CHAPTER - VI
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MATHEMATICAL ANALYSIS OF BLOOD FLOW THROUGH NARROW VESSELS WITH MILD STENOSIS

INTRODUCTION
For the long time it is very well known to medical practitioners that a severe constriction of a coronary artery significantly changes the average flow rate in coronary vessels. This constriction may cause for insufficient flow of blood through coronary arteries into the heart which may result in cardiac unrest. This insufficiency of the blood flow in capillaries is usually due to the deposition of plaque (Atherosclerotic plaque), which builds up in the coronary arteries. This deposition is termed as Stenosis. Therefore the study of blood flow in artery is quite important. In the past various studies had been organized by the researchers in this area, but no specific contribution still have been made particularly to cure the Cardiac- diseases. However, there have been limited studies of the effects of the fluid dynamics on a stenosis in artery using proper modeling techniques (CAVALCANTI, et al [1]). At the low flow rates Stenotic resistance (ratio of pressure drop to flow) is essentially constant and this suggest fully developed laminar flow (MISHRA, et al [2]). When we consider the high rate of flow,
the resistance increases with the flow, suggesting the importance of tubalance flow effects. It was further experienced by the researchers that the resistance of the stenosis was due to its dependency on its minimum cross-sectional area rather than its length (CHAKRAVORTY, et al [3]). These types of the studies were limited to a description of the overall behaviour of blood flow in the presence of a stenosis through experimental investigation.

Some researchers developed the analytical methods to predict the pressure drop, which is actually caused by stenotic area. The minimum lumen areas created in stenosed tube were about 65% and 90%, including a model without stenosis, respectively (ZOHDI, et al [4]). Ischemic heart disease, which results from high grade stenosis, is the single most common cause of death all over the world. High grade stenosis increases flow resistance in arteries, which forces the body to raise the blood pressure in order to maintain the necessary blood supply. Both the high pressure and narrowing vessels cause high flow velocity, high shear stress and low or even negative pressure, at the throat of the stenosis (WILLE, et al [5]). This may be related to the thrombosis process and atherosclerosis growth and plaque cap rupture, leads directly to stroke and heart attack. The exact mechanism of this complicated process is still not well understood. A more comprehensive study in this physiological process is of great importance for diagnosis,
prevention and treatment of stenosis related diseases. A considerable number of experimental and numerical researches have been made to study the flow dynamics and stresses in collapsible elastic tube (TANG, et al [6]).


It has been observed from experiments by previous researchers that blood behaves as a non-Newtonian fluid at low shear rates in arteries. LIU, et al [24] described a numerical simulation of viscous flow in collapsible tubes with stenosis.

In the recent times, CHRISTOFIDIS, et al [32] have shown the influence of a convergent nozzle on the flow fields of a mild stenosis located in T-junction. CUNIBERTI, et al [33] gave the development of mild aortic valve stenosis in a rabbit model with hypertension. JUNG, et al [34] suggested a hemodynamic

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computation using multiphase flow dynamics in right coronary artery. BANKS et al [35] described the turbulence modeling in three-dimensional stenosed arterial bifurcations. LIU, et al [36] has examined the effect of the Reynolds number on the flow pattern in a stenotic right coronary artery. MATAR, et al [37] studied the dynamics and stability of flow down a flexible incline. JAIN, SHARMA and KUMAR et al [38] also worked recently on this topic and have found some fruitful results.

While much work has been reported, the mathematical models for flow in stenotic collapsible tubes were primarily limited. But most researchers were focused on elastic tubes, in which stress, produces its characteristic strain instantaneously, and strain vanishes immediately upon the removal of the stress. In fact for realistic modeling channels have been considered porous as in human physiological tissues in the arteries suck the nutrients flowing within the blood. All the above studies are devoted in the wake of the new models for blood flow over the stenosis. Modeling of blood flow over the mild stenosis with medium degree of constriction through HERCHEL-BULKLEY fluid model for blood flow with oscillating pressure gradient is considered in the present study. Further more we consider blood as non-Newtonian fluid.

**OBJECTIVE OF THIS RESEARCH WORK:**

In this chapter our aim is to find the effect of mild stenosis on blood flow, in an irregular axisymmetric artery with oscillating
pressure gradient. The Herschel-Bulkley fluid model has been utilized for this study. The combined influence of an asymmetric shape and surface irregularities of constriction has been explored in this computational study. We will also try to study the variation of viscosity, shearing stress and velocity over the mild stenosis.

