CHAPTER VI
MATRIX SUMMABILITY OF JACOBI SERIES

In this chapter-VI, a new theorem on matrix summability of Jacobi series is established. This theorem is a generalization of several known results.

6.1. INTRODUCTION

The Nörlund summability \((N,p_n)\) of Jacobi series has been studied by a number of researchers like Gupta [17], Choudhary [12], Thorpe [86], Pandey and Beohar [61], Prasad and Saxena [63], Pandey [64], Beohar and Sharma [7] & Tripathi, Tripathi, and Yadav [90]. After quite a good amount of work in the ordinary Nörlund summability of Jacobi series at the point \(x = 1\), Khare and Tripathi [34] discussed generalized Norlund summability \((N,p,q)\) of Jacobi series. \((N,p,q)\) summability reduces to \((N,p_n)\) summability for \(q_n = 1\) for all \(n\) and \((N,q_n)\) means when \(p_n = 1\) for all \(n\). But nothing seems to have been done so far in the direction of study of Jacobi series by matrix summability method which, as known includes, as special cases, the method of \((N,p_n)\) and \((N,p,q)\) summabilities. In an attempt to make an advance study in this direction we, in chapter-VI, have established a theorem on matrix summability of Jacobi series. So that the results of Thorpe [86] and Beohar and Sharma [7] becomes the particulars cases of our theorem.

6.2. DEFINITIONS AND NOTATIONS

Let \(f(x)\) be defined in closed \([-1,1]\) such that the function

\[(1 - x)^\alpha (1 + x)^\beta f(x) \in L [-1,1]; \alpha > -1, \beta > -1.\]
The Jacobi series corresponding to this function is
\[ f(x) \sim \sum_{n=0}^{\infty} a_n p_n^{(\alpha,\beta)}(x) \quad (6.2.1) \]

where
\[ a_n = \frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \int_{-1}^{1} (1-x)^n (1+x)^\beta f(x) p_n^{(\alpha,\beta)}(x) \, dx \]

and \( p_n^{(\alpha,\beta)} \) are the Jacobi polynomials.

Let \( T = (a_{n,k}) \) be an infinite triangular matrix satisfying the Silverman-Toeplitz [84] condition of regularity i.e.
\[ \sum_{k=0}^{n} a_{n,k} \to 1, \text{as} \, n \to \infty, \]
\[ a_{n,k} = 0 \quad \text{for} \, k > n \]
and \( \sum_{k=0}^{n} |a_{n,k}| \leq M \), a finite constant.

Let \( \Sigma u_n \) be an infinite series with the sequence of partial sums \( \{s_k\} \)
where \( s_k = \sum_{\nu=0}^{k} u_{\nu} \)

The sequence-to-sequence transformation
\[ t_n = \sum_{k=0}^{n} a_{n,k} s_k = \sum_{k=0}^{n} a_{n,n-k} s_{n-k} \quad (6.2.2) \]
defines the sequence \( \{t_n\} \) of matrix means of the sequence \( \{s_n\} \), generated by the sequence of coefficients \( (a_{n,k}) \). The series \( \Sigma u_n \) is said
to be summable to the sum $S$ by matrix method if $\lim_{n \to \infty} t_n$ exists and is equal to $s$ (Zygmund [96]) and we write $t_n \to s(T)$, as $n \to \infty$.

Seven important particular cases of matrix means are

1. (C, 1) means, when $a_{nk} = \frac{1}{n+1}$

2. (C, $\delta$) means, when $a_{nk} = \frac{(n-k+\delta+1)}{\delta-1} \binom{n+\delta}{\delta}$

3. Harmonic means, where $a_{nk} = \frac{1}{(n-k+1)\log n}$

4. (H, $p$) means, when $a_{nk} = \frac{1}{\log^{p-1}(n+1)} \prod_{q=0}^{p-1} \log^q(k+1)$

