CHAPTER 4

SOME SPECIAL OPERATION
A HILBERT SPACE OF
DIRICHLET SERIES
CHAPTER - 4
SOME SPECIAL OPERATORS ON A HILBERT SPACE OF DIRICHLET SERIES

4.1:- Introduction

In the previous chapter, we have studied the space $\Omega^2_u$ of all $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ being a Dirichlet series, satisfying a certain condition, as a Hilbert space.

The analogy of the properties of Banach algebra $B(H)$ of all bounded linear operator on a Hilbert space $H$, with those of the complex plane motivate the study of certain classes of bounded linear operators which are of great practical importance. In our case we are interested to study various types of operators specially the normal and compact operator on $\Omega^2_u$. Many elementary properties of compact/Normal operators are not dependent on the inner-product structure of the space. We present below some of these properties along with many other in the setting of our Hilbert space $\Omega^2_u[132]$ which consists of all those $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ being a Dirichlet series [61, 62] satisfying the condition $\sum \left| \frac{a_k}{a_k} \right|^2 < \infty$ where $u, u(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}, \alpha_k \neq 0$ for
any $k$, is the fixed Dirichlet series as described in chapter 3, satisfying conditions (3.1.1) and (3.1.2) see section 3.1.

In the present chapter we explore the properties of some special operator on this Hilbert space $\Omega^2_u$. Characterization of self adjoint operators, compact, normal and Hilbert-Schmidt and nuclear operators were made on $\Omega^2_u$. Spectral analysis has been obtained in section 4.2 and 4.3. Orthogonal projection on $\Omega^2_u$ and resolution of Identity in section 4.4 and an application in section 4.5 has also been studied.

4.2:- Operators on $\Omega^2_u$

In this section, we have studied some special operator on the Hilbert space $\Omega^2_u$. Characterization of self-adjoint, compact, normal and Hilbert-Schmidt operators were obtained.

Let us first recall the definition of a diagonal-operator on a Hilbert space $H$ with $\{e_k\}$ as an orthonormal basis of $H$. A linear operator $T$ on a Hilbert space $H$ is called a diagonal operator if $A(e_k) = \gamma_k e_k$ for each $= 1, 2, 3, \ldots \ldots$, where $\gamma_k$ is a scalar. That is $A(e_k)$ is a scalar multiple of $e_k$ see [45] and $T$ is called a compact operator if for every bounded sequence $\{x_n\}$ in $H$, the sequence $\{Tx_n\}$ contains a convergent sub-sequence see [123, 125].
Let \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) and \( g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \) belong to the Hilbert spaces \( \Omega_u^2 \) then, we have \( \sum_{k=1}^{\infty} \left| \frac{a_k}{a_k} \right|^2 < \infty \), \( \sum_{k=1}^{\infty} \left| \frac{b_k}{a_k} \right|^2 < \infty \) and the norm generated by the inner product \( \langle ; \rangle \) in \( \Omega_u^2 \) is given by,

\[
\langle f, g \rangle = \sum_{k=1}^{\infty} \left( \frac{a_k}{a_k} \right) \left( \frac{b_k}{a_k} \right) \quad \text{............... (4.2.1)}
\]

so that \( \| f \|^2 = \langle f, f \rangle = \sum_{k=1}^{\infty} \left| \frac{a_k}{a_k} \right|^2 \)

Let there be given a bounded scalar sequence \( E = \{ \gamma_k \} \) of non-zero complex number such that

\[
\sup_k |\gamma_k| = M \quad \text{and} \quad \inf_k |\gamma_k| = m
\]

Define \( T : \Omega_u^2 \to \Omega_u^2 \) as \( (Tf)(s) = \sum_{k=1}^{\infty} \gamma_k a_k e^{s\lambda_k} \) for \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) belonging to \( \Omega_u^2 \).

i.e. \( (Tf)(s) = \sum_{k=1}^{\infty} \gamma_k \langle f, \phi_k \rangle \phi_k(s) \)

or \( Tf = \sum_{k=1}^{\infty} \gamma_k \langle f, \phi_k \rangle \phi_k \quad \text{............... (4.2.2)} \)

Particularly \( T\phi_k = \gamma_k \phi_k \) for each \( k = 1, 2, 3, \ldots \ldots \)

where \( \{ \phi_k \} \) is the orthonormal basis of the Hilbert space \( \Omega_u^2 \) given by \( \phi_k(s) = \alpha_k e^{s\lambda_k} \) for each \( k = 1, 2, 3, \ldots \ldots \ldots \), as described in previous chapter 3 section 3.2 (A).
Clearly \( T f \in \Omega_u^2 \) and it is bounded also as;

\[
\| Tf \|^2 = \sum_{k=1}^{\infty} |\gamma_k \alpha_k|^2 = \sum_{k=1}^{\infty} |\gamma_k|^2 |\alpha_k|^2 \leq M^2 \sum_{k=1}^{\infty} |\phi_k|^2 = M^2 \| f \|^2
\]

So that \( T \) is bounded. Further it gives that

\[
\| T \| \leq M \quad \text{........... (4.2.3)}
\]

On the other hand, if \( T \) be defined on \( \Omega_u^2 \) as in (4.2.1) then

\[
|\gamma_k| = |\gamma_k \varphi_k| = \| T \varphi_k \| = \| T \| | \varphi_k | = | T |
\]

\[
\sup k |\gamma_k| \leq | T | \quad \text{or} \quad M \leq | T | \quad \text{............... (4.2.4)}
\]

Equation (4.2.3) and (4.2.4) together give

\[
| T | = \sup_k |\gamma_k| = M
\]

It is also easy to see that \( T \) is a linear, this way \( T \) defines an operator on \( \Omega_u^2 \) for every bounded sequence \( \{ \gamma_k \} \) of complex number. On the other hand if \( \{ \gamma_k \} \) is unbounded then \( |\gamma_k| > M \) for any given large \( M \) for \( k \geq k_0 \) in that case, evidently \( T f \) will not exist. Thus for every bounded sequence \( \{ \gamma_k \} \) we have a bounded linear operator on our Hilbert space \( \Omega_u^2 \) and conversely.

