_k} \right\} \text{ for each } k \quad \text{and} \quad \frac{d_k}{a_k} = \max \left\{ \frac{a_k}{a_k}, \frac{b_k}{a_k} \right\}$$

Obviously, $h$ and $m \in \Omega_R$ and are the g.l.b. and l.u.b. of $f$ and $g$ respectively. Hence $f \wedge g = h$ and $f \vee g = m$. So $(\Omega_R, \prec)$ is a lattice. Further, it is easy to check that $(\Omega_R, \prec)$ is a vector-lattice [def. 1.41]. For $f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$, and

$$(-f)(s) = \sum_{k=1}^{\infty} (-a_k) e^{s\lambda_k} \quad \text{define}$$

$$|f| = f \vee -f = \max \left\{ \frac{a_k}{a_k}, \frac{-a_k}{a_k} \right\},$$

it is easy to verify that,
\(|f| < |g| \Rightarrow \|f\| \leq \|g\| \) for all \(f, g \in \Omega_R\).

Hence \((\Omega_R, \|\cdot\|, <)\) is a normed vector-lattice. Since \((\Omega_R, \|\cdot\|)\) is a Banach subspace therefore it is Banach-lattice.

Further, we also have the following theorem.

**Theorem - 5.8.2:** \((\Omega_R, <)\) is a modular, Archimedean \(l\)-group and \(l\)-ring.

**Proof:** For the definition of modular, Archimedean, \(l\)-group and \(l\)-ring see the definitions [1.43, 1.45] in part B of Chapter 1.

Suppose \(f < h_1 \Rightarrow \frac{a_k}{\alpha_k} \leq \frac{c'_k}{\alpha_k}\) for all \(k\)

where \(h_1(s) = \sum_{k=1}^{\infty} c'_k e^{s\lambda_k}\) and \(f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}\).

Now for every \(g \in \Omega_R\) defined by \(g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k}\),

consider \(f \lor (g \land h_1) = \max\left(\frac{a_k}{\alpha_k}, \min\left(\frac{b_k}{\alpha_k}, \frac{c'_k}{\alpha_k}\right)\right)\).

But, it is obvious that,

\[
\max\left(\frac{a_k}{\alpha_k}, \min\left(\frac{b_k}{\alpha_k}, \frac{c'_k}{\alpha_k}\right)\right) = \min\left(\max\left(\frac{a_k}{\alpha_k}, \frac{b_k}{\alpha_k}\right), \frac{c'_k}{\alpha_k}\right)
\]

so

\[
f \lor (g \land h_1) = \min\left(\max\left(\frac{a_k}{\alpha_k}, \frac{b_k}{\alpha_k}\right), \frac{c'_k}{\alpha_k}\right) = (f \lor g) \land h_1 \text{ holds for every } g \in \Omega_R.
\]
Thus \((\Omega_R, <)\) is modular.

Next, we shall show that \((\Omega_R, <)\) with group operation + is an \(l\)-group. Since \((\Omega_R, <)\) is a poset and lattice, so to show that it is \(l\)-group, it only remains to show that every group translation i.e. \(f \rightarrow f_1 + f + f_2\) is isotone [def. 1.42].

For this suppose \(f < g \Rightarrow \frac{a_k}{a_k} \leq \frac{b_k}{a_k}\) for each \(k\).

Let \(f_1(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k}\) and \(f_2(s) = \sum_{k=1}^{\infty} a_k'' e^{s \lambda_k}\),

clearly

\[
\frac{a_k'}{a_k} + \frac{a_k''}{a_k} \leq \frac{a_k'}{a_k} + \frac{b_k}{a_k} + \frac{a_k''}{a_k}
\]

for each \(k\), which implies that

\[
f_1 + f + f_2 < f_1 + g + f_2.
\]

So \((\Omega_R, <)\) is an \(l\)-group.

Further, we know that a po-group is called an Archimedean when \(mf < g\) for every integer \(m\) implies \(f \equiv 0\) [def 1.45]. Now suppose

\[
mg < g \Rightarrow m \frac{a_k}{a_k} \leq \frac{b_k}{a_k}
\]

for each \(k\) and every integer \(m\).

Thus

(i) \[\frac{a_k}{a_k} \leq \frac{b_k}{ma_k}\] if \(m > 0\)

(ii) \[\frac{a_k}{a_k} \geq \frac{b_k}{ma_k}\] if \(m < 0\)
Letting $m \to \infty$ in (i) and $m \to -\infty$ in (ii), we get $\frac{a_k}{a_k} = 0$ for each $k \Rightarrow a_k = 0$ for each $k$.

So $f \equiv 0$. Thus, $(\Omega_R, <)$ is an Archimedean.