5. Nörlund means [53] when $a_{nk} = p_{n-k} \frac{p_n}{p}$ when $P_n = \sum_{k=0}^{n} p_k$

6. Riesz means ($N, p_n$) when $a_{nk} = \frac{p_k}{p_n}$

7. (N, $p,q$) means [6] when $a_{nk} = \frac{p_{n-k}q_k}{R_n}$ when $R_n = \sum_{k=0}^{n} p_k q_{n-k}$

We shall use the following notations

\[ F(\phi) = \left\{ f(\cos \phi) - A \right\} \left( \sin \frac{\phi}{2} \right)^{2\alpha+1} \left( \cos \frac{\phi}{2} \right)^{2\beta+1} \]

'A' being fixed constant

\[ \psi(t) = \int_{0}^{t} |F(\phi)| \, d\phi \]
\[ \tau = \text{Integral part of} \quad \frac{1}{\phi} = \left[ \frac{1}{\phi} \right] \]

\[ A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k} \]

\[ \eta = \text{Integral part of} \quad \frac{1}{\delta} = \left[ \frac{1}{\delta} \right] \]

6.3.

Thorpe [86] has proved the following theorem of Nörlund summability of Jacobi series.

**THEOREM A**

If

\[ \int_{-t}^{1} |f(u) - A| \, du = o(t), \quad \text{as} \, t \to 0 \]

and let \( \{q_n\} \) be a non-negative and monotonic non-increasing sequence of numbers, such that

\[ \sum_{k=2}^{n} \frac{Q_k}{k^{(2\alpha+3)/2}} = O \left( \frac{Q_n}{k^{(2\alpha+1)/2}} \right), \quad \text{as} \, n \to \infty \]

then the Jacobi series (6.2.1) of \( f(x) \) is summable \( (N,q_n) \) at the point \( x = +1 \) to the sum \( A \) provided that the condition \( -\frac{1}{2} < \alpha < \frac{1}{2}, \beta > -1 \) and the antipole condition

\[ \int_{-1}^{b} (1 + x)^{(2\beta-1)/4} |f(x)| \, dx < \infty \]

is satisfied.
Beohar and Sharma [7] has proved the following theorem on Nörlund summability of Jacobi series.

**THEOREM B**

Let \( \{p_n\} \) be a non-negative non-increasing sequence such that

\[
\sum_{k=0}^{n} p_k = p_n \quad \text{and} \quad \frac{n^{(2\alpha+1)/2}}{p_n} \to 0 \quad \text{as} \quad n \to \infty
\]

and

\[
\sum_{n} \frac{p_k}{k^{(2\alpha+3)/2}} = O\left(\frac{p_n}{k^{(2\alpha+1)/2}}\right),
\]

(6.3.1)

\( \alpha \) being a fixed positive integer. If

\[
\psi(t) = \int_{0}^{t} |F(\phi)| |d\phi| = o(t^{2\alpha+2}), \quad \text{as} \quad t \to 0
\]

(6.3.2)

then the series (6.2.1) is summable \((N, p_n)\) at \(x = 1\) to the sum \(A\) provided

\[-\frac{1}{2} \leq \alpha < \frac{1}{2}; \quad \beta > -\frac{1}{2}\]

and the antipole condition.

\[
\int_{-1}^{b} (1+x)^{(2\beta-3)/4} |f(x)| \, dx < \infty
\]

'\(b\) fixed,' is satisfied.

**6.4. MAIN THEOREM**

Here, we have established the following theorem on the matrix summability of Jacobi series.

**THEOREM**

Let \( T = (a_{n,k}) \) be an infinite triangular matrix such that the elements \( (a_{n,k}) \) be non-negative non-decreasing with \( k \), \( A_{n,t} = \sum_{k=0}^{t} a_{n,n-k} \) , \( A_{n,n} = 1 \) \( \forall n \),

\[
n^{(2\alpha+1)/2} A_{n,n} \to 0 \quad \text{as} \quad n \to \infty.
\]
If
\[ \psi(t) = \int_0^t |F(\phi)| d\phi = o(t^{2\alpha+2}), \text{ as } t \to 0 \] (6.4.1)

and
\[ \sum_n \frac{A_{n,k}}{k^{(2\alpha+3)/2}} = O\left(\frac{1}{n^{(2\alpha+1)/2}}\right), \] (6.4.2)

\(\alpha\) being a fixed positive integer, then Jacobi series (6.2.1) is matrix summable \((T)\) at \(x = 1\) to the sum \(A\) provided \(-\frac{1}{2} < \alpha < \frac{1}{2}\); \(\beta > \frac{1}{2}\) and the antipole condition

\[ \int_{-1}^{b} (1 + x)^{(2\beta-3)/4} |f(x)| dx < \infty \] (6.4.3)

\(\beta\) fixed, is satisfied.