**DESCRIPTION OF MODEL AND GOVERNING EQUATIONS:**

We consider axisymmetric steady flow in a mild stenotic tube. The flow is assumed to be laminar, non-Newtonian, viscous and incompressible. The shape of the tube is under zero pressure and the tube wall is assumed to have no axial motion, that is, no slipping takes place between the fluid and the wall. The pressure gradient is oscillatory in nature, which is compatible with a pumping heart motion. The complex nature of blood with various parameters is approximated here. The blood is in a uniform circular tube with an axisymmetric mild stenosis takes place whose boundary is specified by KROGH model, (KAPOOR, et al [13]), as

\[
\frac{R}{R_0} = \begin{cases} 
1 - \frac{\delta}{2R_0} \left(1 + \cos \frac{2\pi}{L_0} \left(z - d - \frac{L_0}{2}\right)\right), & d \leq z \leq L_0 + d \\
1, & \text{otherwise}
\end{cases}
\]

Where \( R_0 \) the radius of unobstructed tube and \( R \) is the radius of obstructed tube. \( L_0 \) is the length of the stenosis and 'd' is the
location of the stenosis. The maximum height of stenotic growth is taken as $\delta$. Schematic diagram is shown in figure (6.1),

![Schematic diagram of a mild stenotic tube](image)

**Figure (6.1)**

Schenmatic diagram of a mild stenotic tube

The equation of continuity is,

$$ \nabla \cdot \mathbf{V} = 0 $$

$$ (6.2) $$

The momentum equation of motion is,

$$ \rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \nabla \cdot \mathbf{\tau} $$

$$ (6.3) $$

Where $\rho$ the density, $p$ is the pressure and $\tau$ is the shearing stress tensor.

Herschel-Bulkley law to model the fluid behaviour of blood, flow taking into account two characteristic features, which has emerged from the experimental data namely:

(i) the pressure of a yield stress
the dependence of the viscosity with respect to shear stress rate. (KAPOOR, et al [13]). Let $\tau_0$ be the yield stress, the coefficient of viscosity is $\mu$ and $\gamma'$ be the strain rate.

Then the constitutive equation in one dimensional form for Herschel-Bulkley pulsatile fluid with the shearing stress $\tau$, is given by,

$$\tau = \mu(\gamma') + \tau_0, \quad \tau \geq \tau_0$$  \hspace{1cm} \left\{ \begin{array}{l}
\gamma' = 0, \quad \tau < \tau_0 \quad (TU, et al [31])
\end{array} \right.$$  \hspace{2cm} (6.4)

The governing equation of the motion for the steady incompressible blood flow with pressure gradient through the mild stenosis in an artery reduces to the following form:

$$-P = \frac{1}{r} \frac{\partial (r \tau)}{\partial r}$$  \hspace{1cm} (6.5)

where,

$$\frac{\partial p}{\partial z} = -P$$  \hspace{1cm} (6.6)

$P$ being constant. Integrating Equation (6.5) with respect to $r$ which is the radial co-ordinate, we have

$$\tau = -P \frac{r}{2}$$  \hspace{1cm} (6.7)

From equations (6.4) and (6.7), we have
\[-\frac{P r}{\mu} + \frac{\tau_0}{\mu} = (\gamma')^n\]  \hspace{1cm} \text{(6.8)}

For the Herschel-Bulkley fluid in circular tube, we have \(\gamma' = 0\), when \(\tau \leq \tau_0\) and where is a core region which flows as a plug.

Let the radius of this plug region be \(r_p\). At the surface of this plug, the stress is \(\tau_0\), so that considering the force on the plug, we get,
\[p \times \pi r_p^2 = \tau_0 \times 2\pi r_p \Rightarrow p \times r_p = 2\tau_0\]  \hspace{1cm} \text{(6.9)}

Then the equation (6.8) becomes as,
\[(\gamma')^n = -\left(\frac{1}{2} \frac{pr}{\mu} + \frac{1}{2} \frac{pr_p}{\mu}\right)\]  \hspace{1cm} \text{(6.10)}

Since we know that,
\[
\frac{dv}{dr} = \gamma' \hspace{1cm} \text{(6.11)}
\]

Then using the value from the equation (6.10) in the equation (6.11), we get
\[
\frac{dv}{dr} = -\left(\frac{1}{2} \frac{p}{\mu}\right)^n (-r - r_p)^n \hspace{1cm} \text{(6.12)}
\]

The relevant conditions are,
\[v = 0 \text{ at } r = R \& R_0\]  \hspace{1cm} \text{(6.13)}
Integrating equation (6.10) and using conditions (6.13), we get

\[ v = \frac{n}{n+1} \left( \frac{1}{2} \frac{p}{\mu} \right)^{\frac{1}{n}} \left( -R \right)^{\frac{1+n}{n}} \cdot \left\{ \left( 1 + \frac{r_p}{R} \right)^{\frac{1+n}{n}} - \left( \frac{r}{R} + \frac{r_p}{R} \right)^{\frac{1+n}{n}} \right\} \]

\[ \text{(6.14)} \]

Where \( \frac{r_p}{R} \) can be taken as \( \beta \)  \[ \text{(6.15)} \]

The real part on the right hand side of the equation (6.14) contributes to the fluid velocity. Plug flow exists whenever the shear stress does not exceed yield stress. The velocity of the plug flow can be obtained by putting,

\[ r = \beta R \]

