Chapter 5

A NEW CLASS OF MULTIVALENT FUNCTIONS DEFINED BY
FRACTIONAL INTEGRAL OPERATOR

5.1 INTRODUCTION

Let $S_{k,p}$ denote the class of functions of the form

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n}, a_{p+n} \geq 0,$$  \hspace{1cm} (5.1.1)

which are analytic and $p$-valent in the unit disc $E = \{z : |z| < 1\}$. Let $W_{k,p}(A, B, \alpha)$ denote the class of functions of $S_{k,p}$ which satisfy the condition

$$\left| \frac{f'(z)}{z^{p-1} - p} - \frac{f'(z)}{B z^{p-1} - \alpha} \right| < 1, \quad z \in U$$  \hspace{1cm} (5.1.2)

for some $\alpha (0 \leq \alpha < p), -1 \leq A < B \leq 1, 0 < B \leq 1$.

For the subclass $W_{k,p}(A, B, \alpha)$, we obtain the following results which will be applicable for our further study.

**Lemma 5.1.1** A function $f$ defined by (5.1.1) is in $W_{k,p}(A, B, \alpha, \mu)$ if and only if
\[
\sum_{n=k}^{\infty} \frac{(n+p)(1+B)}{(p-\alpha)(B-A)} a_{p+n} \leq 1.
\] (5.1.3)

**Proof.** Let \(|z|=1\). Then

\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| = \left| B \frac{f'(z)}{z^{p-1}} - \{pB - (B - A)(p - \alpha)\} \right|
\]

\[
\leq \sum_{n=k}^{\infty} (p+n)a_{p+n} - (B - A)(p - \alpha) + \sum_{n=k}^{\infty} Ba_{p+n}(p+n)
\]

\[
= \sum_{n=k}^{\infty} (p+n)(1+B)a_{p+n} - (B - A)(p - \alpha)
\]

\[
\leq 0 \text{ by assumption.}
\]

Hence by maximum modulus principle, \(f\) satisfies the condition (5.1.2). Thus \(f \in W_{k,p} (A,B,\alpha,\mu)\).

Conversely, let

\[
\left| \frac{f'(z)}{z^{p-1}} - p \right|
\]

\[
= \left| \frac{-\sum_{n=k}^{\infty} a_{p+n}(p+n)z^n}{-\sum_{n=k}^{\infty} B(p+n)a_{p+n}z^n + (B - A)(p - \alpha)} \right| < 1, z \in E.
\]

Since \(|Re(z)| \leq |z|\) for all \(z\), we have
\[
\text{Re} \left\{ \frac{\sum_{n=k}^{\infty} a_{p+n}(p+n)z^n}{(B - A)(p - \alpha) - \sum_{n=k}^{\infty} B(p+n)a_{p+n}z^n} \right\} < 1.
\]

Choosing \( z = r \) with \( 0 < r < 1 \), we have

\[
\frac{\sum_{n=k}^{\infty} (p+n)a_{p+n}r^n}{(B - A)(p - \alpha) - \sum_{n=k}^{\infty} B(p+n)a_{p+n}r^n} < 1. \tag{5.1.4}
\]

Let \( S(r) = (B - A)(p - \alpha) - \sum_{n=k}^{\infty} B(p+n)a_{p+n}r^n \).

Clearly \( S(r) \neq 0 \), for \( 0 < r < 1 \), \( S(r) > 0 \) for sufficiently small values of \( r \) and \( S(r) \) is continuous for \( 0 < r < 1 \).

Hence \( S(r) \) can not be negative for any value of \( r \) s.t. \( 0 < r < 1 \).

Upon clearing now the denominator in (5.1.4) and letting \( r \to 1 \), we have

\[
\sum_{n=k}^{\infty} (p+n)a_{p+n} \leq (B - A)(p - \alpha) - \sum_{n=k}^{\infty} B(p+n)a_{p+n}
\]

which gives (5.1.3). This completes the proof.

The function

\[
f(z) = z^p - \sum_{n=k}^{\infty} \frac{(p - \alpha)(B - A)}{(n + p)(1 + B)} z^{p+n}, \ n \geq k \tag{5.1.5}
\]
is an extremal function.