6.5. LEMMAS

The following lemmas are required for the proof of our theorem.

LEMMA 6.5.1.

[Szegö [76]]. If \(\alpha < -1, \beta > -1\), then as \(n \to \infty\).

\[ p_n^{(a,b)}(\cos \phi) = O\left(n^{a}\right), \quad 0 \leq \phi \leq \frac{1}{n} \] (6.5.1.1)

\[ = O\left(n^{b}\right), \quad \pi - \frac{1}{n} \leq \phi \leq \pi \] (6.5.1.2)
\[ = n^{-\frac{1}{2}} k(\phi) \left[ \cos(N\phi + \nu) + \frac{o(1)}{n} \sin \phi \right], \quad \frac{1}{n} \leq \phi \leq \pi - \frac{1}{n} \] (6.5.1.3)

where

\[ k(\phi) = n^{-\frac{1}{2}} \left( \sin \frac{\phi}{2} \right)^{-\frac{\alpha - \frac{1}{2}}{2}} \left( \cos \frac{\phi}{2} \right)^{-\beta - \frac{1}{2}} \]

\[ N = n + \alpha + \beta + 1 \]
\[ \nu = \left( \alpha + \frac{1}{2} \right) \pi \]

**LEMMA 6.5.2.**

[Gupta [17]]. The antipole condition (6.4.3) impiles that

\[ \int_0^\pi \left( \cos \frac{\phi}{2} \right)^{(2\beta - 1)/2} \left| f(\cos \phi) - A \right| d\phi < \infty \] (6.5.2.1)

which further impiles that

\[ \int_0^{1/n} t^{(2\beta - 1)/2} \left| f(-\cos t) - A \right| dt = O(1) \] (6.5.2.2)

**LEMMA 6.5.3.**

[McFadden [50]]. If \( \{p_n\} \) is a non-negative and non-increasing sequence, then for \( 0 \leq a < b \leq \infty \), \( 0 \leq \phi \leq \pi \) and for any \( n \) and \( a \),

\[ \left| \sum_{k \geq a} p_k e^{i(n-k)\phi} \right| = O(P_\tau) \] (6.5.3.1)

where \( P_\tau = P_{\{\phi\}} \) and \( \tau = [1/\phi] \).
LEMMMA 6.5.4.

[Rhodes [73]].

Let \( \{u_n\}, \{v_n\} \) be real sequences and \( \{u_n\} \) be non-negative, if \( \{v_n\} \) is non-increasing, then

\[
\left| \sum_{k=1}^{n} u_k v_k \right| \leq \max_{1 \leq r \leq n} \left| \sum_{k=1}^{r} u_k \right| \tag{6.5.4.1}
\]

If \( \{v_n\} \) is non-decreasing then

\[
\left| \sum_{k=1}^{n} u_k v_k \right| \leq 2 \max_{1 \leq r \leq n} \left| \sum_{k=1}^{r} u_k \right| \tag{6.5.4.2}
\]

LEMMMA 6.5.5.

