CHAPTER 7
Chapter 7

ON THE COEFFICIENTS OF SOME CLASSES OF MULTIVALENT FUNCTIONS RELATED TO COMPLEX ORDER

7.1 INTRODUCTION

Let \( V \) denote the class of functions of the form

\[
w(z) = \sum_{n=1}^{\infty} b_n z^n
\]  

which are analytic in the unit disc \( E = \{ z : |z| < 1 \} \) and satisfying the conditions \( w(0) = 0 \) and \( |w(z)| < 1 \).

The present paper is devoted to a unified study of various subclasses of multivalent and univalent functions. For this purpose we mention the class \( R^b(A,B,p) \) of functions of the form

\[
f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \]

regular in \( E \) and satisfying the condition

\[
p + \frac{1}{b} \left[ \frac{f'(z)}{z^{p-1}} - p \right] = p \frac{(1 + Aw(z))}{1 + Bw(z)} , \quad z \in E
\]  

125
where $A$ and $B$ are fixed numbers such that $-1 \leq B < A < 1$ and $b$ is non-zero complex number or, equivalently (7.1.2) can be expressed as

\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| < 1, \quad z \in E.
\]  

(7.1.3)

The class $R^b(A, B, p)$ was introduced by Dixit and Pathak [22].

Further for a given number $\lambda$, $0 < \lambda \leq 1$, let $R^b_{\lambda}(A, B, p)$ denote the class of functions $g(z)$ analytic and multivalent in $E$ where

\[
g(z) = (1 - \lambda)z^p + \lambda f(z), \quad f(z) \in R^b(A, B, p).
\]  

(7.1.4)

In fact by giving specific values to $p, b, A, B$, and $\lambda$ in (7.1.3), we obtain the following important subclasses studied by various authors in earlier works.

1. For $\lambda = 1$, we obtain the class of functions studied by Dixit and Pathak [22].
2. For $p = 1$, we obtain the class of functions studied by Dixit and Vikas Chandra [21].
3. For $\lambda = 1$ and $p = 1$, we obtain the class of functions studied by Dixit and Pal [17].
4. For \( \lambda = 1, p = 1 \) and \( b = \cos \alpha e^{-i\alpha} \), we obtain the class of functions studied by Dashrath [16].

5. For \( \lambda = 1, p = 1, b = 1, A = (1 - 2\rho)\delta \) and \( B = -\delta \), where \( 0 \leq \rho < 1, 0 < \delta \leq 1 \), we obtain the class of functions studied by Juneja and Mogra [46].

6. For \( \lambda = 1, p = 1, b = 1, A = \delta \) and \( B = -\delta \), we obtain the class of functions studied by Caplinger and Causey [12] and Padmanabhan [68].

7. For \( \lambda = 1, b = 1 \) and \( p = 1 \), we obtain the class of functions studied by Goel and Mehrok [29].

Apart from these, important subclasses can be obtained by giving suitable values to \( p, b, A, B \) and \( \lambda \) studied by Mazur ([61], [62]), Nasr [64] and Janowski [44].

In this paper, results concerning coefficient estimates, sufficient condition in terms of coefficients, distortion theorem for the class \( R_{\lambda}^h(A, B, p) \) and the maximization of \( |a_3 - \mu a_2^2| \) over the class \( R_{\lambda}^h(A, B, p) \) have been obtained systematically.
We state below a lemma that is needed in our investigation. The following is due to Keogh and Merkes [51].

**Lemma 7.1.1** Let \( w(z) = \sum_{k=1}^{\infty} b_k z^k \) be analytic with \( |w(z)| < 1 \) in \( E \). If \( S \) is any complex number then

\[
|b_2 - Sb_1^2| \leq \max(1, |\delta|).
\]

Equality may be obtained for functions \( w(z) = z^2 \) and \( w(z) = z \).

### 7.2 COEFFICIENT ESTIMATES

In this section, method of Clunie [13] and Clunie and Keogh [14] will be used to prove the following theorem.

**Theorem 7.2.1** If \( g(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in R^b_\lambda(A, B, p) \), then

\[
|a_n| \leq \frac{|b| \lambda p(A - B)}{n}.
\]

The results are sharp.

