CHAPTER 3

ON A HILBERT SPACES OF DIRICHLET SERIES
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ON A HILBERT SPACE OF DIRICHLET SERIES

3.1:- Introduction

In this chapter, we study a Hilbert space $\Omega_u^2$ of functions, analytic in a half plane which are represented by Dirichlet series $\sum_{k=1}^{\infty} a_k e^{s \lambda_k}$. In the next section 3.2, we find an orthonormal basis of the space $\Omega_u^2$ and its projection decomposition form. In section 3.3, it is established that $\Omega_u^2$ is isometrically isomorphic to the energy space $l_2 = l_2(N)$. In section 3.4, matrix transformation has been studied on $\Omega_u^2$ and a characterization of compact operators has been obtained. Section 3.5 deals with multipliers on $\Omega_u^2$.

Here we have consider the set $\Omega_u^2$ which consist of all $f$ such that $f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k}$ is a Dirichlet series [7] where $\{a_k\}$ is a complex sequence and $s = \sigma + it$ ($\sigma, t$ real variables) a complex variable and $\lambda_k^*$ is a fixed sequence positive real number such that, $0 < \lambda_k < \lambda_{k+1} \to \infty$ as $k \to \infty$, satisfying $\sum_{k=1}^{\infty} \left| \frac{a_k}{\lambda_k} \right|^2 < \infty$,

where $u, u(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k}, a_k \neq 0$ for any $k = 1, 2, 3, \ldots \ldots$ is the fixed Dirichlet series as described in chapter - 2 section 2.2 satisfying,
\[
\lim_{k \to \infty} \frac{\log k}{\lambda_k} = D = 0 \quad \ldots \ldots \ldots (3.1.1)
\]

and

\[
- \lim_{k \to \infty} \sup \frac{\log |\alpha_k|}{\lambda_k} = \alpha \quad \ldots \ldots \ldots (3.1.2)
\]
as a Hilbert space [29] or [see Def. (1.14)]. We study its various properties and applications in section (3.1) and (3.2). In view of conditions (3.1.1) and (3.1.2), it is clear that \( u(s) \), the fixed Dirichlet series has its abscissa of absolute convergence \( \sigma_a = \sigma_c = \alpha \) (see [7]) and the function \( u \) has its region of convergence, the half-plane \( R_u \) given by \( \sigma < \alpha \), and in its region of convergence, it represents an analytic function as explained earlier in Section [2.2]. Note that \( u \notin \Omega_u^2 \) as \( \sum_{k=1}^{\infty} \left| \frac{a_k}{\alpha_k} \right|^2 = \sum 1 = \infty \). In the following we study the space \( \Omega_u^2 \) as a Hilbert space along with some of its important properties. Some other Hilbert spaces on Dirichlet series may be seen in [47, 105, 108, 121, 128, 130] etc.

**Theorem - 3.1.1:** \( \Omega_u^2 \) is a Hilbert space.

**Proof:** Let \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) and \( g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \) belong to \( \Omega_u^2 \), then we can see that \( \Omega_u^2 \) is a linear space w.r.t. point-wise linear operations and it contains all the function \( f \) whose abscissa of absolute convergence \( \sigma_a(f) \geq \alpha \) if \( \alpha < \infty \). If \( \alpha = \infty \) then it contain entire functions only
representable by a Dirichlet series but not all entire function as 'u' itself in this case is an entire but does not belong to $\Omega_u^2$.

Further as $f, g \in \Omega_u^2$, we have $\sum_{k=1}^{\infty} \left| \frac{a_k}{a_k} \right|^2 < \infty, \sum_{k=1}^{\infty} \left| \frac{b_k}{b_k} \right|^2 < \infty$. Define in $\Omega_u^2$, a complex valued function $<, >$ on the pair of elements $f, g$ of $\Omega_u^2$ as follows,

$$<f, g> = \sum_{k=1}^{\infty} \left( \frac{a_k}{a_k} \cdot \frac{b_k}{b_k} \right)$$

(3.1.3)

Since $2 \left| \frac{a_k}{a_k} \right| \left| \frac{b_k}{a_k} \right| \leq \left| \frac{a_k}{a_k} \right|^2 + \left| \frac{b_k}{b_k} \right|^2$ for each $k = 1, 2, 3, \ldots$

Therefore $\sum_{k=1}^{\infty} \left( \frac{a_k}{a_k} \cdot \frac{b_k}{b_k} \right) \leq \frac{1}{2} \left( \sum_{k=1}^{\infty} \left| \frac{a_k}{a_k} \right|^2 + \sum_{k=1}^{\infty} \left| \frac{b_k}{b_k} \right|^2 \right) < \infty$ so that series $\sum_{k=1}^{\infty} \left( \frac{a_k}{a_k} \cdot \frac{b_k}{b_k} \right)$ convergence absolutely and $<, >$ is well defined.

