ALMOST (N,p,q) SUMMABILITY OF CONJUGATE FOURIER SERIES*

In this chapter-III, a new theorem on almost generalized Nörlund summability of conjugate series of Fourier series has been established under a very general condition.

3.1. INTRODUCTION

Lorentz [35], for the first time in 1948, defined almost convergence of a bounded sequence. It is easy to see that convergent sequence is almost convergent [52]. The idea of almost convergence led to the formulation of almost generalized Nörlund summability method. Here almost generalized Nörlund summability method is considered. In 1913, Hardy [21] established \((C,\alpha)\) \(\alpha > 0\) summability of the series. Later on in 1948, harmonic summability which is weaker than the summability of \((C,\alpha)\), \(\alpha > 0\) of the series, was discussed by Siddiqui [75]. The generalization of Siddiqui has been obtained by several workers for example, Singh [79,80], Iyenger [26], Pati [60], Tripathi and Lal [88], Rajagopal [72] etc. for Nörlund mean. But nothing seems to have been done so far in the direction of study of conjugate Fourier series.

by almost generalized Nörlund summability method. Almost generalized Nörlund summability includes almost Nörlund, Riesz, harmonic and Cesàro as particular cases. In an attempt to make an advance study in this direction, in present chapter-III, we [42] have established a theorem on almost generalized Nörlund summability of conjugate Fourier series.

3.2. DEFINITIONS AND NOTATIONS

Let $\Sigma a_n$ be an infinite series with $\{S_n\}$ as the sequence of its $n^{th}$ partial sums. Lorentz [35] has given the following definition.

A bounded sequence $\{S_n\}$ is said to be almost convergent to a limit $S$, if

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{v=m}^{n+m} S_v = S \quad (3.2.1)$$

uniformly with respect to $m$.

Let $\{p_n\}$ and $\{q_n\}$ be the two sequences of non-zero real constants such that

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n, \quad P_{-1} = p_{-1} = 0 \quad (3.2.2)$$

$$Q_n = q_0 + q_1 + q_2 + \cdots + q_n, \quad Q_{-1} = q_{-1} = 0 \quad (3.2.3)$$

Given two sequences $\{p_n\}, \{q_n\}$, convolution $(p \cdot q)$ is defined by

$$R_n = (p \ast q)_n = \sum_{k=0}^{n} p_k q_{n-k}$$

It is familiar and can be easily verified that the operation of convolution is commutative and associative, and

$$(p \ast 1)_n = \sum_{k=0}^{n} p_k$$
The series $\Sigma a_n$ or the sequence $\{S_n\}$ is said to be almost generalized Nörlund $(N,p,q)$ (Qureshi [66]) summable to $S$, if

$$t_{n,m} = \frac{1}{R_n} \sum_{\nu=0}^{n} p_{n-\nu} q_{\nu} S_{\nu,m}$$

(3.2.4)

tends to $S$, as $n \to \infty$, uniformly with respect to $m$, where

$$S_{\nu,m} = \frac{1}{\nu + 1} \sum_{k=m}^{\nu+m} S_k$$

(3.2.5)

**PARTICULAR CASES**

1. Almost $(N,p,q)$ method reduces to almost Nörlund method $(N,p_n)$ if $q_n=1$ for all $n$.

2. Almost $(N,p,q)$ method reduces to almost Riesz method $(\bar{N},q_n)$ if $p_n = 1$ for all $n$.

3. In the special case when $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$, the method $(N,p_n)$ reduces to the well known method of summability $(C,\alpha)$.