If we put $\alpha = 0$, the undermentioned result of Pal and Dixit [69] follows from the above theorem:

**COROLLARY 5.1.1** A function $f(z)$ defined by (5.1.1) is in $W_{k,p}(A,B)$ if and only if
\[
\sum_{n=k}^{\infty} \frac{(n+p)(1+B)}{p(B-A)} a_{p+n} \leq 1.
\]

From Lemma 5.1.1, we easily have following corollary.

**COROLLARY 5.1.2** If $f \in W_{k,p}(A,B,\alpha)$, then
\[
a_{p+n} \leq \frac{(p-\alpha)(B-A)}{(n+p)(1-B)}
\]
the equality holds only for functions of the form (5.1.5).

There are several definitions of fractional integral. In [40], [54], [66], [69] fractional integral is defined as follows:

**DEFINITION 5.1.1** The fractional integral of order $\mu$ is defined by
\[
D_{z}^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_{0}^{z} \frac{f(\xi)}{(z-\xi)^{1-\mu}} d\xi
\]
where $0 < \mu < 1, f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of
$(z - \xi)^{n-1}$ is removed by requiring $\log(z - \xi)$ to be real when $(z - \xi) > 0$.

A function $F$ belongs to the class $W_{k,p}(A, B, \alpha, \mu)$ if it satisfies

$$F(z) = \frac{\Gamma(\mu + p + 1)}{\Gamma(p + 1)} z^{-\mu} D_z^{-n} f(z)$$

for some $f$ belonging to $W_{k,p}(A, B, \alpha, \mu)$. After a simple calculation, we have

$$F(z) = z^p - \sum_{n=k}^{\infty} \frac{\Gamma(n + p + 1) \Gamma(\mu + p + 1)}{\Gamma(n + p + \mu + 1) \Gamma(p + 1)} a_{p+n} z^{p+n}.$$  

The subclass $S^*_{k,p}(A, B, \alpha, \mu)$ and $C_{k,p}(A, B, \alpha, \mu)$ of $S_{k,p}$ obtained by replacing $\frac{f'(z)}{z^{p-1}}$ with $\frac{zf'(z)}{f(z)}$ and $\left[1 + \frac{zf''(z)}{f'(z)}\right]$ respectively in (5.1.2) can be studied in analogous manner.

It is worthy to note that some interesting results on subclasses of multivalent functions involving the fractional differintegral operator have been investigated by Patel and Mishra [70] and also by El-Ashwah and Aouf [27].

In this paper our aim is to obtain a necessary and sufficient conditions in terms of coefficients for a function $F$ to be in
and consequently, we show that

\[ W_{k,p}(A, B, \alpha, \mu) \subseteq W_{k,p}(A, B, \alpha). \]

Then we extend the above inclusion relationship.

Applying the results of coefficient estimates we obtain class preserving integral operators of the form

\[ G(z) = \frac{r + p}{z^r} \int_0^z u^{r-1} F(u) du, \quad r > -p \text{ for } F \in W_{k,p}(A, B, \alpha, \mu). \]

Conversely when \( G(z) \in W_{k,p}(A, B, \alpha, \mu) \), radius of \( p \)-valence of \( F(z) \) has been investigated. Further sharp results concerning distortion theorem and radius of convexity for the class \( W_{k,p}(A, B, \alpha, \mu) \) have been obtained. It is also shown that the class \( W_{k,p}(A, B, \alpha, \mu) \) is closed under "arithmetic mean" and "convex linear combinations”. Also some distortion theorems, for the fractional integral of the elements of \( W_{k,p}(A, B, \alpha) \) have been determined.