Further it can be easily be seen that (3.1.3) defines an inner product on $\Omega_u^2$ and the norm $\| \cdot \|$ on $\Omega_u^2$, generated by the inner product, is given by

$$\|f^2\| = <f, f> = \sum_{k=1}^{\infty} \left| \frac{a_k}{a_k} \right|^2 < \infty$$

(3.1.4)

Next we show that $\Omega_u^2$ is complete as a normed linear space. For that let $\{f_p\}$ be Cauchy sequence in $\Omega_u^2$ such that

$$f_p(s) = \sum_{k=1}^{\infty} a_{pk} e^{s\lambda_k}, p = 1, 2, 3, \ldots.$$
Then \[ \sum_{k=1}^{\infty} \left| \frac{a_{pk}}{a_k} \right|^2 < \infty \] for each \( p = 1, 2, 3, \ldots \).

Now for any given \( \varepsilon > 0 \) there must exist a positive integer \( k_0 = k_0(\varepsilon) \) such that

\[ \|f_p - f_q\| < \varepsilon \text{ for } p, q > k_0 \]

\[ \Rightarrow \sum_{k=1}^{\infty} \left| \frac{a_{pk} - a_{qk}}{a_k} \right|^2 < \varepsilon^2 \text{ for } p, q \geq k_0 \] \hspace{1cm} \text{ ................... (3.1.5)}

\[ \Rightarrow \left| \frac{a_{pk} - a_{qk}}{a_k} \right|^2 < \varepsilon^2 \text{ for } p, q \geq k_0 \text{ and for each } k \]

\[ \Rightarrow \left| \frac{a_{pk} - a_{qk}}{a_k} \right| < \varepsilon \text{ for each } k \text{ and } p, q \geq k_0 \]

This shows that for each \( k, \left\{ \frac{a_{pk}}{a_k} \right\} \) is a Cauchy sequence of the complex number therefore it must be convergent. Let

\[ \left\{ \frac{a_{pk}}{a_k} \right\} \to \frac{a_{ok}}{a_k} \text{ as } p \to \infty. \]

Let \( f \) be defined by \( f(s) = \sum_{k=1}^{\infty} a_{ok} e^{s\lambda k} \)

Then we show that \( f \in \Omega_u^2 \) and \( f_p \to f \text{ as } p \to \infty. \)

Now, from (3.1.5) we have for all \( p, q > k_0 \) (a fixed integer)

\[ \sum_{k=1}^{n} \left| \frac{a_{pk} - a_{qk}}{a_k} \right|^2 < \varepsilon^2, \text{ } p, q \geq k_0 \]

Letting \( q \to \infty \), we get for \( p \geq k_0 \)
\[ \sum_{k=1}^{n} \left| \left( \frac{a_{pk} - a_{ok}}{\alpha_k} \right) \right|^2 < \varepsilon^2 \]

We may now let \( n \to \infty \) then for all \( p \geq k_0 \), we get

\[ \sum_{k=1}^{\infty} \left| \left( \frac{a_{pk} - a_{ok}}{\alpha_k} \right) \right|^2 < \varepsilon^2 \] ........................... (3.1.6)

This shows that \((f_p - f) \in \Omega_u^2\). Since \( f_p \in \Omega_u^2 \), it follows therefore that

\[ f = (f - f_p) + f_p \in \Omega_u^2 \]

Further more the series in (3.1.6) represents \( \|f_p - f\|^2 \), hence (3.1.6) also implies that \( f_p \to f \) as \( p \to \infty \). Since \( \{f_p\} \) was an arbitrary Cauchy sequence in \( \Omega_u^2 \) which is convergent, this proves the completeness of \( \Omega_u^2 \). Hence \( \Omega_u^2 \) is a Hilbert space [def 1.14].