The operator \( T \) as defined in (4.2.2) is called a diagonal operator with diagonal entries as \( \gamma_k^S \) see [45]. Further as we know that every bounded linear operator \( A \) on a separable Hilbert space \( H \) can be represented by an infinite
matrix \{\xi_{nk}\} see [123], where \(\xi_{nk} = \langle Ae_k, e_n\rangle\) and \{e_k\} is an orthonormal basis of the Hilbert space.

Here in our case T is a linear and bounded operator on the separable Hilbert space \(\Omega_u^2\) [129] therefore we must have

\[\xi_{nk} = \langle T\varphi_k, \varphi_n\rangle = \frac{\gamma_k \alpha_k}{\alpha_k} \cdot 0 = 0 \text{ if } n \neq k\]

and

\[\xi_{nk} = \langle T\varphi_k, \varphi_n\rangle = \frac{\gamma_k \alpha_k}{\alpha_k} \cdot \frac{\bar{\alpha}_k}{\alpha_k} = \gamma_k \text{ if } n = k\]

Thus

\[\xi_{nk} = \begin{cases} 0 & \text{if } n \neq k \\ \gamma_k & \text{if } n = k \end{cases}\]

i.e. \(\xi_{nk} = \gamma_k\) for each \(n = k\) and zero otherwise.

so T has the matrix representation as

\[T = \text{dia}\{\gamma_1, \gamma_2, \gamma_3, \ldots \ldots \ldots, \gamma_k, \ldots \ldots\} = \text{dia}\{\gamma_k\}\]

Next, if we define another bounded linear operator S on \(\Omega_u^2\) as follows,

\[S(f)(s) = \sum_{k=1}^{\infty} \overline{\gamma}_k (f, \varphi_k) \varphi_k(s)\]

Then we see that \(S = \text{dia}\{\overline{\gamma}_k\}\) is adjoint of the operator T as
Further

\[
(S\varphi_k,\varphi_n) = (\varphi_k, T\varphi_n) = \overline{(T\varphi_n, \varphi_k)} = \overline{\gamma_k}
\]

\[
(TS)(f)(s) = (ST)(f)(s) = \sum_{k=1}^{\infty} |\gamma_k|^2 a_k e^{s \lambda k} = \sum_{k=1}^{\infty} |\gamma_k|^2 \langle f, \varphi_k \rangle \varphi_k(s) \ldots (4.2.5)
\]

So that TS = ST. Let us write S as T* thus we see from (4.2.5) that TT* = T*T. Hence T is a normal operator on $\Omega_\mu^2$ as well. As we know that the set of all bounded sequence \{\gamma_k\} of complex number with point wise linear operation and unit denoted by 1 = \{1, 1, 1, ......., 1, ........\} and conjugation \{\gamma_k\} \rightarrow \{\overline{\gamma_k}\} under sup norm i.e. $||\{\gamma_k\}|| = sup_k |\gamma_k|$ is a commutative algebra with unity 1 (see [68]). A bounded sequence \{\gamma_k\} in this algebra is called invertible if it has an inverse in this algebra i.e. if there exist a bounded sequences \{\eta_k\} such that $\gamma_k \eta_k = 1$ for all k. A necessary and sufficient condition for that is \{\gamma_k\} should be bounded away from zero i.e. there must exist a positive number $\delta'$ such that $|\gamma_k| \geq \delta$ for all k. Now the correspondence \{\gamma_k\} \rightarrow T where T is the operator on $\Omega_\mu^2$, as defined in (4.2.2) above is an isomorphism (an embedding) of sequence algebra into the algebra of operators on $\Omega_\mu^2$. The correspondence preserves not only the algebraic operation but also conjugation i.e. \{\gamma_k\} \rightarrow T then \{\overline{\gamma_k}\} \rightarrow T^* (the conjugate of T). This correspondence preserves the norm also. Thus we have established the following theorem.
Theorem - 4.2.1:- A necessary and sufficient condition that a scalar sequence \( \{ y_k \} \) be the diagonal of a diagonal operator \( T \) on the Hilbert space \( \Omega_u^2 \) is that it be bounded with
\[
\|T\| = \sup_k |y_k|.
\]

We now have the following theorem about the operator \( T \).

Theorem - 4.2.2:- \( T : \Omega_u^2 \to \Omega_u^2 \) as defined above is

(i) Self adjoint iff \( \{ y_k \} \) is real sequence.

(ii) Positive iff \( y_k > 0 \) for each \( k = 1, 2, 3, \ldots \).

(iii) Unitary iff \( y_k \) lies on the boundary of unit disc for each \( k = 1, 2, 3, \ldots \).

That is \( |y_k| = 1 \).

See definitions of these operators in definitions 1.17 to 1.20 of Part B, chapter - 1.

Proof (i):- We know that the operator \( T \) on a Hilbert space \( H \) is self-adjoint if
\( T = T^* \) or equivalently if \( \langle Tx, y \rangle = \langle x, Ty \rangle \) for all \( x, y \in H \) see [96].

For \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}, g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \) belonging to \( \Omega_u^2 \), we see that,
\[
\langle Tf, g \rangle = \sum_{k=1}^{\infty} \left( y_k a_k \right) \left( \frac{b_k}{\bar{a_k}} \right) = \sum \gamma_k \left( \frac{a_k}{\bar{a_k}} \cdot \frac{b_k}{ \bar{a_k}} \right)
\]
and \[ \langle f, Tg \rangle = \sum_{k=1}^{\infty} \left( \frac{a_k}{a_k} \right) \left( \frac{b_k}{a_k} \right) = \sum \frac{\sqrt{a_k} b_k}{a_k} \left( \frac{\sqrt{a_k}}{a_k} \right) \]

so \[ \langle Tf, g \rangle = \langle f, Tg \rangle \text{ iff } \]

i.e. \[ y_k = \overline{y_k} \text{ for each } k = 1, 2, 3, \ldots \ldots \]

i.e. when \( \{y_k\} \) is a real sequence. Hence \( T \) is self-adjoint i.e. \( T = T^* \) if \( \{y_k\} \) is a bounded sequence of real numbers.