We now prove the following theorem which will be used in the next section.
THEOREM 5.1.1 A function $F(z) = z^p - \sum_{n=k}^{\infty} b_{p+n} z^{p+n}$ belongs to $W_{k,p}(A,B,\alpha,\mu)$ if and only if

$$\sum_{n=k}^{\infty} b_{p+n} / T(n,\mu) \leq 1,$$

(5.1.7)

where $b_{p+n} = \frac{\Gamma(n+p+1)\Gamma(\mu+p+1)}{\Gamma(n+p+\mu+1)\Gamma(p+1)} a_{p+n}$ and

$$T(n,\mu) = \left[ \frac{(p-\alpha)(B-A)}{(n+p)(1+B)} \frac{\Gamma(n+p+1)\Gamma(\mu+p+1)}{\Gamma(n+p+\mu+1)\Gamma(p+1)} \right].$$

Proof. By definition, $F \in W_{k,p}(A,B,\alpha,\mu)$ if it satisfies the relation (5.1.6) for some $f \in W_{k,p}(A,B,\alpha)$. Let

$$f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n}.$$

Then from (5.1.6), we have

$$F(z) = z^p - \sum_{n=k}^{\infty} b_{p+n} z^{p+n},$$

where

$$b_{p+n} = \frac{\Gamma(n+p+1)\Gamma(\mu+p+1)}{\Gamma(n+p+\mu+1)\Gamma(p+1)} a_{p+n}$$

or

$$a_{p+n} = \frac{\Gamma(n+p+\mu+1)\Gamma(p+1)}{\Gamma(n+p+1)\Gamma(\mu+p+1)} b_{p+n}, n \geq k,$$
with the help of (5.1.3), the required result follows:

The function

\[ F(z) = z^p - T(n, \mu)z^{p+n} \]  

(5.1.8)
is an extremal function.

**COROLLARY 5.1.3** If \( F \in W_{k,p}(A,B,\alpha,\mu) \), then \( b_{p+n} \leq T(n,\mu) \), with equality only for the functions of the form (5.1.8).

### 5.2 MAIN RESULTS

Let \( F \in W_{k,p}(A,B,\alpha,\mu) \). Then

\[ F(z) = z^p - \sum_{n=k}^{\infty} b_{p+n} z^{p+n}, \]

where

\[ b_{p+n} = \frac{\Gamma(n + p + 1)\Gamma(\mu + p + 1)}{\Gamma(n + p + \mu + 1)\Gamma(p + 1)} a_{p+n}. \]

Clearly \( b_{p+n} < a_{p+n} \), for all \( n \geq k \), and therefore

\[ \sum_{n=k}^{\infty} \frac{(n + p)(1 + B)}{(p - \alpha)(B - A)} b_{p+n} < \sum_{n=k}^{\infty} \frac{(n + p)(1 + B)}{(p - \alpha)(B - A)} a_{p+n} \leq 1, \]

since \( f \in W_{k,p}(A,B,\alpha) \).

Hence \( F \in W_{k,p}(A,B,\alpha) \) and thus we have the inclusion relation

\[ W_{k,p}(A,B,\alpha,\mu) \subset W_{k,p}(A,B,\alpha). \]  

(5.2.1)
Since

\[ \lim_{\beta \to 0} W_{k,p}(A, B, \alpha, \beta) = W_{k,p}(A, B, \alpha). \]

The relation (5.2.1) is equivalent to

\[ W_{k,p}(A, B, \alpha, \mu) \subset \lim_{\beta \to 0} W_{k,p}(A, B, \alpha, \beta). \]

The following theorem is an extension of the above relation.

**THEOREM 5.2.1** If \( 0 < \beta \leq \mu \), then

\[ W_{k,p}(A, B, \alpha, \mu) \subset W_{k,p}(A, B, \alpha, \beta). \]

*Proof.* Let \( F(z) = z^p - \sum_{n=k}^{\infty} b_{p+n} z^{p+n} \in W_{k,p}(A, B, \alpha, \mu). \)

Then from (5.1.7), we have

\[ \sum_{n=k}^{\infty} b_{p+n} / T(n, \mu) \leq 1. \]  \hfill (5.2.2)

Since \( \beta < \mu \), we have

\[ \frac{\Gamma(n + p + \beta + 1)\Gamma(p+1)}{\Gamma(n + p + 1)\Gamma(\beta + p + 1)} < \frac{\Gamma(n + p + \mu + 1)\Gamma(p+1)}{\Gamma(n + p + 1)\Gamma(\mu + p + 1)}. \]

Therefore

\[ \sum_{n=k}^{\infty} b_{p+n} / T(n, \beta) \leq \sum_{n=k}^{\infty} b_{p+n} / T(n, \mu). \]  \hfill (5.2.3)

Using (5.2.2) in (5.2.3), we obtain
\[ \sum_{n=k}^{\infty} b_{p+n} / T_{(n,\beta)} \leq 1. \]

Hence \( F \in W_{k,p}(A,B,\alpha,\beta) \).