4. $p_n = (n+1)^{\alpha}$ of the Nörlund means is known as harmonic means and is written as $\left(N,\frac{1}{n+1}\right)$

Let $f(t)$ be a periodic with period $2\pi$ and integrable in the sense of Lebesgue over an interval $(-\pi, \pi)$. 
Let its Fourier series be given by

\[ f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \]  

(3.2.6)

\[ = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t) \]

and then conjugate series of (3.2.6) is given by

\[ \sum_{n=1}^{\infty} (a_n \sin nt - b_n \cos nt) = \sum_{n=1}^{\infty} B_n(t) \]  

(3.2.7)

We shall use the following notations.

\[ \phi(t) = f(x+t) + f(x-t) - 2f(x) \]

\[ \psi(t) = f(x+t) - f(x-t) \]

\[ \Phi(t) = \int_0^t |\phi(u)| \, du \]

\[ \Psi(t) = \int_0^t |\psi(u)| \, du \]

\[ N_{n,m}(t) = \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \frac{\sin(v+1) \frac{t}{2} \left\{ \cos(v+2m+1) \frac{t}{2} - \cos \frac{t}{2} \right\}}{(v+1) \sin^2(t/2)} \]

\[ \overline{N}_{n,m}(t) = \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \frac{\cos(v+2m+1) \frac{t}{2} \sin(v+1) \frac{t}{2}}{(v+1) \sin^2(t/2)} \]
\[ \tau = \left[ \frac{1}{t} \right] = \text{The integral part of } \frac{1}{t} \]

3.3.

Pati [60] has established the following theorem for Nörlund summability of Fourier series.

**THEOREM A**

If \((N, p_n)\) be a regular Nörlund method defined by a real non-negative monotonic non-increasing sequence of coefficients \(\{p_n\}\) such that

\[ P_n = \sum_{v=0}^{n} p_v \rightarrow \infty, \text{ as } n \rightarrow \infty \quad (3.3.1) \]

and

\[ \log n = O(P_n), \text{ as } n \rightarrow \infty \quad (3.3.2) \]

then if

\[ \Phi(t) = \int_{0}^{t} |\phi(u)| \, du = o\left[ \frac{t}{P_\tau} \right], \text{ as } t \rightarrow +0 \quad (3.3.3) \]

the series \((3.2.6)\) is summable \((N, p_n)\) to \(f(x)\) at the point \(t = x\).

3.4. MAIN THEOREM

Our object of this chapter-III is to generalize the above result for almost \((N, p, q)\) summability of conjugate Fourier series.
we [42] prove the following theorem.

**THEOREM**

Let \( \{p_n\} \) and \( \{q_n\} \) be the monotonic non-increasing sequences of real constants such that

\[
R_n = \sum_{v=0}^{n} p_v q_{n-v} \to \infty, \text{ as } n \to \infty
\]

If

\[
\Psi(t) = \int_0^t |\psi(u)| \, du = o \left( \frac{t \alpha(1/t)}{R(\psi(t))} \right), \text{ as } t \to +0
\]

(3.4.1)

\[
\int_{\psi(t(n+m))}^{\psi(n+m)} \frac{|\psi(t)|}{t^2} \, dt = o(n), \text{ as } n \to \infty
\]

(3.4.2)

where \( 0 < \delta < \frac{1}{2} \) uniformly with respect to \( m \), and \( \alpha(t) \) is positive monotonic non-increasing function of \( t \) such that

\[
\alpha(n+m) \log(n+m) = O(R_{n+m}), \text{ as } n \to \infty
\]

(3.4.3)

and

\[
\sum_{v=0}^{n} \left( \frac{p_{n-v} q_v}{v+1} \right) = O \left( \frac{R_n}{n} \right)
\]

(3.4.4)

then the conjugate Fourier series (3.2.7) is almost \((N,p,q)\) summable to

\[
-\frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{t}{2} \, dt
\]

at point \( t = x \).
3.5. LEMMAS

For the proof of our theorem following lemmas are required:

**LEMMA 3.5.1.**

For $0 < t < \frac{1}{n+m}$ we have

\[
|N_{n,m}(t)| = \frac{1}{2\pi R_n} \left| \sum_{v=0}^{n} p_{n-v} q_v \right|
\]

\[
= \frac{1}{2\pi R_n} \left| \sum_{v=0}^{n} \frac{\sin(v+1) \cdot t \cdot \left\{ \cos(v+2m+1) \cdot \frac{t}{2} - \cos \frac{t}{2} \right\}}{(v+1) \sin^2(t/2)} \right|
\]

\[
\leq \frac{1}{2\pi R_n} \left| \sum_{v=0}^{n} \frac{\sin(v+1) \cdot t \cdot \left\{ 2 \sin \left( \frac{v+2m+2}{2} \right) \cdot \frac{t}{2} \sin \left( \frac{v+2m}{2} \right) \frac{t}{2} \right\}}{(v+1) \sin^2(t/2)} \right|
\]

\[
\leq \frac{1}{2\pi R_n} \left| \sum_{v=0}^{n} \frac{\left( \frac{v+2m+2}{2} \right) \cdot \sin \frac{t}{2} \sin \left( \frac{v+2m}{2} \right) \frac{t}{2} \right| \right|
\]

\[
= \frac{1}{2\pi R_n} \left\{ \sum_{v=0}^{n} p_{n-v} q_v \right\} (n+2m+2)
\]

\[
= O(n+m) \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v
\]

\[
|N_{n,m}(t)| = O(n+m)
\]

by (3.4.4)
LEMMA 3.5.2.

For $\frac{1}{n + m} < t < \pi$

$$|N_{n,m}(t)| = \frac{1}{2\pi R_n} \sum_{v=0}^{n} \frac{p_{n-v}q_v}{(v+1) \sin^2(t/2)} \cos\left(m + \frac{v+1}{2}\right) t \sin\left(\frac{v+1}{2}\right) t$$

$$|N_{n,m}(t)| \leq \frac{1}{2\pi R_n} \sum_{v=0}^{n} \frac{p_{n-v}q_v}{(v+1) \sin^2(t/2)}$$

$$\leq \frac{1}{2\pi R_n} \sum_{v=0}^{n} \frac{p_{n-v}q_v}{(v+1) \sin^2(t/2)} \cos\left(m + \frac{v+1}{2}\right) t \sin\left(\frac{v+1}{2}\right) t$$

$$= \mathcal{O}\left(\frac{1}{t^2} \right) \frac{1}{R_n} \sum_{v=0}^{n} \left(\frac{p_{n-v}q_v}{(v+1)}\right)$$

$$|N_{n,m}(t)| = \mathcal{O}\left(\frac{1}{t^2n}\right)$$

by (3.4.4)

3.6. PROOF OF THE MAIN THEOREM

Let $S_k(x)$ denotes the $n^{th}$ partial sum of the series (3.2.7). Then we have

$$s_k(x) = \frac{1}{2\pi} \int_{0}^{x} \frac{\cos\left(k + \frac{1}{2}\right) t - \cos\frac{t}{2}}{\sin t/2} \psi(t) dt$$

(3.6.1)

$$= \frac{1}{2\pi} \int_{0}^{x} \frac{\cos\left(k + \frac{1}{2}\right) t}{\sin t/2} \psi(t) dt - \frac{1}{2\pi} \int_{0}^{x} \cot\frac{t}{2} \psi(t) dt$$
Now by using (3.2.5) we get

$$S_{v,m} = \frac{1}{v+1} \sum_{k=m}^{v+m} \left\{ \frac{1}{2\pi} \int_0^\pi \cos \left( \frac{k+1}{2} t \right) \sin \frac{t}{2} \psi(t) \, dt - \frac{1}{2\pi} \int_0^\pi \cot \frac{t}{2} \psi(t) \, dt \right\}$$

so that by using (3.2.4) we have

$$t_{n,m} = \frac{1}{R_n} \sum_{v=0}^{n} P_{n-v} q_v \left\{ \frac{1}{2\pi} \int_0^\pi \cos \left( \frac{k+1}{2} t \right) \sin \frac{t}{2} \psi(t) \, dt - \frac{1}{2\pi} \int_0^\pi \cot \frac{t}{2} \psi(t) \, dt \right\}$$

$$= \frac{1}{2\pi R_n} \sum_{v=0}^{n} P_{n-v} q_v \int_0^\pi \frac{\sin(v+m+1)t - \sin mt}{2(v+1) \sin^2(t/2)} \psi(t) \, dt$$