**Proof.** Since \( g(z) \in R^b_\lambda(A, B, p) \), then from the definition of the classes \( R^b_\lambda(A, B, p) \) and \( R^b(A, B, p) \), it follows that there exists a function \( w(z) \) satisfying

\[
\frac{g'(z)}{z^{p-1}} = p(1 - \lambda) + \lambda \left[ p + \{bp(A - B) + pB\}w(z) \right]
\]
which gives
\[ w(z)\left[ \lambda b p(A - B) + p B - \frac{B g'(z)}{z^{p-1}} \right] = \frac{g'(z)}{z^{p-1}} - p \]
where \( w(0) = 0, \quad |w(z)| < 1 \) for \( |z| < 1 \), that is
\[ w(z)\left[ \lambda b p(A - B) - B \lambda \sum_{n=p+1}^{\infty} na_n z^{n-p} \right] = \lambda \sum_{n=p+1}^{\infty} na_n z^{n-p}. \quad (7.2.1) \]

Equating corresponding coefficients on both sides of (7.2.1) we observe that the coefficient \( a_n \) on the right hand side of (7.2.1) depends only on \( a_{p+1}, a_{p+2}, \ldots, a_{n-p} \) on the left side of (7.2.1). Hence it follows from (7.2.1) that
\[ w(z)\left[ \lambda b p(A - B) - B \lambda \sum_{k=p+1}^{n-1} k a_k z^{k-p} \right] = \sum_{k=p+1}^{n} k a_k z^{k-p} + \sum_{k=n+1}^{\infty} c_k z^{k-p} \]
\( c_{k_0} \) being some complex number. Since \( |w(z)| < 1 \), we have by means of Parseval's identity
\[
|\lambda|^2 |b|^2 p^2 (A - B)^2 + B^2 \lambda^2 \sum_{k=p+1}^{n-1} k^2 \left| a_k \right|^2 r^{2k-2p} \]
\[ \geq \sum_{k=p+1}^{n} k^2 \left| a_k \right|^2 r^{2k-2p} + \sum_{k=n+1}^{\infty} \left| c_k \right|^2 r^{2k-2p} \]
if we take limit as \( r \) approaches 1, then
\[ |b|^2 p^2 \lambda^2 (A - B)^2 + B^2 \sum_{k=p+1}^{n-1} k^2 |a_k|^2 \geq \sum_{k=p+1}^{n} k^2 |a_k|^2 \]

or

\[ |b|^2 p^2 \lambda^2 (A - B)^2 + B^2 \sum_{k=p+1}^{n-1} k^2 |a_k|^2 \geq \sum_{k=p+1}^{n} k^2 |a_k|^2 \]

\[ = \sum_{k=p+1}^{n-1} k^2 |a_k|^2 + n^2 |a_n|^2 \]

or

\((1 - B^2) \sum_{k=p+1}^{n-1} k^2 |a_k|^2 + n^2 |a_n|^2 \leq |b|^2 \lambda^2 p^2 (A - B)^2 \)

or

\[ |a_n| \leq \frac{|b| \lambda (A - B)}{n}, n = p + 1, p + 2, \ldots \]

The sharpness of the result follows for the functions

\[ f(z) = (1 - \lambda)z^p + \lambda \int_0^z \left[ pt^{p-1} + \frac{(A - B)t^{n-1}}{1 - Bt^{n-1}} \right] dt \]

for \( n \geq p + 1 \) and \( z \in E \).

**7.3 A SUFFICIENT CONDITION FOR A FUNCTION TO BE IN**

\[ R^b_\lambda(A, B, p) \]

**Theorem 7.3.1** Let \( g(z) = z + \sum_{n=2}^\infty a_n z^n \) be analytic in \( E \).
If for some $A, B (-1 \leq B < A \leq 1)$ and

$$\sum_{n=p+1}^{\infty} n |a_n| [1+|B|] \leq (A - B) \lambda p |b|.$$  \hspace{1cm} (7.3.1)

Then $g(z) \in R_{A,B}^{p}(A, B, p)$.