3.2:- Orthonormal Basis in \( \Omega_u^2 \)

In this section, we find an orthonormal basis in \( \Omega_u^2 \) and its projection decomposition form.

3.2(A):- let \( \varphi_k(s) = \alpha_k e^{s\lambda_k} \), \( k = 1, 2, 3, \ldots \) then observe that \( \varphi_k \in \Omega_u^2 \) for each \( k \) and by definition (3.1.3) of inner production in \( \Omega_u^2 \), we get,

(i) \( \langle \varphi_k, \varphi_j \rangle = 0 \) if \( k \neq j \)

(ii) \( \langle \varphi_k, \varphi_k \rangle = \|\varphi_k\|^2 = 1 \) for each \( k \) and for any \( f \in \Omega_u^2 \)
(iii) \(< f, \varphi_k > = 0 \Rightarrow \frac{a_k}{a_k} = 0 \) for each \( k = 1, 2, 3, \ldots \ldots \) which in turn further implies that \( a_k = 0 \) for each \( k \). Therefore \( f = 0 \) where \( O(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) such that \( a_k = 0 \) for all \( k \), is the zero element of the space \( \Omega^2_u \).

Therefore \{\varphi_k\} is an orthonormal basis [123], a special case of a schauder basis [see Def. (1.5)] for the Hilbert space \( \Omega^2_u \) but not a Hamel basis [see def. 1.5] for \( \Omega^2_u \). For example \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) of \( \Omega^2_u \) cannot be written as a finite linear combination of \( \varphi_k^s \) but an infinite combination as \( \sum_{k=1}^{\infty} a_k \varphi_k \), explained below,

Any \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \) in \( \Omega^2_u \) can be written as

\[
f(s) = \sum_{k=1}^{\infty} \left( \frac{a_k}{a_k} \right) a_k e^{s\lambda_k}
\]

\[
= \sum_{k=1}^{\infty} a_k \varphi_k(s) \quad \text{where} \quad \frac{a_k}{a_k} = a_k' = < f, \varphi_k >
\]

\[
\Rightarrow \quad f = \sum_{k=1}^{\infty} a_k' \varphi_k = \sum_{k=1}^{\infty} < f, \varphi_k > \varphi_k \quad \text{.................. (3.2.1)}
\]

Now, we study uniform convergence in the Hilbert space \( \Omega^2_u \) in the following theorem.

**Theorem - 3.2.1:** Let \( f_p \rightarrow f \) in \( \Omega^2_u \) then \( f_p(s) = f(s) \) uniformly where \( s \in R_u \), the region of convergence of \( u \).

**Proof:** \( f_p \rightarrow f \) in \( \Omega^2_u \) means that \( \| f_p - f \| < \varepsilon \) for \( p \geq p_0 (\varepsilon) \).
or \[ \| f_p - f \| ^2 < \varepsilon^2 \text{ for } p \geq p_0(\varepsilon) \]

\[ \sum_{k=1}^{\infty} \left| \frac{a_{p_k} - a_{o_k}}{a_k} \right|^2 < \varepsilon^2 \text{ for each } k = 1, 2, 3, \ldots \ldots \]

\[ \Rightarrow \left| \frac{a_{p_k} - a_{o_k}}{a_k} \right|^2 < \varepsilon^2 \text{ for each } k = 1, 2, 3, \ldots \ldots \]

\[ \Rightarrow |a_{p_k} - a_{o_k}| < \varepsilon \left| a_k \right| \]

Now, \[ |f_p(s) - f(s)| \leq \sum_{k=1}^{\infty} |a_{p_k} - a_{o_k}| e^{\sigma \lambda_k} \]

or \[ |f_p(s) - f(s)| \leq \varepsilon \sum_{k=1}^{\infty} |a_k| e^{\sigma \lambda_k} \to 0 \text{ as } p \to \infty \]

### 3.2 (B):
\[ \Omega^2_u \text{ as a Direct Sum} \]

For a fix \( k \) let us define

\[ K = \text{span} \{ \varphi_k \} = \{ \lambda \alpha_k e^{s \lambda_k} : \lambda \in \mathcal{C} \} \] and

\[ M = \left\{ f \in \Omega^2_u, f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k} : a_k = 0 \right\} \]

Then both \( K \) and \( M \) are seen to be closed subspaces of \( \Omega^2_u \) such that \( M = K^\perp \).