Under the condition of the theorem on \((a_{n,k})\), for large \( n \), uniformly in \( 0 < \phi \leq \pi, 0 \leq a \leq b \leq n, \)

\[
\left| \sum_{k=a}^{b} a \cos\{(n-k+\rho)\phi - r\} (n-k)^{\alpha + \frac{1}{2}} \right| = O \left| n^{\alpha + \frac{1}{2}} A_{n,r} \right| \tag{6.5.5.1}
\]

where \( \rho = \frac{\alpha + \beta + 2}{2}, \gamma = -\left(\frac{\alpha + 3}{2}\right) \frac{\pi}{4} \)

PROOF

\[
\left| \sum_{k=a}^{b} a_{n,n-k} \cos\{(n-k+\rho)\phi - r\} (n-k)^{\alpha + \frac{1}{2}} \right|
\]
by lemma 6.5.4 we have,

$$\mathcal{O}\left(n^{\alpha+\frac{1}{2}}\right) \text{ Real part of } \sum_{k=a}^{b} a_{n,n-k} e^{i[(n-k+p)\phi-r]}$$

$$= \mathcal{O}\left(n^{\alpha+\frac{1}{2}}\right) \left| \sum_{k=a}^{b} a_{n,n-k} e^{i(n-k)\phi} e^{i(\rho-r)} \right|$$

$$= \mathcal{O}\left(n^{\alpha+\frac{1}{2}}\right) \left| \sum_{k=a}^{b} a_{n,n-k} e^{i(n-k)\phi} \right|$$

by using lemma 6.5.3 we have,

$$= \mathcal{O}\left(n^{\alpha+\frac{1}{2}} A_{n,\alpha} \right)$$

which proves the result.

**LEMMA 6.5.6.**

Under the hypothesis of the theorem,

$$\sum_{k=0}^{n-1} a_{n,n-k} (n-k)^{\alpha-\frac{1}{2}} = \mathcal{O}\left(n^{\alpha-\frac{1}{2}}\right)$$

**PROOF**

$$\sum_{k=0}^{n-1} a_{n,n-k} (n-k)^{\alpha-\frac{1}{2}} = \mathcal{O}\left(n^{\alpha-\frac{1}{2}}\right) \left[ \sum_{k=0}^{n-1} a_{n,n-k} \right]$$

by lemma 6.5.4
which proves the result.

**LEMMA 6.5.7.**

Let

\[ M_n(\phi) = 2^{\alpha+\beta+1} \sum_{k=0}^{n-1} a_{n,n-k} \lambda_{n-k} P^{(\alpha+\beta)}(\cos \phi) \]

where

\[ \lambda_n = \frac{2^{-(\alpha+\beta+1)} \Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} = \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)} n^{\alpha+1} \]

then, for \( \frac{1}{2} > \alpha > -\frac{1}{2}, \beta > -\frac{1}{2} \) and if \((a_{n,k})\) satisfies the hypothesis of the theorem,

\[ M_n(\phi) = O(n^{2\alpha+2}) \quad \text{if} \quad 0 \leq \phi < \frac{1}{n} \quad (6.5.7.1) \]

\[ = O(n^{\alpha+\beta+1}) \quad \text{if} \quad \pi - \frac{1}{n} \leq \phi \leq \pi \quad (6.5.7.2) \]
\[
M_n(\phi) = O(2^{\alpha+\beta+1}) \sum_{k=0}^{n-1} a_{n,n-k} O(2^{-(\alpha+\beta+1)}(n-k)^{\alpha+1}(n-k)^{\alpha+1}) \\
= O(1) \left[ \sum_{k=0}^{n-1} a_{n,n-k} O(n-k)^{2\alpha+2} \right] \quad \text{by (6.5.1.1)}
\]

by using lemma 6.5.4, we have

\[
= O(n^{2\alpha+2}) \sum_{k=0}^{n-1} a_{n,n-k}
\]

\[
= O(n^{2\alpha+2}) \left[ \sum_{k=0}^{n} a_{n,n-k} = O(n^{2\alpha+2})(A_{n,n}) \right]
\]

\[
= O(n^{2\alpha+2})O(1)
\]

\[
= O(n^{2\alpha+2})
\]