**Proof.** We prove this theorem by the technique of Clunie and Keogh [14]. Suppose that (7.3.1) holds and that

$$g(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

then for $|z| < 1$,

$$\left| \frac{g'(z)}{z^{p-1} - p} - \left[ \lambda b p (A - B) - B \left\{ \frac{g'(z)}{z^{p-1} - p} \right\} \right] \right|$$

$$= \left| \sum_{n=p+1}^{\infty} n a_n z^{n-p} \right| - \left| \lambda b p (A - B) - B \lambda \sum_{n=p+1}^{\infty} n a_n z^{n-p} \right|$$

$$\leq \sum_{n=p+1}^{\infty} n |a_n| r^{n-p} - |b| |\lambda p (A - B)| + |B| |\lambda \sum_{n=p+1}^{\infty} n |a_n| r^{n-p}$$

$$\leq \sum_{n=p+1}^{\infty} n |a_n| - |b| |\lambda p (A - B)| + |B| |\lambda \sum_{n=p+1}^{\infty} n |a_n|$$

$$= \sum_{n=p+1}^{\infty} n |a_n| (1+|B|) - |b| |\lambda p (A - B)| \leq 0$$

Hence it follows that
\[
\begin{vmatrix}
g'(z) \\
z^{p-1} - p \\
\lambda bp(A - B) - B \left( \frac{g'(z)}{z^{p-1}} - p \right)
\end{vmatrix} < 1, \quad z \in E,
\]

therefore \( g \in \mathcal{R}^b_\lambda(A, B, p) \).

The following functions shows that the result is sharp.

\[
g(z) = z^p + \frac{(A - B)\lambda b p z^n}{(1 + |B|)n} \text{ for } n \geq p + 1 \text{ and } z \in E.
\]

### 7.4 DISTORTION THEOREM

**Theorem 7.4.1** If \( g(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{R}^b_\lambda(A, B, p) \), then

\[
\Re \frac{g'(z)}{z^{p-1}} \geq (1 - \lambda)p + \lambda \frac{p - AB r^2 p \Re(b) - B^2 r^2 \Re(1-b)p - (A - B)p |b|r}{1 - B^2 r^2}
\]  

(7.4.1)

and

\[
\Re \frac{g'(z)}{z^{p-1}} \leq (1 - \lambda)p + \lambda \frac{p - AB r^2 p \Re(b) - B^2 r^2 \Re(1-b)p + (A - B) |b||r|}{1 - B^2 r^2}
\]

(7.4.2)

**Proof.** Since \( f \in \mathcal{R}^b_\lambda(A, B, p) \). Therefore by Theorem 7.3.1 of Dixit and Pathak [22], we have
Re \frac{f'(z)}{z^{p-1}} \geq \frac{p - AB r^2 p \Re(b) - B^2 r^2 \Re(1 - b)p - (A - B)p \mid b \mid r}{1 - B^2 r^2}

and

Re \frac{f'(z)}{z^{p-1}} \leq \frac{p - AB r^2 p \Re(b) - B^2 r^2 \Re(1 - b)p + (A - B)p \mid b \mid r}{1 - B^2 r^2}

using (7.1.4)

\frac{g'(z)}{z^{p-1}} = (1 - \lambda)p z^{p-1} + \lambda f'(z)

\frac{g'(z)}{z^{p-1}} = (1 - \lambda)p + \frac{\lambda f'(z)}{z^{p-1}}

Re \frac{g'(z)}{z^{p-1}} \geq (1 - \lambda)p + \lambda \left[ \frac{p - AB r^2 p \Re(b) - B^2 r^2 \Re(1 - b)p - (A - B)p \mid b \mid r}{1 - B^2 r^2} \right]

**Theorem 7.4.2** If \( g(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in R^b_{\lambda}(A, B, p) \) and \( \mu \) is any complex number then

\[ |a_{p+2} - \mu a_{p+1}^2| \leq \frac{\lambda p \mid b \mid (A - B)}{p + 2} \max \left\{ 1, \frac{|B(p + 1)^2 + (p + 2)\mu pb\lambda(A - B)|}{(p + 1)^2} \right\} \]

(7.4.3)

The result is sharp.

**Proof.** Since \( g \in R^b_{\lambda}(A, B, p) \), we have

\[ \frac{g'(z)}{z^{p-1}} = p(1 - \lambda) + \lambda \left[ \frac{p + \{bp(A - B) + pB\} w(z)}{1 + Bw(z)} \right] \]
where \( w(z) = \sum_{k=1}^{\infty} b_k z^k \) is analytic in \( E \) and satisfying the conditions

\[ w(0) = 0, \quad |w(z)| < 1 \quad \text{for} \quad z \in E. \]

\[
w(z) = \frac{\sum_{n=p+1}^{\infty} n a_n z^{n-p}}{\lambda bp(A - B)} \left[ 1 + \frac{B}{\lambda bp(A - B)} \sum_{n=p+1}^{\infty} n a_n z^{n-p} + \ldots \right]
\]