(ii):- We know that \( T \) is positive iff \( \langle Tf, f \rangle > 0 \). We have here

\[ \langle Tf, f \rangle = \sum_{k=1}^{\infty} y_k \left| \frac{a_k}{a_k} \right|^2 > 0 \text{ for each } f \in \Omega^2_u \text{ that is iff } y_k > 0 \text{ for each } k. \]

(iii):- A bounded linear operator \( T \) on a Hilbert space \( H \) is **Unitary** see [96] iff

\[ TT^* = T^*T = I \]

Here we have \( (TT^*)(f)(s) = \sum_{k=1}^{\infty} \overline{y_k} a_k e^{s \lambda_k} \)

\[ = \sum_{k=1}^{\infty} (y_k \overline{y_k}) a_k e^{s \lambda_k} \]

\[ = \sum_{k=1}^{\infty} |y_k|^2 a_k e^{s \lambda_k} \]

\[ = \sum |y_k|^2 \langle f, \varphi_k \rangle \varphi_k(s) \]

Similarly, we get \( (T^*T)f(s) = \sum_{k=1}^{\infty} |y_k|^2 \langle f, \varphi_k \rangle \varphi_k(s) \) and

\[ f(s) = f(s) = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \varphi_k(s) \]

so \( TT^* = T^*T = I \) iff \( |y_k|^2 = 1 \) for each \( k = 1, 2, 3, \ldots \ldots \ldots \)

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\[ |\gamma_k| = 1 \text{ for each } k \geq 1. \]

Hence \( \gamma_k \) should lie on the boundary of unit disk \(|z| = 1\) of the complex plane.

**Remark - 4.1:** It is further to add that if \( T \) is self-adjoint then
\[ e^{iT} = \sum_{k=1}^{\infty} \frac{(iT)^k}{k!} \]
will be an unitary operator.

From the definition of \( T \) on \( \Omega_u^2 \), it is clear that \( \text{Range of } T = R(T) = T(\Omega_u^2) \) is a close subspace of \( \Omega_u^2 \) and it is contained in \( \Omega_u^2 \) iff \{\gamma_k\} is bounded. Further let \( \gamma_k \neq 0 \), for any \( k \) and
\[ (Tf)(s) = \sum_{k=1}^{\infty} \gamma_k a_k e^{s\lambda_k} = g(s) \text{ where } g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \text{ then } b_k = \gamma_k a_k \]
for each \( k \). Now \( T^{-1} \) can be defined on \( R(T) \) as follows;
\[ (T^{-1}g)(s) = \sum_{k=1}^{\infty} \left( \frac{b_k}{\gamma_k} \right) e^{s\lambda_k} \]
and it is easy to see that \( T^{-1} \) is also a bounded linear operator on \( \Omega_u^2 \).

We next state the following theorem to characterize the compact and Hilbert-Schmidt operator on \( \Omega_u^2 \).

**Theorem - 4.2.3:** \( T : \Omega_u^2 \to \Omega_u^2 \) as defined above is a

(i) Compact operators iff \( \lim_{k \to \infty} |\gamma_k| = 0 \)

(ii) Hilbert-Schmidt operator iff \( \sum_{k=1}^{\infty} |\gamma_k|^2 < \infty \) i.e. iff \{\gamma_k\} \( \in l_2 \)

(iii) Nuclear operator iff \( \sum_{k=1}^{\infty} |\gamma_k|^2 < \infty \) i.e. iff \{\gamma_k\} \( \in l_1 \)
Proof (i):- Let us first assume that $\gamma_k \to 0$ as $k \to \infty$. Taking $|\gamma_k| < \varepsilon$ for $k \geq n$. For each $n$, define,

$$(T_n f)(s) = \sum_{k=1}^{\infty} \gamma_k a_k e^{s\lambda_k} \quad \text{or} \quad T_n(f) = \sum_{k=1}^{\infty} \gamma_k \langle f, \varphi_k \rangle \varphi_k$$

Then clearly $T_n$ is linear and bounded for each $n = 1, 2, 3, \ldots \ldots$ also as $\dim[T_n(\Omega^2_{\alpha})] < \infty$, therefore each $T_n$ is compact [89] or see [Def. 1.16]. Further more,

$$\|(T - T_n)\|^2 = \sum_{k=n+1}^{\infty} \left| \frac{\gamma_k a_k}{\alpha_k} \right|^2 = \sum \left| \gamma_k \right|^2 \left| \frac{a_k}{\alpha_k} \right|^2 \leq \varepsilon^2 \|f\|$$

Taking the supremum over all $f$ of norm 1, we find that

$$\|T - T_n\| \leq \varepsilon$$

Thus $T_n \to T$ as $n \to \infty$, hence $T$ is compact, being limit of compact operators see [96] or [Def. theorem 1.7]. For the converse suppose that $\{\gamma_k\}$ does not go to zero, how large $k$ may be, then for some $\varepsilon > 0$, we get $\gamma_k \geq \varepsilon$ for an infinite numbers of $k$, say, for $k = k_1, k_2, k_3, \ldots \ldots, k_n, \ldots \ldots$

By the definition (4.2.2) of $Tf$, we get,

$$(T \varphi_k)(s) = \gamma_k a_k e^{s\lambda_k} \quad \text{for each} \quad k = 1, 2, 3, \ldots \ldots \ldots\ldots \ldots \ldots$$

Therefore

$$\|T(\varphi_{k_i}) - T(\varphi_{k_j})\|^2 = \left\| \frac{\gamma_{k_i} \alpha_{k_i}}{\alpha_{k_i}} - \frac{\gamma_{k_j} \alpha_{k_j}}{\alpha_{k_j}} \right\|^2 = \left| \gamma_{k_i} - \gamma_{k_j} \right|^2 = \left| \gamma_{k_i} \right|^2 + \left| \gamma_{k_j} \right|^2 \geq 2 \varepsilon^2$$
This is true for all $i, j$ ($i \neq j$). This show that $\{T(\varphi_k)\}$ cannot have a Cauchy sub-sequence even. Hence T cannot be compact since $\{\varphi_k\}$ is a bounded sequence.