Then we get

\[ v_p = \frac{n}{n+1} \left( \frac{1}{2} \frac{p}{\mu} \right)^{\frac{1}{n}} \left( -R \right)^{\frac{1+n}{n}} \cdot \left\{ \left( 1 + \beta \right)^{\frac{1+n}{n}} - \left( 2 \beta \right)^{\frac{1+n}{n}} \right\} \]

\[ \text{(6.17)} \]

Flow rate 'Q' is obtained as,

\[ Q = \frac{n\pi}{3n+1} \left( \frac{1}{2} \frac{p}{\mu} \right)^{\frac{1}{n}} c^3 \cdot \left\{ 1 - \left( \frac{3n+1}{n(2n+1)} \beta \right) \right\} \]

\[ \text{(6.18)} \]

Where, \( c = \left( -R \right)^{\frac{1+n}{n}} \)  \[ \text{(6.19)} \]

Apparent fluidity \( \varphi_\alpha = \left( \frac{1}{\mu} \right)^2 \left( 1 - 3n \frac{\delta}{c} \right) \left\{ 1 - \left( \frac{3n+1}{(2n+1) \beta} \right) \right\} \]

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From equation (6.16), we get the shear stress $\tau_\omega$ as follows:

$$\tau_\omega = \left( \frac{3n+1}{n \pi c^3} \right)^n \left( 1 + 3n \frac{\delta}{c} \right) \left\{ 1 + \left( \frac{3n+1}{(2n+1) \beta} \right) \right\}$$

------ (6.21)
RESULT AND DISCUSSION

In this section we present the numerical results for velocity profile, volumes flow rate, apparent fluidity and wall shear stress. All these profiles provide detailed description of flow field. In the presence of mild stenosis the flow shows resistance and increases the shear stress. These are the quantities of physiological relevance. Computations were organized through computers. Here we take default values for,

\[ \beta = 1.29, \quad r_0 = 0.01, \quad r_w = 0.02, \quad n = 2, \quad r = 0.15, \quad r_p = 0.003, \quad P = 0.5 \quad \text{and} \quad \delta = 0.2. \]

These values have been chosen by consulting medical professionals of pathologies, having long clinical experience.

Figure (6.2) explains that the velocity profiles of fluid flow with respect to the radius of the obstructed tube for different values of \( \mu \). It is observed that the velocity of the fluid decreases with increasing \( r \) in the presence of the mild stenosis. If we increase the value of \( \mu \), the velocity decreases. We also observe that for lower values of \( \mu \) the graph between velocity and stenosis is having sharp decrease.

In figure (6.3) we see the trend of flow rate in the plug region for different values of \( \mu \). It is observed that the velocity in the plug region increases gradually at first and then it
Figure (6.2)
Profile of velocity vs. radius of obstructed tube (r) for different values of $\mu$
Figure (6.3)
Profile of velocity of plug region vs. radius of obstructed tube (r) for different values of \( \mu \)
Figure (6.4)

Variation of Apparent Fluidity $\varphi_a$ vs. radius of obstructed tube $(r)$ for different values of $\mu$. 

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becomes rapid with the increase in \( r \), by increasing the values of \( \mu \), the flow rate decreases.

For different value of \( \mu \), the pattern of the apparent fluidity in the direction of radius is shown in Fig (6.4). It is observed that the apparent fluidity slightly increases first with \( r \) and then attains almost constant value. All the finding in this experiment are very close to the experiment results (CUNIBERTI, et al [33]), done on rabbit.

**CONCLUSIONS**

A mathematical model of blood flow through an irregular mild-stenotic artery, is developed. All the numerical investigations are showing that the shape of the velocity is strongly perturbed by the stenosis and disturbances are clearly created by these obstructions. The flow of blood is sharper for narrowing constricted channel. It is also realized that if the velocity of fluid increases, the velocity of fluid decreases in the presence of stenosis, which is desirable in the physical situations. If we put \( n = 0 \) our results match with the ZOHDI, et al [4]. All these results are very much similar to those experimental outputs which support the critical role of hemodynamic factors. This investigation may be useful to medical practitioners to analyse their understanding about the blood flow in human cardiovascular system and which may further be affected by stenosis in different blood vessels. Putting this model for a long time on the quantitative analysis may bring certain
experimental tool for the medical research and may benefit the pathology of arterial diseases.

**COMPARISON WITH EARLIER STUDIES:**

In our entire study we found that our works many times qualify the standard results obtained by many researchers. Our work is factually supported by the research work done by CHAKRAVORTY and DUTTA [3]. Our findings are also matching with YOUNG and TSAI [8]. We have also found our work is in coherence with the work done by SHARMA and KAPOOR [19]. We have also found that our work is also matching with sole recent work done by JAIN, SHARMA and KUMAR [38]
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