\[
= \frac{1}{\lambda bp(A - B)} \left[ \sum_{n=p+1}^{\infty} n a_n z^{n-p} + \frac{B}{\lambda bp(A - B)} \left( \sum_{n=p+1}^{\infty} n a_n z^{n-p} \right)^2 + \ldots \right]
\]

\[
= \frac{1}{\lambda bp(A - B)} \left[ (p + 1)a_{p+1} z + (p + 2)a_{p+2} z^2 + \ldots \right]
\]

\[
+ \frac{B}{\lambda bp(A - B)} (p + 1)^2 (a_{p+1})^2 z^2 + \ldots \right]
\]

and then comparing the coefficient of \( z \) and \( z^2 \) on both sides, we have

\[
b_1 = \frac{(p + 1)a_{p+1}}{\lambda bp(A - B)}, \quad b_2 = \frac{1}{\lambda bp(A - B)} \left[ (p + 2)a_{p+2} + \frac{B(p + 1)^2 (a_{p+1})^2}{\lambda bp(A - B)} \right]
\]

Thus

\[
a_{p+1} = \frac{\lambda bp(A - B)b_1}{p + 1}
\]

and

\[
(p + 2)a_{p+2} = \lambda bp(A - B)b_2 - \frac{B(p + 1)^2 (a_{p+1})^2}{\lambda bp(A - B)}
\]

134
or

\[ a_{p+2} = \frac{\lambda bp(A - B)b_2}{(p + 2)} - \frac{B(p + 1)^2(a_{p+1})^2}{(p + 2)\lambda bp(A - B)}. \]

Hence

\[ a_{p+2} - \mu(a_{p+1})^2 = \frac{\lambda bp(A - B)b_2}{(p + 2)} - \frac{B(p + 1)^2(a_{p+1})^2}{(p + 2)\lambda bp(A - B)} - \mu(a_{p+1})^2 \]

\[ a_{p+2} - \mu(a_{p+1})^2 = \frac{\lambda bp(A - B)b_2}{(p + 2)} - \left[ \frac{B(p + 1)^2}{(p + 2)\lambda bp(A - B)} + \mu \right] (a_{p+1})^2 \]

\[ = \frac{\lambda bp(A - B)b_2}{(p + 2)} - \left[ \frac{B(p + 1)^2 + \mu(p + 2)\lambda bp(A - B)}{(p + 2)\lambda bp(A - B)} \right] \frac{\lambda^2 b^2 p^2 (A - B)^2 b_2^2}{(p + 1)^2} \]

\[ = \frac{\lambda bp(A - B)}{(p + 2)} \left[ b_2 - \{B(p + 1)^2 + \mu(p + 2)\lambda bp(A - B)\} \right] \frac{b_2^2}{(p + 1)^2} \]

\[ = \frac{\lambda bp(A - B)}{(p + 2)} \left[ b_2 - \left\{ B + \frac{(p + 2)\mu \lambda bp(A - B)}{(p + 1)^2} \right\} b_1^2 \right] \]

Using Lemma 7.1.1, we obtain

\[ \left| a_{p+2} - \mu(a_{p+1})^2 \right| \leq \frac{\lambda p |b| (A - B)}{(p + 2)} \max \left\{ 1, \frac{(p + 1)^2 B + (p + 2)\mu \lambda bp(A - B)}{(p + 1)^2} \right\}. \]

Which is (7.4.3) of Theorem 7.4.2, when

\[ \left| \frac{B(p + 1)^2 + (p + 2)\mu \lambda bp(A - B)}{(p + 1)^2} \right| > 1. \]

135
We choose the function

\[ g(z) = \frac{Bp + (A - B)\lambda bp}{Bp} z^p - \frac{(A - B)\lambda bp}{B^2} \left[ \frac{Bz^p}{p} - \frac{B^2 z^{p+1}}{p+1} + \frac{B^3 z^{p+2}}{p+2} - \ldots \right] \]

and when

\[ \left| \frac{B(p + 1)^2 + (p + 2)\mu \lambda bp(A - B)}{(p + 1)^2} \right| < 1, \]

we have

\[ g(z) = \frac{Bp + (A - B)\lambda bp}{Bp} z^p - \frac{(A - B)\lambda bp}{B^2} \left[ \frac{z^p}{p} + \frac{Bz^{p+1}}{p+1} + \frac{B^2 z^{p+2}}{p+2} + \ldots \right]. \]