(ii):- An operator $T$ on a H.S. $H$ is a *Hilbert-Schmidt* operator if 

$$\sum_{k=1}^{\infty} \|T(e_k)\|^2 < \infty$$

where $\{e_k\}$ is an orthonormal basis for the Hilbert space see [59].

Here we have $\{\varphi_k\}$ as an orthonormal basis in our case and

$$(T\varphi_k)(s) = \gamma_k a_k e^{s \lambda_k}, \quad k = 1, 2, 3, \ldots \ldots$$

Therefore, 

$$\sum_{k=1}^{\infty} \|T(\varphi_k)\|^2 = \sum_{k=1}^{\infty} \|\frac{\gamma_k a_k}{a_k}\|^2 = \sum_{k=1}^{\infty} |\gamma_k|^2$$

Since $\{\varphi_k\}$ is an orthonormal basis this show that $T$ is a *Hilbert-Schmidt* operator on $\Omega_\mu^2$ if $\sum_{k=1}^{\infty} |\gamma_k|^2 < \infty$ that is when $\{\gamma_k\} \in l_2$.

(iii):- An operator $T$ on a Hilbert space $H$ is called *Nuclear* iff there is an orthonormal basis $\{e_k\}$ which satisfies $\sum_{k=1}^{\infty} \|Te_k\| < \infty$; see [59].

Here in our case, we see that $\{\varphi_k\}$ is an orthonormal basis in $\Omega_\mu^2$ and

$$\|T\varphi_k\|^2 = \langle T\varphi_k, T\varphi_k \rangle$$

where $\varphi_k(s) = \alpha_k e^{s \lambda_k}; \quad k = 1, 2, 3, \ldots \ldots$

But 

$$T\{\varphi_k(s)\} = \gamma_k a_k e^{s \lambda_k}$$

$$\Rightarrow \quad \|T\varphi_k\|^2 = \langle T\varphi_k, T\varphi_k \rangle = \frac{\gamma_k a_k}{a_k} \cdot \frac{\gamma_k a_k}{a_k} = \gamma_k \overline{\gamma_k} = |\gamma_k|^2$$

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\[ \Rightarrow \|T \varphi_k\| = |\gamma_k| \]

Therefore T is \textit{Nuclear} iff

\[ \sum_{k=1}^{\infty} \|T \varphi_k\| = \sum_{k=1}^{\infty} |\gamma_k| < \infty \quad \text{that is when} \ \{\gamma_k\} \in l_1 \]

**Note - 1:** If T is \textit{Nuclear} then 
\textit{trace} of T is defined as follows,

\[ \text{tr}(T) = \sum_{k=1}^{\infty} \langle T \varphi_k, T \varphi_k \rangle = \sum_{k=1}^{\infty} \gamma_k \]

**Note - 2:** If T is compact and Hermitian (self-adjoint) then \( \gamma_k > 0 \). Now if we define for \( f \in \Omega^2_u \), \((Sf)(s) = \sum \gamma_k^{1/2}\langle f, \varphi_k \rangle \varphi_k \) then clearly

\[ S^2 = T \Rightarrow \text{square root of} \ T = \sqrt{T} = S. \]

Since self adjoint operators T on a Hilbert space are always normal, therefore operator T is normal on \( \Omega^2_u \) iff \( \{\gamma_k\} \) is a real sequence as stated above. Also taking \( E = \{\gamma_k : k = 1, 2, 3, \ldots \} \) all \( \gamma_k \) being real numbers so that T become self adjoint on \( \Omega^2_u \) and therefore normal.

We now study in the next section 4.3 about the \textit{uniform} and \textit{strong} stability of the operator T.

**4.3:** \textbf{Uniform and Strong Stability}

We first give below the definitions of uniform and strong \textit{stability} as given in [120].
1. An operator $T$ on a Normed Linear Space (NLS) $X$ is *Uniformly stable* if the power sequence \( \{T^n\} \) converges *uniformly* to the null operator ‘0’ i.e.

\[
T^n \xrightarrow{u} 0 \quad \text{(equivalently, if } \|T^n\| = 0)\]

2. An operator $T$ on a NLS $X$ is *strongly stable* if $\{T^n\}$ converges *strongly* to the null operator i.e.

\[
T^n \xrightarrow{s} 0 \quad \text{(equivalently, if } \|T^n x\| \to 0 \text{ for every } x \in X)\]

Now we have the following theorem;

**Theorem - 4.3.1:-** \( T : \Omega^2_u \to \Omega^2_u \) as defined above is such that $T$ is bounded and \( \|T\| = \sup_k |\gamma_k| < \infty \). Then

(i) $T$ is uniformly-stable iff \( \|T\| < 1 \) (i.e. \( \sup_k |\gamma_k| < 1 \))

(ii) $T$ is strongly-stable iff \( |\alpha_k| < 1 \) for every $k \geq 1$

(iii) Exhibit an example of strongly stable but not uniformly stable diagonal operator.