$$= \int_0^\pi \left\{ \frac{1}{2\pi R_n} \sum_{v=0}^{n} P_{n-v} q_v \frac{\cos(v+2m+1)\frac{t}{2} - \sin(v+1)\frac{t}{2}}{(v+1)\sin^2(t/2)} \right\} \psi(t) \, dt$$

$$= \int_0^\pi \overline{N}_{n,m}(t) \psi(t) \, dt$$

$$= \left\{ \int_0^{l/(n+m)} + \int_{l/(n+m)}^{l/(n+m)+l} + \int_{l/(n+m)+l}^\pi \right\} \overline{N}_{n,m}(t) \psi(t) \, dt$$

$$= I_1 + I_2 + I_3 , \quad \text{(say)}$$

(3.6.2)
First we consider,

\[ I_1 = \int_{0}^{1} N_{n,m}(t) \psi(t) \, dt \]

\[ = \int_{0}^{1} \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \frac{\cos(v + 2m + 1) t \sin(v + 1) t}{(v + 1) \sin^2(t/2)} \psi(t) \, dt \]

\[ = \int_{0}^{1} \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \frac{\sin(v + 1) t}{2} \left\{ \cos(v + 2m + 1) t - \cos t \right\} \frac{1}{(v + 1) \sin^2(t/2)} \psi(t) \, dt \]

\[ + \int_{0}^{1} \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \frac{\sin(v + 1) t \cdot \cot \frac{t}{2}}{(v + 1) \sin(t/2)} \psi(t) \, dt \]

\[ = I_{1.1} + I_{1.2}, \quad \text{(say)} \]  

\[ \text{(3.6.3)} \]

Now,

\[ |I_{1.1}| \leq \int_{0}^{1} \frac{1}{2\pi R_n} \left| \sum_{v=0}^{n} p_{n-v} q_v \frac{\sin(v + 1) t}{2} \left\{ \cos(v + 2m + 1) t - \cos t \right\} \right| \left| \psi(t) \right| \, dt \]

\[ = \int_{0}^{1} \left| N_{n,m}(t) \right| \left| \psi(t) \right| \, dt \]

\[ = O(n + m) \int_{0}^{1} \left| \psi(t) \right| \, dt \quad \text{by lemma 3.5.1} \]

\[ = O(n + m) o \left( \frac{\alpha(n + m)}{(n + m) R_{n+m}} \right) \quad \text{by (3.4.1)} \]
\[
\begin{align*}
1_{1,1} &= o(1), \text{ as } n \to \infty, \text{ uniformly with respect to } m. \\
\text{Next,} \\
\text{For } 0 < t \leq \frac{1}{n + m} \\
\left| I_{1,2} \right| &\leq \frac{1}{2\pi R_a} \sum_{\nu = 0}^{n} p_{\nu} q_{\nu} \int_{0}^{\frac{1}{n + m}} \frac{\sin(v + 1) t}{(v + 1) \sin(t/2)} \cot \frac{t}{2} \psi(t) \, dt \\
&\leq \frac{1}{2\pi R_a} \sum_{\nu = 0}^{n} p_{\nu} q_{\nu} \int_{0}^{\frac{1}{n + m}} (v + 1) \sin \frac{t}{2} \cot \frac{t}{2} \psi(t) \, dt - \frac{1}{2\pi} \int_{0}^{\frac{1}{n + m}} \cot \frac{t}{2} \psi(t) \, dt
\end{align*}
\]

since the conjugate function exists, therefore

\[
\frac{1}{2\pi} \int_{0}^{\frac{1}{n + m}} \cot \frac{t}{2} \psi(t) \, dt = o(1), \text{ as } n \to \infty,
\]

uniformly with respect to m.