Further, it is easy to verify that $(\Omega_R, <)$ is an l-ring. [def. 1.46]

**Theorem - 5.8.3:** $(\Omega_R, <)$ is an f-ring.

**Proof:** For the definition of f-ring see [def. 1.46].

Let $f \wedge g = 0$ and $h' > 0$ be given, that is

$$\min\left(\frac{a_k}{a_k}, \frac{b_k}{a_k}\right) = 0 \text{ and } 0 \leq \frac{c'_k}{a_k} \text{ where}$$

$$f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \text{ and } h'(s) = \sum_{k=1}^{\infty} c'_k e^{s\lambda_k} \text{ belong to } \Omega_R.$$ 

Then

$$h' \circ f \wedge g = \min\left(\frac{a_k}{a_k}, \frac{b_k}{a_k}\right) = 0.$$ 

Similarly $f \circ h' \wedge g = 0$. So $(\Omega_R, <)$, being an l-ring, it also satisfies the condition

$$f \wedge g = 0 \text{ and } 0 < h' \Rightarrow h' \circ f \wedge g = f \circ h' \wedge g = 0.$$ 

So $(\Omega_R, <)$ is an f-ring.

**Theorem - 5.8.4:** $(\Omega_R, \|\|, <)$ is an Am-space.

**Proof:** To prove it, we first prove two lemmas.
See the definition of an AM-space in [def. 1.47].

Lemma – 5.8.1:- For two real sequences \( \{s'_k\} \) and \( \{t'_k\} \) if \( s'_k, t'_k \geq 0 \) then

\[
\sup_k (\max(s'_k, t'_k)) = \max \left( \sup_k s'_k, \sup_k t'_k \right).
\]

Proof:- Since \( s'_k \leq \max(s'_k, t'_k) \) so \( \sup_k s'_k \leq \sup_k (\max(s'_k, t'_k)) \).

Similarly \( \sup_k t'_k \leq \sup_k (\max(s'_k, t'_k)) \)

Thus combining both, we get,

\[
\max \left( \sup_k s'_k, \sup_k t'_k \right) \leq \sup_k (\max(s'_k, t'_k)).
\]

Conversely, it is obvious that,

\[
\sup_k (\max(s'_k, t'_k)) \leq \max \left( \sup_k s'_k, \sup_k t'_k \right).
\]

This completes the proof of the Lemma.

Lemma – 5.8.2:- Let \( \{s'_k\} \) and \( \{t'_k\} \) be sequence of real numbers such that

\[
\min(s'_k, t'_k) = 0 \text{ for every } k\text{, then}
\]

\[
\sup_k (s'_k + t'_k) = \sup_k |s'_k - t'_k|.
\]

Proof is straight forward so we omit it.

Proof of Theorem – 5.8.4:- Since \((\Omega_R, \|\cdot\|, <)\) is a Banach lattice. So to show it an AM-space, it is required to prove-
(i) If $f \land g = 0$ then $\|f + g\| = \|f - g\|$, and

(ii) If $0 < f$, $0 < g$ then $\|f \lor g\| = \max(\|f\|, \|g\|)$.

Condition (i) follows immediately by Lemma (5.8.2).

To show the condition (ii), we have,

$$0 < f, 0 < g \Rightarrow 0 \leq \frac{a_k}{a_k}, 0 \leq \frac{b_k}{a_k} \text{ for all } k.$$ 

Then by Lemma (5.8.1), we get,

$$\sup_k \left( \max \left( \frac{a_k}{a_k}, \frac{b_k}{a_k} \right) \right) = \max \left( \sup_k \frac{a_k}{a_k}, \sup_k \frac{b_k}{a_k} \right)$$

i.e.

$$\|f \lor g\| = \max(\|f\|, \|g\|).$$

Thus, $(\Omega_R, \|\|, \langle\rangle)$ is an AM-space.

**Theorem - 5.8.5:** $(\Omega_R, \|\|, \langle\rangle)$ is not an abstract-lebesque space (AL-space).

**Proof:** For the definition of AL-space see [def 1.48]. Now, let $f$ be defined by

$$f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \text{ where } a_k = \alpha_k \text{ when } k \text{ is odd and } a_k = 0 \text{ otherwise and}$$

let $g$ be defined by $g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \text{ where } b_k = \alpha_k \text{ when } k \text{ is even and } b_k = 0 \text{ otherwise.}$

Clearly, $f, g \in (\Omega_R, \|\|)$ and $\|f\| = \|g\| = 1$.

also $f \land g = 0$ but $\|f + g\| = 1$. 

[130]
This violates the condition, i.e.

\[ 0 < f, 0 < g \implies \|f + g\| = \|f\| + \|g\| . \]

Thus \( \Omega_R \) is not an Abstract Lebesgue-space or AL-space.