**Proof (i):-** It can readily be checked that $n^{th}$ power of $T$ is again a diagonal operator on $\Omega^2_u$. In fact,

\[
T^n f = \sum_{k=1}^{\infty} y^n a_k e^{s \lambda_k} \text{ where } f, f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k} \in \Omega^2_u
\]
So that \( \|T^n\| = \sup_{k} |\gamma_k|^n = (\sup_{k} |\gamma_k|)^n = M^n \) for every \( n \geq 1 \). Hence \( T \) is uniformly-stable iff \( M < 1 \) that is

\[
T^n \overset{u}{\to} 0 \quad \text{iff} \quad M = \sup_{k \geq 1} |\gamma_k| < 1.
\]

(ii): If \( \|T^n f\| \to 0 \) for every \( f \) in \( \Omega_u^2 \), then in particular \( \|T^n \phi_k\| = |\gamma_k|^n \to 0 \) as \( n \to \infty \) and hence \( |\gamma_k| < 1 \) for every \( k \geq 1 \). On the other hand take an arbitrary \( f, f(s) = \sum_{k=1}^{\infty} \xi_k e^{s \lambda_k} \) in \( \Omega_u^2 \) so that

\[
0 \leq \|T^n f\|^2 = \sum_{k=1}^{\infty} |\gamma_k|^{2n} \left| \frac{\xi_k}{\alpha_k} \right|^2 = \sum_{k=1}^{m} |\gamma_k|^{2n} \left| \frac{\xi_k}{\alpha_k} \right|^2 + \sum_{k=m+1}^{\infty} |\gamma_k|^{2n} \left| \frac{\xi_k}{\alpha_k} \right|^2
\]

\[
\leq \max_{1 \leq k \leq m} |\gamma_k|^{2n} \sum_{k=1}^{m} \left| \frac{\xi_k}{\alpha_k} \right|^2 + \sup_{k \geq m} |\gamma_k|^{2n} \sum_{k=m+1}^{\infty} \left| \frac{\xi_k}{\alpha_k} \right|^2
\]

For every integer \( m \geq 0 \) if \( |\gamma_k| < 1 \) for every \( k \geq 1 \), then \( \sup_{k} |\gamma_k|^{2n} \leq 1 \) for all \( n \geq 1 \). Thus as \( \sum_{k=1}^{m} \left| \frac{\xi_k}{\alpha_k} \right|^2 \leq \|f\|^2 \) for all \( m \geq 1 \),

\[
0 \leq \|T^n f\|^2 \leq \max_{1 \leq k \leq m} |\gamma_k|^{2n} \|f\|^2 + \sum_{k=m+1}^{\infty} \left| \frac{\xi_k}{\alpha_k} \right|^2 \text{ for every } m, n \geq 1.
\]

Now take an arbitrary \( \varepsilon > 0 \).

Since \( \sum_{k=m+1}^{\infty} \left| \frac{\xi_k}{\alpha_k} \right|^2 \to 0 \) as \( \to \infty \left( \text{for } \sum_{k=1}^{\infty} \left| \frac{\xi_k}{\alpha_k} \right|^2 < \infty \right) \).

It follows that \( \exists \) a positive integer \( m = m(\varepsilon) \) s.t.
\[ m \geq m(\varepsilon) \Rightarrow \sum_{k=m+1}^{\infty} \left| \frac{\xi_k}{a_k} \right|^2 < \frac{\varepsilon^2}{2} \]

Moreover, if \(|\gamma_k| < 1\) for every \(k \geq 1\), then \(\max |\gamma_k|^2 < 1\), so that

\[ \lim_{n \to \infty} (\max_{1 \leq k \leq m} |\gamma_k|^{2n}) = \lim_{n \to \infty} (\max_{1 \leq k \leq m} |\gamma_k|^2)^n = 0 \] for every integer \(m \geq 1\).

In particular, \(\lim_{n \to \infty} (\max_{1 \leq k \leq m} |\gamma_k|^{2n}) = 0\) then there exists a positive integer \(n(\varepsilon)\) s.t.

\[ \max_{1 \leq k \leq m(\varepsilon)} |\gamma_k|^{2n} < \frac{\varepsilon^2}{2} \text{ for } n \geq n(\varepsilon) \]

But

\[ 0 \leq \|T^n f\|^2 \leq \max_{1 \leq k \leq m} |\gamma_k|^{2n} \|f\|^2 + \sum_{k=m(\varepsilon)+1}^{\infty} \left| \frac{\xi_k}{a_k} \right|^2 \]

And so

\[ n \geq n(\varepsilon) \Rightarrow 0 \leq \|T^n f\| < \varepsilon \]

Therefore, if \(|\gamma_k| < 1\) for every \(k \geq 1\) then \(\|T^n f\| \to 0\) for every \(f \in \Omega^2_u\). Hence the bounded linear operator \(T\) on \(\Omega^2_u\) is *strongly stable* if \(|\gamma_k| < 1\) for every \(k \geq 1\) that is \(T^n \to 0\) if \(|\gamma_k| < 1\) for every \(k \geq 1\).

(iv):- Particularly choosing \(\gamma_k = \frac{k+1}{k+2} = \left(1 - \frac{1}{k+2}\right)\) then we see that \(T\) is *strongly-stable* (since \(|\gamma_k| < 1\) for every \(k \geq 1\)) but \(T\) is not *uniformly-stable* because \(\sup_{k \geq 1} |\gamma_k| = 1\).

In the next section, we shall study the spectral analysis of the operator \(T\) in \(\Omega^2_u\).
In this section, we studied spectral analysis of the operator \( T \) on \( \Omega_u^2 \) as defined in (4.2.2). Since \( \Omega_u^2 \) is a separable Hilbert space \([111]\) and \( \{\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_k, \ldots\} \) is an orthonormal basis for \( \Omega_u^2 \). Taking \( E = \{\gamma_k\} \), a bounded sequence of real numbers so that \( T \) is a self adjoint hence normal on \( \Omega_u^2 \). Then operator \( T \) as defined by (4.2.2) above given by:

\[
T(f) = \sum_{k=1}^{\infty} \gamma_k \langle f, \varphi_k \rangle \varphi_k, \quad f \in \Omega_u^2
\]

is a normal operator.

As the spectrum \( \sigma(T) \) of a normal operator consists of approximate eigen values see \([114]\), therefore here in our case too, the spectrum of \( T \) will consist of approximate eigen values of \( T \).