Hence

\[
I_{1,2} = o(1)
\]

(3.6.5)
then from (3.6.3), (3.6.4) and (3.6.5)

\[ I_1 = o(1) \]  \hspace{1cm} (3.6.6)

Now we take

\[ |I_2| \leq \int_{(l/(n+m))^b}^{(l/(n+m))^b} \left| N_{n,m}(t) \right| |\psi(t)| dt \]

\[ = O \left( \frac{1}{(n+m)^b} \right) \frac{|\psi(t)|}{t^n} dt \hspace{1cm} \text{by lemma 3.5.2} \]

\[ = O \left( \frac{1}{n} \right) \int_{(l/(n+m))^b}^{(l/(n+m))^b} \frac{|\psi(t)|}{t^n} dt \]

\[ = O \left( \frac{1}{n} \right) o(n) \hspace{1cm} \text{by (3.4.2)} \]

I_2 = o(1), as \( n \to \infty \),

uniformly with respect to \( m \).

Lastly we have

\[ |I_3| \leq \int_{(l/(n+m))^b}^{(l/(n+m))^b} \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{v-\nu} q_v \left| \cos(v+2m+1)t \sin(v+1) \frac{t}{2} \right| |\psi(t)| dt \]

\[ = \int_{(l/(n+m))^b}^{(l/(n+m))^b} \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{v-\nu} q_v \left| \frac{\sin(v+m+1)t - \sin mt}{2(v+1)\sin^2(t/2)} \right| |\psi(t)| dt \]
\[
I_3 = I_{3,1} + I_{3,2}, \quad \text{(say)}
\]

Now by using the second mean value theorem, we have

\[
|I_{3,1}| \leq \frac{1}{2\pi R_n} \sum_{\nu=0}^{n} p_{n,\nu} q_{\nu} \left[ \int_{L(n+m)^{\delta}}^{E} \left| \frac{\sin (\nu + m + 1)t}{2(\nu + 1) \sin^2 (t/2)} \right| |\psi(t)| \, dt \right].
\]

where \( \frac{1}{(n + m)^{\delta}} < \varepsilon < \pi \) and \( 0 < \delta < \frac{1}{2} \)

\[
= O \left( \frac{1}{n} \right) (n + m)^{2\delta} \left( \frac{1}{2(n + m)^{\delta}} \right)^2 \int_{L(n+m)^{\delta}}^{E} \left| \frac{\sin mt}{2(\nu + 1) \sin^2 (t/2)} \right| |\psi(t)| \, dt
\]

\( I_{3,1} = o(1), \) as \( n \to \infty, \) \quad (3.6.8.1)

uniformly with respect to \( m. \)

Now,

\[
|I_{3,2}| \leq \frac{1}{2\pi R_n} \int_{L(n+m)^{\delta}}^{E} \sum_{\nu=0}^{n} p_{n,\nu} q_{\nu} \left| \frac{\sin mt}{2(\nu + 1) \sin^2 (t/2)} \right| |\psi(t)| \, dt
\]
\[ I_3 = o(1), \quad \text{as } n \to \infty, \]

uniformly with respect to \( m \). Hence,

\[ I_3 = o(1), \quad \text{as } n \to \infty, \quad (3.6.8.3) \]

Now by combining (3.6.2) (3.6.6), (3.6.7) and (3.6.8.3) we have,

\[ \int_0^s N_{n,m}(t) \psi(t) \, dt = o(1), \quad \text{as } n \to \infty \]

uniformly with respect to \( m \).

Thus the theorem is established.

### 3.7. APPLICATIONS

In this section we deduce some corollaries from our theorem.

**COROLLARY 3.7.1.**

If the conditions

\[ \Psi(t) = \int_0^t |\psi(u)| \, du = o \left[ \frac{t}{R_{(1)(1)}} \right], \]

\[ \log (n+m) = O(R_{n+m}), \quad \text{as } n \to \infty, \]
(3.4.2) and (3.4.4) of the main theorem are satisfied, then the conjugate Fourier series is almost \((N,p,q)\) summable to

\[-\frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{t}{2} dt\]

**COROLLARY 3.7.2.**

If

\[\Psi(t) = \int_0^t |\psi(u)| \, du = o\left(\frac{t}{\log(1/t)}\right)\]

conditions (3.4.2) and (3.4.4) of our theorem hold, then the conjugate Fourier series is almost \((N,p,q)\) summable to

\[-\frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{t}{2} dt\]

without employing (3.4.3).