If \( \pi - \frac{1}{n} \leq \phi \leq \pi \), using (6.5.1.2), we have

\[
M_n(\phi) = O \left[ \sum_{k=0}^{n-1} a_{n,n-k} O(n-k)^{\beta} O(n-k)^{\alpha+1} \right]
\]
\[
= O\left[ \sum_{k=0}^{n-1} a_{n,n-k} O(n-k)^{\alpha+\beta+1} \right]
\]

\[
= O(n^{\alpha+\beta+1}) \sum_{k=0}^{n-1} a_{n,n-k} \quad \text{by lemma 6.5.4}
\]

\[
= O(n^{\alpha+\beta+1}) \sum_{k=0}^{n} a_{n,n-k} = O(n^{\alpha+\beta+1})A_{n,n}
\]

\[
= O(n^{\alpha+\beta+1}) O(1)
\]

\[
= O(n^{\alpha+\beta+1})
\]

If \( \frac{1}{n} \leq \phi \leq \pi - \frac{1}{n} \), we have with notation as in lemma 6.5.5 and using (6.5.1.3)

\[
M_n(\phi) = O(1) \sum_{k=0}^{n-1} a_{n,n-k} (n-k)^{\alpha+\frac{1}{2}} \left( \sin \frac{\phi}{2} \right)^{-\alpha-\frac{3}{2}} \left( \cos \frac{\phi}{2} \right)^{-\beta-\frac{1}{2}} \cos \{ (n-k)\phi + \rho \phi - r \} + O(1) \frac{1}{(n-k)\sin \phi}
\]

\[
= O(1) \sum_{k=0}^{n-1} a_{n,n-k} (n-k)^{\alpha+\frac{1}{2}} \left( \sin \frac{\phi}{2} \right)^{-\alpha-\frac{3}{2}} \left( \cos \frac{\phi}{2} \right)^{-\beta-\frac{1}{2}} \cos \{ (n-k)\phi + \rho \phi - r \} + O(1) \sum_{k=0}^{n-1} a_{n,n-k} (n-k)^{\alpha+\frac{1}{2}} \left( \sin \frac{\phi}{2} \right)^{-\alpha-\frac{5}{2}} \left( \cos \frac{\phi}{2} \right)^{-\beta-\frac{3}{2}}
\]
by using lemma 6.5.4, we get

\[
= O(1) \left[ \sum_{k=0}^{n-1} a_{n,n-k} \left( \sin \frac{\phi}{2} \right)^{-\frac{\alpha-3}{2}} \left( \cos \frac{\phi}{2} \right)^{-\frac{\beta-1}{2}} (n-k)^{\alpha+1} \cos((n-k)\phi + \rho \phi - r) \right] \\
+ O \left( n^{\frac{\alpha-1}{2}} \sum_{k=0}^{n-1} a_{n,n-k} \left( \sin \frac{\phi}{2} \right)^{-\frac{\alpha-5}{2}} \left( \cos \frac{\phi}{2} \right)^{-\frac{\beta-3}{2}} \right)
\]

by using lemmas 6.5.5 and 6.5.6, we have

\[
= O \left( \left( \sin \frac{\phi}{2} \right)^{-\frac{\alpha-1}{2}} \left( \cos \frac{\phi}{2} \right)^{-\frac{\beta-1}{2}} \right) O \left( n^{\frac{\alpha-1}{2}} A_{n,n} \right) + O \left( \left( \sin \frac{\phi}{2} \right)^{-\frac{\alpha-5}{2}} \left( \cos \frac{\phi}{2} \right)^{-\frac{\beta-3}{2}} \right) O \left( n^{\frac{\alpha-1}{2}} \right)
\]

\[
= O \left[ n^{\frac{\alpha-1}{2}} A_{n,n} \left( \sin \frac{\phi}{2} \right)^{-\frac{\alpha-3}{2}} \left( \cos \frac{\phi}{2} \right)^{-\frac{\beta-1}{2}} \right] + O \left[ n^{\frac{\alpha-1}{2}} \left( \sin \frac{\phi}{2} \right)^{-\frac{\alpha-5}{2}} \left( \cos \frac{\phi}{2} \right)^{-\frac{\beta-3}{2}} \right]
\]

In this way lemma 6.5.7 is proved.