We show below that the eigen values of \( T \) are given by \( E = \{\gamma_k\} \) along with every approximate eigen value of \( T \) which is a limit of these eigen values. In fact we have the following theorem.

**Theorem - 4.4.1:** The spectrum of the operator \( T \) consist of approximate eigen values of \( T \) i.e. \( \sigma(T) = \overline{E} \).

**Proof:** Since \( T(\varphi_k) = \gamma_k \varphi_k \), it follows that every \( \gamma_k \) is an eigen value of \( T \) see \([111, p-189]\). Conversely let \( \lambda \) be an eigen value of \( T \) with an eigen vector \( f \neq 0 \) in \( \Omega_u^2 \), then
\[
\lambda \sum_{k=1}^{\infty} (f, \varphi_k) \varphi_k = \lambda f = T(f) = \sum_{k=1}^{\infty} \gamma_k (f, \varphi_k) \varphi_k
\]

\[
\Rightarrow \quad \lambda (f, \varphi_k) = \gamma_k (f, \varphi_k) \text{ for each } k
\]
or
\[
(\lambda - \gamma_k) (f, \varphi_k) = 0 \text{ for all } k
\]

This will hold if \( \lambda = \gamma_k \) since \( f \neq 0 \), \( (f, \varphi_k) \neq 0 \) so \( \gamma_k \) are eigen values. If on the other hand \( \lambda \neq \gamma_k \) then we must have \( f = \varphi_k \) for all values of \( k \) which is not possible unless \( f = 0 \).

This shows that \( \lambda = \gamma_k \) are eigen values of \( T \). Thus spectrum of \( T = E = \{ \gamma_k : k = 1, 2, 3, \ldots \} \)

Next if \( \lambda \) is an approximate eigen value of \( T \) see [111, p-189] or [def 1.31], we show that \( \lambda \) belongs to the closure of the set \( E = \{ \gamma_k : k = 1, 2, 3, \ldots \} \) for, if this were not the case, there would be \( a, \delta > 0 \) with \( |\lambda - \gamma_k| \geq \delta \) for all \( k \). Then for any \( f \in \Omega_2^2 \),

\[
\|Tf - \lambda f\|^2 = \|\sum_{k=1}^{\infty} (\lambda - \gamma_k) (f, \varphi_k) \varphi_k\|^2
\]

\[
= \sum_{k=n}^{\infty} |\gamma_k - \lambda|^2 |(f, \varphi_k)|^2 \geq \delta^2 \sum_{k=1}^{\infty} |(f, \varphi_k)|^2
\]

\[
= \delta^2 \|f\|^2
\]

and \( \lambda \) would not be an approximate eigen value of \( T \). Thus the set of all approximate eigen values is the closure of \( E \) i.e. \( E = \text{closure} \{ \gamma_k \} \).
4.5:- Orthogonal Projection on $\Omega_u^2$ and Resolution of Identity

In this section, we obtain the orthonormal projection on $\Omega_u^2$ and resolution of Identity.

Let $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ belong to $\Omega_u^2$. Then $f$ can also be expressed as follows;

$$f(s) = \sum_{k=1}^{\infty} a_{2k} e^{s\lambda_{2k}} + \sum_{k=1}^{\infty} a_{2k-1} e^{s\lambda_{2k-1}}$$

$$= f_e(s) + f_o(s)$$

where $f_e(s) = \sum_{k=1}^{\infty} a_{2k} e^{s\lambda_{2k}}$ and $f_o(s) = \sum_{k=1}^{\infty} a_{2k-1} e^{s\lambda_{2k-1}}$

i.e

$$f = f_e + f_o \quad \ldots \ldots \ldots \quad (4.5.1)$$

Let $M = \{f \in \Omega_u^2 : f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ with $a_{2k-1} = 0, k = 1, 2, 3, \ldots \ldots \}$

that is $M$ consists of only $f_e$'s and

$N = \{f \in \Omega_u^2 : f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ with $a_{2k} = 0, k = 1, 2, 3, \ldots \ldots \}$ that is $N$ consists of only $f_o$'s.

Clearly $M, N$ are closed sub-spaces of $\Omega_u^2$ such that $M^\perp = N$. Because if $f \in M$ and $g \in N$ then clearly $\langle f, g \rangle = \sum_{k=1}^{\infty} \left( \frac{a_k}{\lambda_k} \right) \left( \frac{\partial_k}{\partial_k} \right) = 0 \text{ by the definition of } M \text{ and } N$ and $M \cap N = \{0\}$ for if $f \in M \cap N$ where $f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ then $f \in M$ and $f \in N$ implies
\[ a_{2k-1} = 0 = a_{2k} \quad \text{for each } k = 1, 2, 3, \ldots \ldots \]

Thus \[ a_k = 0 \quad \forall k \implies f = 0 \]

Hence \[ \Omega_u^2 = M \oplus N = M \oplus M^\perp. \] Further if we define

\[ (p_k(f))(s) = f_k(s) = a_k e^{s\lambda_k}, \quad k = 1, 2, 3, \ldots \ldots \]

i.e. \[ p_k(f) = (f, \varphi_k)\varphi_k = a'_k \varphi_k \quad \text{where} \quad a'_k = \frac{a_k}{a_k} \quad \text{for each } k \]

Then for \( i \neq j \) \[ P_i P_j = 0 \quad \text{as} \quad P_i (P_j f) = P_i (a_j e^{s\lambda_j}) = 0 \quad \text{so that} \quad P_i \perp P_j. \]