Now, we study the class preserving integral operator for \( W_{k,p}(A,B,\alpha,\mu) \).

**THEOREM 5.2.2** Let \( r \) be a real number such that \( r > -p \). If \( F \in W_{k,p}(A,B,\alpha,\mu) \), then the function \( G \) defined by

\[ G(z) = \frac{r + p}{z^r} \int_0^z u^{r-1} F(u) du \quad (5.2.4) \]

is also an element of \( W_{k,p}(A,B,\alpha,\mu) \).

**Proof.** Let

\[ F(z) = z^p - \sum_{n=k}^{\infty} b_{p+n} z^{p+n}, \]

then

\[ G(z) = z^p - \sum_{n=k}^{\infty} C_{p+n} z^{p+n} \]

where

\[ C_{p+n} = \frac{(r + p)}{(r + p + n)} b_{p+n} < b_{p+n}. \]

Therefore
\[
\sum_{n=k}^{\infty} C_{p+n} / T(n, \mu) < \sum_{n=k}^{\infty} b_{p+n} / T(n, \mu) \leq 1.
\]

Hence

\[G \in W_{k,p}(A, B, \alpha, \mu).\]

The following theorem is the converse problem of the above theorem.

**THEOREM 5.2.3** Let \( r \) be a real number such that \( r > -p \). If \( G(z) \in W_{k,p}(A, B, \alpha, \mu) \), then the function \( F \) defined in (5.2.4) is \( p \)-valent in \(|z| < R^*\), where

\[R^* = \inf_{n \geq k} \left[ \frac{(r + p)}{(r + p + n)} \cdot \frac{p}{(p + n)} \cdot \frac{1}{T(n, \mu)} \right]^{1/n}.
\]

The result is sharp for the function \( G(z) = z^p - T(n, \mu) z^{p+n}, \ n \geq k.\)

**Proof.** Let

\[ G(z) = z^p - \sum_{n=k}^{\infty} b_{p+n} z^{p+n}. \]

It follows from (5.2.4) that

\[ F(z) = z^p - \sum_{n=k}^{\infty} \left( \frac{r + p + n}{r + p} \right) b_{p+n} z^{p+n}. \]

In order to establish the required result, it suffices to prove that

\[ \left| \frac{F'(z)}{z^{p-1}} - p \right| < p \text{ for } |z| < R^*. \]

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Now
\[
\left| \frac{F'(z)}{z^{p-1}} - p \right| = \left| \sum_{n=k}^{\infty} (n + p) \left( \frac{r + p + n}{r + p} \right) b_{p+n} z^n \right|
\]
\[
\leq \sum_{n=k}^{\infty} (n + p) \left( \frac{r + p + n}{r + p} \right) b_{p+n} |z|^n,
\]
thus \[
\left| \frac{F'(z)}{z^{p-1}} - p \right| < p \text{ if }
\]
\[
\sum_{n=k}^{\infty} \frac{(n+p)}{p} \left( \frac{r + p + n}{r + p} \right) b_{p+n} |z|^n < 1. \tag{5.2.5}
\]
Since \( G \in W_{k,p}(A,B,\alpha,\mu) \), then
\[
\sum_{n=k}^{\infty} b_{p+n} / T(n,\mu) \leq 1.
\]
Therefore (5.2.5) will be satisfied if
\[
\left( \frac{n + p}{p} \right) \left( \frac{r + p + n}{r + p} \right) b_{p+n} |z|^n \leq b_{p+n} / T(n,\mu) \quad \forall n \geq k.
\]
or if
\[
|z| \leq \left[ \frac{p}{(n+p)} \frac{r+p}{r+p+n} \frac{1}{T(n,\mu)} \right]^{1/n} \quad \forall n \geq k.
\]
Hence \( F \) is \( p \)-valent in \( |z| < R^* \).