6.6. PROOF OF THE MAIN THEOREM

Following Obrechkoff [54], the \( n^{th} \) partial sum of the series (6.2.1) at the point \( x=1 \) is given by

\[
S_n(1) = 2^{\alpha+\beta+1} \int_0^\pi \left( \sin \frac{\phi}{2} \right)^{2\alpha+1} \left( \cos \frac{\phi}{2} \right)^{2\beta+1} f(\cos \phi) S_n(1, \cos \phi) d\phi
\]
where $S_n'(1, \cos \phi)$ denotes the $n^{th}$ partial sum of the series

$$\sum_m \frac{P_m^{(\alpha, \beta)}(1) P_m^{(\alpha, \beta)}(\cos \phi)}{g_m}$$

where

$$g_m = \frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1} \Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$$

Rao [71] has shown that $S_n'(1, \cos \phi) = \lambda_n P_n^{(\alpha+1, \beta)}(\cos \phi)$

Therefore

$$S_n(1) - A = 2^{\alpha+\beta+1} \lambda_n \int_0^\pi \left( \sin \left( \frac{\phi}{2} \right) \right)^{(2\alpha+1)} \left( \cos \left( \frac{\phi}{2} \right) \right)^{(2\beta+1)} \{f(\cos \phi - A)\} P_n^{(\alpha+1, \beta)}(\cos \phi) d\phi$$

$$= 2^{\alpha+\beta+1} \lambda_n \int_0^\pi F(\phi) P_n^{(\alpha+1, \beta)}(\cos \phi) d\phi$$

where $\lambda_n$ is defined as in lemma 6.5.7.

The matrix means of the series (6.2.1) at $x = 1$ is given by

$$t_n = \sum_{k=0}^n a_{n,k} S_k(1)$$

$$= \sum_{k=0}^n a_{n,n-k} S_{n-k}(1)$$

or

$$t_n - A = \sum_{k=0}^n a_{n,k} \{ S_{n-k}(1) - A \}$$

$$= \int_0^\pi F(\phi) M_n(\phi) d\phi + (a_{n,0}) O(1) \int_0^\pi F(\phi) d\phi$$

since $\int_0^\pi F(\phi) d\phi$ is a finite constant, by assumption, second term on the
right is \( o(1) \), as \( n \to \infty \), hence in order to prove the theorem we have to show that

\[
I = \int_0^\pi F(\phi) M_n(\phi) \, d\phi = o(1) \quad \text{as } n \to \infty
\]

Let us denote

\[
I = \left[ \int_0^1 \phi + \int_1^5 \phi^{-1} + \int_{\pi-1}^{\pi} \phi \right] F(\phi) M_n(\phi) \, d\phi
\]

\[
= I_1 + I_2 + I_3 + I_4, \quad \text{(say)} \tag{6.6.1}
\]

\( \delta \) being a suitable constant.

\[
I_1 = \int_0^{1/n} F(\phi) M_n(\phi) \, d\phi
\]

\[
|I_1| = \int_0^{1/n} |F(\phi)| O\left(n^{(2\alpha+2)}\right) \, d\phi \quad \text{by (6.5.7.1)}
\]

\[
= O\left(n^{(2\alpha+2)}\right) \int_0^{1/n} |F(\phi)| \, d\phi
\]

\[
= O\left(n^{2\alpha+2}\right) o\left(\frac{1}{n^{2\alpha+2}}\right) \quad \text{by (6.4.1)}
\]

\[
I_1 = o(1), \quad \text{as } n \to \infty \tag{6.6.2}
\]

In order to estimate \( I_2 \), we employ the asymptotic relation given in (6.5.7.3). Thus

\[
I_2 = O\left[ \int_{1/n}^\delta |F(\phi)| n^{(2\alpha+1)/2} A_{n,\tau} \left(\sin \frac{\phi}{2}\right)^{-(2\alpha+3)/2} \, d\phi \right]
\]

\[
+ O\left[ \int_{1/n}^\delta |F(\phi)| n^{(2\alpha-1)/2} \left(\sin \frac{\phi}{2}\right)^{-(2\alpha+5)/2} \, d\phi \right]
\]