Let for \( k = 1, 2, 3, \ldots \ldots \)

\[ M_k = \{ f_k : f_k(s) = a_k e^{s\lambda_k}, \quad a_k \in C \} \] then evidently \( M_i \cap M_j = \{0\}. \) Let us now define \( P : \Omega_u^2 \rightarrow \Omega_u^2 \) as follows:

for \( f \in \Omega_u^2, \quad P(f) = f_e \) where

\[ f_e = \sum_{k=1}^{\infty} a_{2k} e^{s\lambda_{2k}} = \sum_{k=1}^{\infty} P_{2k}(f) \]

\[ \Rightarrow \quad P(f) = \sum_{k=1}^{\infty} P_{2k}(f) \]

\[ \therefore (I - P)(f) = f_o \quad \text{where} \quad f_o(s) = \sum_{k=1}^{\infty} a_{2k-1} e^{s\lambda_{2k-1}} = \sum_{k=1}^{\infty} P_{2k-1}(f) \]

\[ \therefore P + (I - P) = I \equiv \sum_{k=1}^{\infty} P_{2k} + \sum_{k=1}^{\infty} P_{2k-1} \]

\[ I \equiv \sum_{k=1}^{\infty} P_k \quad \text{............ (4.5.2)} \]
Since r.h.s. of (4.5.2) is an infinite series so the validity of the above operator relation should be established. For that, we show that sequence $\{\sum_{k=1}^{n} P_k\}$ converges strongly to I. That is

$$\sum_{k=1}^{\infty} P_k \to I$$

We have by Fourier series theorem

$$If - \sum_{k=1}^{n} P_k (f) = \sum_{k=1}^{\infty} (f, \varphi_k) \varphi_k - \sum_{k=1}^{n} (f, \varphi_k) \varphi_k$$

$$= \sum_{k=n+1}^{\infty} (f, \varphi_k) \varphi_k \to 0 \text{ as } n \to \infty,$$

because the infinite series $\sum_{k=1}^{\infty} (f, \varphi_k) \varphi_k$ converges in $\Omega_u^2$.

Relation (4.5.2) is a *resolution of identity* on $\Omega_u^2$ and

$$T(f)(s) = \sum_{k=1}^{\infty} \gamma_k a_k e^{s\lambda_k} = \sum_{k=1}^{\infty} \gamma_k P_k (f)(s)$$

$$\Rightarrow T = \sum_{k=1}^{\infty} \gamma_k P_k = \gamma_1 P_1 + \gamma_2 P_2 + \ldots + \gamma_k P_k + \ldots \infty$$

and

$$\Omega_u^2 = M_1 \oplus M_2 \oplus M_3 \oplus \ldots \oplus M_k \oplus \ldots$$

$$= \oplus_{k \in N} M_k$$

Remark: Though the sequence $\{\sum_{k=1}^{\infty} P_k\}$ converges strongly to I but it does not converge to I under uniform operator convergence see [103, p-200], may be seen in the following lines;
Thus \( \| I - \sum_{k=1}^{n} P_k \| = \sup_{\| f \| = 1} \| (I - \sum_{k=1}^{n} P_k) f \| \geq 1 \) for every \( n \geq 1 \).

Showing that \( \{ \sum_{k=1}^{n} P_k \} \) does not converge uniformly under operator convergence [def. theorem 1.7].

Further we know that \( \sigma(T) = E \) i.e. closure of the set of \( \gamma_k^{is} \) and the \( \gamma_k^{is} \) form part of \( \sigma(T) \). The resolution of Identity corresponding to T is defined by

\[
Tf = g \text{ where } f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \text{ belongs to } \Omega^2_u. \]

We have

\[
g(s) = \sum_{k=1}^{\infty} a_k \gamma_k e^{s\lambda_k},
\]

and \( \lambda \) is any real number in \([m, M]\)

\[
\langle P_\lambda f, g \rangle = \sum_{\gamma_k \leq \lambda} a_k^i \eta_k \quad \text{where} \quad \eta_k = \frac{a_k \gamma_k}{\lambda}
\]

\[
= \sum_{\gamma_k \leq \lambda} \left( \frac{a_k \gamma_k}{\lambda} \right) = \sum_{\gamma_k \leq \lambda} \left| \frac{a_k}{\lambda} \right|^2 \gamma_k \quad \text{(as } \gamma_k \text{ are reals)}
\]

This means the matrix representing \( P_\lambda \) is a diagonal matrix with \( \gamma_i \) in the \( i^{th} \) diagonal position if \( \gamma_k \leq \lambda \) and \( \text{zero} \) otherwise.

4.6:- Applications

In this last section, we shall study some of the applications of \( \Omega^2_u \).
We know that $\varphi_k(s) = a_k e^{s\lambda_k}$, $k = 1, 2, 3, \ldots$ is an orthonormal basis for the Hilbert space $\Omega^2_\mu$. Replacing for convenience, the already defined bounded linear operator $T$ by $A$, we have then $A : \Omega^2_\mu \rightarrow \Omega^2_\mu$, given by

$$A(f)(s) = \sum_{k=1}^{\infty} \gamma_k a_k e^{s\lambda_k} = \sum_{k=1}^{\infty} \gamma_k a'_k \varphi_k(s)$$

Where $a'_k = \frac{a_k}{\gamma_k}$ and $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ belong to $\Omega^2_\mu$.

or

$$A(f) = \sum_{k=1}^{\infty} \gamma_k a'_k \varphi_k$$

We observe from (4.6.1) that

$$A(\varphi_k) = \gamma_k \varphi_k$$

for each $k = 1, 2, 3, \ldots \ldots$ and $\langle f, \varphi_k \rangle = a'_k$. That is basis $\{\varphi_k\}$ are eigen vector of eigen values $\{\gamma_k\}$ of the operator $A$. Therefore an unique solution to the operator equation

$$Ag = f$$

$f \in \Omega^2_\mu$ is possible in $\Omega^2_\mu$.