The result is sharp with extremal function
\[
G(z) = z^p - T(n,\mu)z^{p+n}, \quad n \geq k.
\]
Now we determine the radius of convexity of the class $W_{k,p}(A,B,\alpha,\mu)$.

**Theorem 5.2.4** If $F \in W_{k,p}(A,B,\alpha,\mu)$ then $F$ is $p$-valently convex in the disc $|z| < R^{**}$, where

$$R^{**} = \inf_{n \geq k} \left( \frac{p}{n + p} \right)^2 \frac{1}{T(n,\mu)} \right)^{1/n}.$$ 

The result is sharp.

**Proof.** In order to establish the required result, it suffices to show that

$$\left| 1 + \frac{zF''(z)}{F'(z)} \right| - p < p \text{ for } |z| < R^{**}. \quad (5.2.6)$$

Now

$$\left| 1 + \frac{zF''(z)}{F'(z)} \right| - p = \left| \frac{-\sum_{n=k}^{\infty} n(n + p)b_{p+n}z^n}{p - \sum_{n=k}^{\infty} (n + p)b_{p+n}z^n} \right| \leq \frac{\sum_{n=k}^{\infty} n(n + p)b_{p+n} |z|^n}{p - \sum_{n=k}^{\infty} (n + p)b_{p+n} |z|^n},$$

thus

$$\left| 1 + \frac{zF''(z)}{F'(z)} \right| - p < p \text{ if}.$$
Since $F \in W_{k,p}(A,B,\alpha,\mu)$, then \( \sum_{n=k}^{\infty} b_{p+n} / T(n,\mu) \leq 1 \).

Therefore (5.2.7) will be satisfied if

\[
\left( \frac{n + p}{p} \right)^2 b_{p+n} \left| z \right|^n \leq b_{p+n} / T(n,\mu)
\]

or if \( |z| \leq \left[ \left( \frac{p}{n + p} \right)^2 \frac{1}{T(n,\mu)} \right]^{1/n} \) for each \( n \geq k \).

Hence \( F \) is \( p \)-valently convex in \( |z| < R^{**} \).

The result is sharp with extremal function

\[
F(z) = z^p - T(n,\mu)z^{p+n}, \quad n \geq k.
\]

**THEOREM 5.2.5** Let a function \( f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n} z^{p+n} \) be in the class \( W_{k,p}(A,B,\alpha,\mu) \), then we have

\[
\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} r^{p+\mu} \left[ 1 - T(k,\mu)r^k \right] \leq \left| D_z^{-\mu} f(z) \right| \leq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} r^{p+\mu} \left[ 1 + T(k,\mu)r^k \right],
\]

the bounds are sharp.
Proof. Let \( f(z) = z^p - \sum_{n=k}^{\infty} a_{p+n}z^{p+n} \). Then it follows from the Lemma 5.1.1.

\[
\frac{(k + p)(1 + B)}{(p - \alpha)(B - A)} \sum_{n=k}^{\infty} a_{p+n} \leq \sum_{n=k}^{\infty} \frac{(n + p)(1 + B)}{(p - \alpha)(B - A)} a_{p+n} \leq 1.
\]

Therefore

\[
\sum_{n=k}^{\infty} a_{p+n} \leq \frac{(B - A)(p - \alpha)}{(k + p)(1 + B)}.
\]

(5.2.9)

Let us consider the function

\[
F(z) = \frac{\Gamma(p + \mu + 1)}{\Gamma(p + 1)} z^{-\mu} D_z^{-\mu} f(z)
\]

\[
= z^p - \sum_{n=k}^{\infty} \frac{\Gamma(p + n + 1)\Gamma(\mu + p + 1)}{\Gamma(p + n + \mu + 1)\Gamma(p + 1)} a_{p+n}z^{p+n}
\]

then, by using (5.2.9), we get

\[
|F(z)| \leq r^p + \sum_{n=k}^{\infty} \frac{\Gamma(p + n + 1)\Gamma(\mu + p + 1)}{\Gamma(p + n + \mu + 1)\Gamma(p + 1)} a_{p+n}r^{p+n}
\]

\[
\leq r^p + T(k, \mu) r^{p+k}
\]

and

\[
|F(z)| \geq r - \sum_{n=k}^{\infty} \frac{\Gamma(p + n + 1)\Gamma(\mu + p + 1)}{\Gamma(p + n + \mu + 1)\Gamma(p + 1)} a_{p+n}r^{p+n}
\]

\[
\geq r^p - T(k, \mu) r^{p+k}.
\]
The required inequalities follows at once.