\[
= I_{2,1} + I_{2,2} \quad \text{(say)} \tag{6.6.3}
\]
\[ I_{2,1} = O\left(n^{(2\alpha+1)/2}\right) \left[ \int_{1/n}^{\delta} \frac{F(\phi) \, |A_{n,\tau}| \, d\phi}{\phi^{(2\alpha+3)/2}} \right] \]

\[ = O\left(n^{(2\alpha+1)/2}\right) \left[ \int_{1/n}^{\delta} \frac{o\left(\phi^{(2\alpha+2)}\right) \, A_{n,\tau} \, d\phi}{\phi^{(2\alpha+3)/2}} \right] \]

\[ = O\left(n^{(2\alpha+1)/2}\right) \left[ o\left(\left(\frac{\phi^{(2\alpha+2)}}{\phi^{(2\alpha+3)/2}}\right) A_{n,\tau}\right) \right] + o\left(\int_{1/n}^{\delta} \left(\frac{\phi^{(2\alpha+2)}}{\phi^{(2\alpha+3)/2}}\right) d\phi \right) \]

\[ = o\left(n^{(2\alpha+1)/2} A_{n,\eta}\right) + o\left(A_{n,n}\right) + o\left(n^{(2\alpha+1)/2}\right) \int_{1/n}^{\delta} \left(\frac{\phi^{(2\alpha+2)}}{\phi^{(2\alpha+3)/2}}\right) A_{n,\tau} \, d\phi \]

\[ = o(1) + o(1) + o\left(n^{(2\alpha+1)/2}\right) \int_{1/n}^{\delta} \phi^{\alpha-\frac{1}{2}} A_{n,\tau} \, d\phi \]

\[ = o(1) + o\left(n^{(2\alpha+1)/2}\right) \int_{1/n}^{\delta} \frac{A_{n,u}}{u^{(2\alpha+3)/2}} \, du \]

\[ \frac{1}{\phi} = u \quad \text{and by the hypothesis of the theorem} \]

\[ = o(1) + o\left(n^{(2\alpha+1)/2}\right) \sum_{a} A_{n,k} \frac{1}{k^{(2\alpha+3)/2}} \quad \text{where} \quad a = \left[\frac{1}{\delta}\right]+1, \quad n \geq \left[\frac{1}{t}\right] \]

by using (6.4.2) we have

\[ I_{2,1} = o(1) \quad (6.6.4) \]
Now we take,

\[ I_{2.2} = O \left( \int_{1/n}^{\delta} |F(\phi)| \, n^{(2\alpha-1)/2} \left( \sin \frac{\phi}{2} \right)^{-(2\alpha+5)/2} \, d\phi \right) \]

\[ = O \left( n^{(2\alpha-1)/2} \right) \left[ \int_{1/n}^{\delta} \frac{F(\phi)}{\phi^{(2\alpha+5)/2}} \, d\phi \right] \]

\[ = O \left( n^{(2\alpha-1)/2} \right) \left[ \int_{1/n}^{\delta} \frac{o\left( \phi^{(2\alpha+2)} \right)}{\phi^{(2\alpha+5)/2}} \, d\phi \right] \]

\[ = O \left( n^{(2\alpha-1)/2} \right) \left[ \int_{1/n}^{\delta} \frac{1}{\phi^{(2\alpha+5)/2}} \, o\left( \phi^{(2\alpha+2)} \right) \, d\phi \right] + o \left( \int_{1/n}^{\delta} \phi^{(2\alpha+2)} \frac{d}{d\phi} \left( \phi^{-(2\alpha+5)/2} \right) \, d\phi \right) \]

\[ = o\left( n^{(2\alpha-1)/2} \right) + o(1) + o\left( n^{(2\alpha-1)/2} \right) \int_{1/n}^{\delta} \phi^{(2\alpha-3)/2} \, d\phi \]

\[ = o(1) + o(1) + o\left( n^{(2\alpha-1)/2} \right) \left[ \frac{\phi^{(2\alpha-1)/2}}{\left( \frac{2\alpha-1}{2} \right)} \right]_{1/n}^{\delta} \left( \because -\frac{1}{2} \leq \alpha < \frac{1}{2} \right) \]

\[ = o(1) + o\left( n^{(2\alpha-1)/2} \right) + o\left( n^{(2\alpha-1)/2} \right) \left( \frac{1}{n^{(2\alpha-1)/2}} \right) \]

\[ = o(1) + o(1) \]

\[ I_{2.2} = o(1) \quad \text{as } n \to \infty \quad \text{(6.6.5)} \]