Further as each \( f, f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k} \) can be represented uniquely as

\[ f(s) = a_n e^{s \lambda_n} + \sum_{k \neq n}^{\infty} a_k e^{s \lambda_k} \]

\[ = \begin{pmatrix} a_n \\ \alpha_n \end{pmatrix} e^{s \lambda_n} + \sum_{k \neq n}^{\infty} a_k e^{s \lambda_k} \]

Therefore \( \Omega^2_u = K \oplus M = K \oplus K^\perp \). Particularly \( \Omega^2_u = K + M \)
But, there exist a subspace $U$ of $\Omega_u^2$ such that $\Omega_u^2 \neq U + U^\perp$ and therefore $\Omega_u^2 \neq U \oplus U^\perp$. For this we consider a subset $U$ of $\Omega_u^2$ as follows,

$$U = \left\{ f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} : a_k = 0 \text{ for all but a finite number of } k \right\}$$

Then $U$ is a subspace of $\Omega_u^2$ with inner product inherited from $\Omega_u^2$, $U$ is an inner product space. Consider the sequence $\{f_n\} \subset U$ where

$$f_n(s) = \sum_{k=1}^{n} \frac{\alpha_k}{2^{k-1}} e^{s\lambda_k} = \sum_{k=1}^{n} \frac{1}{2^{k-1}} \varphi_k(s) \text{ for each } n = 1, 2, 3, \ldots$$

Now we see that it is a Cauchy sequence, since for $n > m$, we have,

$$\|f_n - f_m\|^2 = \sum_{k=m+1}^{\infty} \frac{1}{2^{2k}} < \sum_{k=m+1}^{\infty} \frac{1}{2^{k}} \to 0 \text{ as } m \to \infty.$$ 

Since $\{f_n\}$ is a Cauchy sequence, it must converge to some element $f \in \Omega_u^2$

since

$$< f, \varphi_k > = < \lim_{n \to \infty} f_n, \varphi_k >$$

$$= \lim_{n \to \infty} < f_n, \varphi_k >$$

$$= \frac{1}{2^{k-1}}.$$ 

Therefore

$$f = \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \varphi_k$$

and

$$f(s) = \sum_{k=1}^{n} \frac{\alpha_k}{2^{k-1}} e^{s\lambda_k}$$

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Thus $f_n \to f$ where $f(s) = \sum_{k=1}^{\infty} \frac{\alpha_k}{2^{k-1}} e^{s\lambda_k}$, clearly $f \in \Omega_u^2$, but $f \notin U$, therefore $U$ is not a closed subspace of $\Omega_u^2$. Next we claim that $U^\perp = \{0\}$. For that let $g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k}$ belong to $U^\perp$. Then for each $k = 1, 2, 3, \ldots \ldots$

$$0 = \langle g, \varphi_k \rangle = \frac{b_k}{\alpha_k} \implies b_k = 0 \implies g = 0$$

Thus $U^\perp = \{0\}$ and hence $\Omega_u^2 \neq U + U^\perp$ implying $\Omega_u^2 \neq U \oplus U^\perp$. Also as $U$ is a subspace of $\Omega_u^2$ such that $U^\perp = \{0\}$, therefore $U$ is dense in $\Omega_u^2$ follows (coro 6.87) from a result in [126].

3.3:- **Isometric Image of $\Omega_u^2$**

In this section, it has been also established that $\Omega_u^2$ is isometrically isomorphic to the energy space $l_2 = l_2(N)$.

**Theorem - 3.3.1:** $\Omega_u^2$ is isometrically isomorphic to the space $l_2$ hence separable.

**Proof:** As $\Omega_u^2$ has a countable orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ and every Hilbert space with countable orthonormal basis is always isometrically isomorphic to the space $l_2$ [113]. Therefore $\Omega_u^2$ is isometrically isomorphic to $l_2$ under the isomorphism $f \to \{a_k^* e_k\}$ where $e_k = \{\delta_n^k\}_{n=1}^{\infty} \in l_2$ is Kronecker Delta, by the use of Parseval’s identity [92]. Since $l_2$ is a separable, so is $\Omega_u^2$. 