To establish the sharpness of the bounds in (5.2.8), we take

\[
f(z) = z^p - \frac{(B - A)(p - \alpha)}{(k + p)(1 + B)} z^{p+k}.
\]

In (5.2.8), the left hand side inequality is obtained at \( z = r \), whereas, the right hand side equality is attained at \( z = re^{i\pi/k} \).

Hence the bounds are sharp.

This completes the proof of the theorem.

**THEOREM 5.2.6** If \( F \in W_{k,p}(A, B, \alpha, \mu) \) and \( |z| = r \), then

\[
r^p|1 - T(k, \mu)r^k| \leq |F(z)| \leq r^p[1 + T(k, \mu)r^k]
\]

and

\[
r^{p-1}[p - (p + k)T(k, \mu)r^k] \leq |F'(z)| \leq r^{p-1}[p + (p + k)T(k, \mu)r^k].
\]

All these inequalities are sharp.

**Proof.** Let \( F(z) = z^p - \sum_{n=k}^{\infty} b_{p+n}z^{p+n} \). Then in the view of

Theorem 5.1.1,
\[
\frac{1}{T(k, \mu)} \sum_{n=k}^{\infty} b_{p+n} \leq \sum_{n=k}^{\infty} b_{p+n} / T(n, \mu) \leq 1,
\]

we have \( \sum_{n=k}^{\infty} b_{p+n} \leq T(k, \mu). \)

Now

\[
|F'(z)| \leq r^p + \sum_{n=k}^{\infty} b_{p+n} r^{p+n}
\]

\[
\leq r^p + T(k, \mu) r^{p+n}
\]

and

\[
|F'(z)| \geq r^p - \sum_{n=k}^{\infty} b_{p+n} r^{p+n}
\]

\[
\geq r^p - T(k, \mu) r^{p+k}.
\]

Hence the inequalities in (5.2.10) are proved. Further

\[
|F'(z)| \leq pr^{p-1} + \sum_{n=k}^{\infty} b_{p+n} (p + n) r^{p+n-1}
\]

\[
\leq pr^{p-1} + r^{k+p-1} \sum_{n=k}^{\infty} (p + n) b_{p+n}
\]

\[
|F'(z)| \geq pr^{p-1} - \sum_{n=k}^{\infty} b_{p+n} (p + n) r^{p+n-1}
\]

\[
\geq pr^{p-1} - r^{k+p-1} \sum_{n=k}^{\infty} (p + n) b_{p+n}
\]

since
\[
\frac{1}{(p + k)T(k, \mu)} \sum_{n=k}^{\infty} (p + n)b_{p+n} \leq \sum_{n=k}^{\infty} b_{p+n} / T(n, \mu) \leq 1.
\]

we have

\[
\sum_{n=k}^{\infty} (p + n)b_{p+n} \leq (p + k)T(k, \mu). \tag{5.2.14}
\]

The inequalities in (5.2.11) follows by using (5.2.14) in (5.2.12) and (5.2.13).

Equalities are obtained in (5.2.10) and (5.2.11) by taking

\[
F(z) = z^p - T(k, \mu)z^{p+k}.
\]

We note that for the above defined function \( F \), equalities on the left handsie of (5.2.10) and (5.2.11) are obtained at \( z = r \), where as the equalities on the right hand side are attained at \( z = re^{in/k} \).

Lastly, we show that the class \( W_{k,p}(A,B,\alpha,\mu) \) is closed under “arithmetic mean” and “convex linear combination.”