Hence

\[ I_2 = o(1), \quad \text{as } n \to \infty. \quad \text{(6.6.6)} \]
Now we take,

\[ I_3 = O \left( \int_\delta^{\pi-1/n} \frac{|F(\phi)| A_{n,n} n^{(2\alpha+1)/2}}{\left( \sin \frac{\phi}{2} \right)^{(2\alpha+3)/2} \left( \cos \frac{\phi}{2} \right)^{(2\beta+1)/2}} d\phi \right) \]

\[ + O(n^{(2^{-1}/2)} \left[ \int_\delta^{\pi-1/n} \frac{|F(\phi)| d\phi}{\left( \sin \frac{\phi}{2} \right)^{(2\alpha+5)/2} \left( \cos \frac{\phi}{2} \right)^{(2\beta+3)/2}} \right] \]

\[ = O(n^{(2\alpha+1)/2} A_{n,n}) \int_\delta^{\pi-1/n} |f(\cos \phi) - A| \left( \cos \frac{\phi}{2} \right)^{(2\beta-1)/2} \cos \frac{\phi}{2} d\phi \]

\[ + O(n^{(2\alpha+1)/2}) \int_\delta^{\pi-1/n} |f(\cos \phi) - A| \left( \cos \frac{\phi}{2} \right)^{(2\beta-1)/2} d\phi \]

\[ = O(n^{(2\alpha+1)/2} A_{n,n}) + O(n^{(2\alpha-1)/2}) \quad \text{by (6.5.2.1)} \]

\[ = o(1) \quad \text{as } n \to \infty \]

\[ I_3 = o(1), \quad \text{as } n \to \infty \quad (6.6.7) \]

Lastly we consider \( I_4 \),

\[ I_4 = O(n^{\alpha+\beta+1}) \int_{\pi-1/n}^{\pi} |F(\phi)| d\phi \quad \text{by (6.5.7.2)} \]

\[ = O(n^{\alpha+\beta+1}) \int_{\pi-1/n}^{\pi} |f(\cos \phi) - A| \left( \sin \frac{\phi}{2} \right)^{(2\alpha+1)} \left( \cos \frac{\phi}{2} \right)^{(2\beta+1)} d\phi \]
\[= O(n^{\alpha + \beta + 1}) \int_0^{1/n} |f(-\cos t) - A| \left( \cos \frac{t}{2} \right)^{(2\alpha + 1)} \left( \sin \frac{t}{2} \right)^{(2\beta + 1)} \, dt,\]

taking \(\pi - \phi = t\)

\[= O(n^{\alpha + \beta + 1}) \int_0^{1/n} |f(-\cos t) - A| t^{2\beta + 1} \, dt\]

\[= O(n^{(2\alpha - 1)/2}) \int_0^{1/n} |f(-\cos t) - A| t^{(2\beta - 1)/2} \, dt\]

by using (6.5.2.2), we have

\[L_4 = o(1), \quad \text{as } n \rightarrow \infty \quad (6.6.8)\]

Hence

\[I = o(1), \quad \text{as } n \rightarrow \infty\]

Thus the theorem is completely established.

6.7. PARTICULAR CASES

1. Result of Thorpe [86] becomes the particular case of our theorem

\[\text{if } a_{n,k} = \frac{q_{n-k}}{Q_n} \quad \text{where } Q_n = \sum_{v=0}^{n} q_v\]

2. If \(a_{n,k} = \frac{P_{n-k}}{P_n}\), where \(P_n = \sum_{v=0}^{n} q_v\), then the result of Beohar and Sharma [7] becomes the particular case of main theorem.