Let us assume that $\gamma_k \neq 0$ for any $k$. We have

$$f = \sum_{k=1}^{\infty} a'_k \varphi_k$$

Let $g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} = \sum_{k=1}^{\infty} b'_k \varphi_k(s)$

Then $Ag = A \sum_{k=1}^{\infty} b'_k \varphi_k = \sum_{k=1}^{\infty} b'_k A(\varphi_k) = \sum_{k=1}^{\infty} b'_k \gamma_k \varphi_k$
Now \( Ag = f \)

\[ \sum b'_k \gamma_k \varphi_k = \sum a'_k \varphi_k \]

\[ b'_k \gamma_k = a'_k \quad \text{for each} \quad k = 1, 2, 3, \ldots \]

\[ b'_k = \frac{a'_k}{\gamma_k} \quad \text{for each} \quad k = 1, 2, 3, \ldots \]

Consequently, an unique solution of the equation (4.6.2) has the form

\[ g = \sum_{k=1}^{\infty} \gamma_k^{-1} a'_k \varphi_k = A^{-1}(f) \]

(1) Let \( \{ \gamma_k \} \) be all real and positive, then \( T \) will be self adjoint, positive definite operator see [103, p-272], on the Hilbert space \( \Omega^2_u \) into itself with its domain \( D(T) = \Omega^2_u \). The operator equation

\[ Af = g \quad \text{............... (4.6.4)} \]

has a solution 'h' (for f) i.e. \( Ah = g \) in \( \Omega^2_u \)

where \( h(s) = \sum_{k=1}^{\infty} c_k e^{s\lambda_k} \) with \( c_k = \frac{b_k}{\gamma_k}, \quad k = 1, 2, 3, \ldots \quad \text{therefore, the functional equation given by,} \)

\[ F(w) = \langle Aw, w \rangle - 2\langle w, g \rangle = \sum_{k=1}^{\infty} \left( \gamma_k - 2 \left( \frac{b_k}{a_k} \right) \right) \left| \frac{c_k}{\alpha_k} \right|^2 \]

Attain its minimum at 'h' i.e.
\( F(h) < F(w) \) for all \( w \in \Omega_u^2 \) then 'h’ is the solution of equation (4.6.4) i.e. 
\( Ah = g \).

(2) Next, we have here a non-zero compact self adjoint operator \( A \) on the Hilbert space \( \Omega_u^2 \) with the representation

\[
A(f) = \sum_{k=1}^{\infty} \gamma_k \langle f, \varphi_k \rangle \varphi_k, \quad f \in \Omega_u^2
\]

for \( \lambda \neq 0 \) consider the operator equation

\[
Af - \lambda f = g
\]

\[
\text{............... (4.6.5)}
\]

If \( \lambda = \gamma_k \) for any \( k \), then for every \( g \in \Omega_u^2 \) there exist a unique solution \( f \in \Omega_u^2 \) of equation (4.6.5). The solution is given by,

\[
f = \sum_{k=1}^{\infty} \left( \frac{b_k / \alpha_k}{\alpha_k - \lambda_k} \right) \varphi_k
\]

with \( ||f|| < \alpha ||g|| \), \( \alpha \) being a constant independent of \( g \in \Omega_u^2 \).

Fix \( g \), where \( g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \in \Omega_u^2 \). If \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) is a solution of (4.6.5) then this implies

\[
f = \frac{1}{\lambda} (A(f) - g) = \frac{1}{\lambda} \left[ \sum_{k=1}^{\infty} (\gamma_k \langle f, \varphi_k \rangle \varphi_k) - g \right]
\]
so that \( \langle f, \varphi_k \rangle = \frac{1}{\lambda} (\gamma_k \langle f, \varphi_k \rangle - \langle g, \varphi_k \rangle) \)

or \((\gamma_k - \lambda)\langle f, \varphi_k \rangle = \langle g, \varphi_k \rangle\) for all \(k\).

Now assume that \(\lambda \neq \gamma_k\) for any \(k\). If \(f\) is a solution of equation (4.6.5) then

\[
\langle f, \varphi_k \rangle = \frac{(g, \varphi_k)}{(\gamma_k - \lambda)} \text{ for all } k \text{ and we must have then }
\]

\[
f = \frac{1}{\lambda} \left[ \sum_{k=1}^{\infty} \left( \frac{\gamma_k}{(\gamma_k - \lambda)} \langle g, \varphi_k \rangle \varphi_k \right) - g \right]
\]

\[
= \sum_{k=1}^{\infty} \frac{(b_k/\alpha_k)}{(\gamma_k - \lambda)} \varphi_k
\]

Thus (4.6.5) admits at most one solution for every \(g \in \Omega_u^2\). But since \(\gamma_k \to 0\) (as \(T\) is compact) whenever the sum on the r.h.s. is infinite, we see that there exist a constant \(\beta\) such that

\[
\left| \frac{\gamma_k}{\gamma_k - \lambda} \right| \leq \beta \text{ for all } k \text{ and we have by Bessel's inequality }
\]

\[
\sum_{k=1}^{\infty} \left| \left( \frac{\gamma_k}{\gamma_k - \lambda} \right) \langle g, \varphi_k \rangle \right|^2 \leq \beta^2 \|g\|^2
\]

hence the series \(\sum_{k=1}^{\infty} \left( \frac{\gamma_k}{\gamma_k - \lambda} \right) \langle g, \varphi_k \rangle \varphi_k\) converge by Riesz-Fischer theorem [111, p-238], and it can easily be verified that

\[
f = \frac{1}{\lambda} \left[ \left( \sum_{k=1}^{\infty} \left( \frac{\gamma_k}{\gamma_k - \lambda} \right) \langle g, \varphi_k \rangle \varphi_k \right) - g \right] = \sum_{k=1}^{\infty} \frac{(b_k/\alpha_k)}{(\gamma_k - \lambda)} \varphi_k
\]

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does satisfy the equation (4.6.5). Moreover, we also have

\[ \|f\| = \frac{1}{|\lambda|} \left( \left\| \sum_{k=1}^{\infty} \left( \frac{\gamma_k}{\gamma_k - \lambda} \right) \langle g, \varphi_k \rangle \varphi_k \right\| + \|g\| \right) \]

\[ \leq \frac{1}{\lambda} (\beta \|g\| + \|g\|) \]

\[ = \frac{(\beta + 1)}{|\lambda|} \|g\| = \alpha \|g\| \text{ where } \alpha = \frac{(\beta + 1)}{|\lambda|} \text{ etc.} \]