**THEOREM 5.2.7** Let \( F_j(z) = z^p - \sum_{n=k}^{\infty} a_{j,p+n}z^{p+n}, \quad a_{j,p+n} \geq 0, \)

\( j = 1,2,3,...m \). If \( F_j \in W_{k,p}(A,B,\alpha,\mu) \) for each \( j = 1,2,3,...m \), then the function

\[
H(z) = z^p - \sum_{n=k}^{\infty} C_{p+n}z^{p+n}, \text{ where } C_{p+n} = \frac{1}{m} \sum_{j=1}^{m} a_{j,p+n}
\]
also belongs to \( W_{k,p}(A, B, \alpha, \mu) \).

**Proof.** Since \( F_j \in W_{k,p}(A, B, \alpha, \mu) \), then

\[
\sum_{n=k}^{\infty} a_{j,p+n} / T(n, \mu) \leq 1 \text{ for each } j = 1, 2, \ldots m.
\]

Therefore

\[
\sum_{n=k}^{\infty} C_{p+n} / T(n, \mu) = \sum_{n=k}^{\infty} \left[ \frac{1}{m} \sum_{j=1}^{m} a_{j,p+n} / T(n, \mu) \right]
\]

\[
= \frac{1}{m} \sum_{j=1}^{m} \left[ \sum_{n=k}^{\infty} a_{j,p+n} / T(n, \mu) \right] \leq 1.
\]

which implies that \( H(z) \in W_{k,p}(A, B, \alpha, \mu) \).

**Theorem 5.2.8** Let \( F_p(z) = z^p \) and \( F_{p+n}(z) = z^p - T(n, \mu)z^{p+n}, \)

\( (n = k, k+1, \ldots) \).

Then \( F \in W_{k,p}(A, B, \alpha, \mu) \) if and only if it can be expressed in the form

\[
F(z) = \mu_p F_p(z) + \sum_{n=k}^{\infty} \mu_{p+n} F_{p+n}(z)
\]

(5.2.15)

where \( \mu_{p+n} \geq 0 \) and \( \mu_p + \sum_{n=k}^{\infty} \mu_{p+n} = 1 \).

**Proof.** Suppose that \( F(z) \) can be expressed as in (5.2.15). Then
\[ F(z) = \mu_F F_p(z) + \sum_{n=k}^{\infty} \mu_{p+n} F_{p+n}(z) \]

\[ = z^p - \sum_{n=k}^{\infty} \mu_{p+n} T(n, \mu) z^{p+n}. \]

Now \[ \sum_{n=k}^{\infty} \frac{1}{T(n, \mu)} T(n, \mu) \mu_{p+n} = \sum_{n=k}^{\infty} \mu_{p+n} \leq 1. \]

Hence by theorem 5.1.1, \( F \in W_{k,p}(A, B, \alpha, \mu) \).

Conversely, suppose that \( F \in W_{k,p}(A, B, \alpha, \mu) \) and

\[ F(z) = z^p - \sum_{n=k}^{\infty} b_{p+n} z^{p+n}. \] (5.2.16)

Setting

\[ \mu_{p+n} = \frac{b_{p+n}}{T(n, \mu)}, (n = k, k + 1, \ldots) \]

and \( \mu_p = 1 - \sum_{n=k}^{\infty} \mu_{p+n} \) from (5.2.16), we have

\[ F(z) = \mu_F F_p(z) + \sum_{n=k}^{\infty} \mu_{p+n} F_{p+n}(z). \]

This completes the proof of the theorem.

**REMARK** It is worth mentioning here that

1. If we put \( \alpha = 0 \) in above theorems we get results of Pal and Dixit [69].

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2. If $\mu = 0, \alpha = 0$, and $k = 1$ in above theorems we obtain results of Shukla and Dashrath [83].

3. If we put $\mu = 0, \alpha = 0$ and $p = 1$ in above theorems we obtain the results of Vinod Kumar [54].

4. If we put $\mu = 0$, $\alpha = 0$, $p = 1$, $k = 1$, $B = \beta$ and $A = (2\alpha - 1)^{\beta}$ where $0 < \beta < 1$, $0 \leq \alpha < 1$ in above theorems, we get the results of Gupta and Jain [34].