[71]
3.4:- Matrix Transformation on $\Omega_u^2$

In this section, a matrix transformation on the space $\Omega_u^2$ has been considered and characterization of compact operators on it has been obtained. Let $\{\xi_{nk}\}$ be an infinite matrix of complex entries and $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ belong to $\Omega_u^2$. Define a map

$$A : \Omega_u^2 \rightarrow \Omega_u^2$$

as follows;

$$(Af)(s) = \sum_{k=1}^{\infty} (Af)_n e^{s\lambda_n}$$

where $(Af)_n = \sum_{k=1}^{\infty} \xi_{nk} a_k$ exist for each $n = 1, 2, 3, \ldots$.

We observe that,

$$(A\phi_k)(s) = \sum_{n=1}^{\infty} (A\phi_k)_n e^{s\lambda_n} = \sum_{n=1}^{\infty} \xi_{nk} a_k e^{s\lambda_n}$$

as $$(A\phi_k)_n = \xi_{nk} a_k ; \ n = 1, 2, 3, \ldots$$

$$\therefore \quad \langle A\phi_k, \phi_n \rangle = \frac{\xi_{nk} a_k}{a_n} \frac{\overline{a_n}}{\overline{a_n}} = \frac{\xi_{nk} a_k}{a_n}$$

**Theorem - 3.4.1:-** $A : \Omega_u^2 \rightarrow \Omega_u^2$ defines a bounded linear compact operator if

$$(1) \quad \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\xi_{nk} a_k}{a_n} \right|^2 = M^2 < \infty$$

**Proof:-** We first show that $A$ is bounded linear operator on $\Omega_u^2$ provided condition (1) holds. For that, we have:
\[
\sum_{n=1}^{\infty} \left| (Af)_n \right|^2 = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} \frac{\xi_{nk}a_k}{\alpha_n} \right|^2 \\
\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}a_k}{\alpha_n} \right|^2 \\
\leq \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}a_k}{\alpha_n} \right|^2 \cdot \sum_{k=1}^{\infty} \left| \frac{a_k}{\alpha_k} \right|^2 \right) \\
\leq \|f\|^2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}a_k}{\alpha_n} \right|^2 < \infty \{\text{by (1)}\}
\]

So that \(Af \in \Omega_u^2\) whenever \(f \in \Omega_u^2\). Further from above, we have

\[
\|Af\|^2 = \sum_{n=1}^{\infty} \left| \frac{(Af)_n}{\alpha_n} \right|^2 \leq M^2 \|f\|^2,
\]

showing that \(A\) is bounded and

\[
\|A\|^2 \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}a_k}{\alpha_n} \right|^2 = M^2
\]

\(A\) is linear, this much is trivial. Next we show that \(A\) is compact. For that let us define a sequence of map \(\{A_m\}\) on \(\Omega_u^2\) as follows;

\[
(A_mf)(s) = \sum_{n=1}^{m} (Af)_n e^{s\lambda_n} \quad \text{for} \quad m = 1, 2, 3, \ldots \ldots
\]

Then each \(A_m\) is linear and bounded and \(\text{dim}[A_m(\Omega_u^2)]\) is clearly finite. Therefore \(A_m\) is compact for each \(m\) see [def. (1.16)]. Further
\[ \|(A - A_m)f\|^2 = \sum_{n=m+1}^{\infty} \left| \frac{(Af)_n}{\alpha_n} \right|^2 = \sum_{n=m+1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}a_k}{\alpha_n} \right|^2 \]

\[ \leq \|f\|^2 \sum_{n=m+1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}a_k}{\alpha_n} \right|^2 \]

Taking the supremum over all \( f \) of norm 1, we see that:

\[ \|A - A_m\|^2 \leq \sum_{n=m+1}^{\infty} \left( \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}a_k}{\alpha_n} \right|^2 \right) \rightarrow 0 \]

by the given condition (1) and therefore \( A \) is compact being the uniform operator convergent limit of compact operators see \([def.- (theorem 1.7)]\) or \([109]\).

Further this condition is sufficient for compactness but not necessary, because if we choose particularly \( \xi_{nk} = \frac{1}{\sqrt{n}} \) if \( k = n \) and '0' (zero) if \( k \neq n \) then

\( (Af)(s) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} e^{s\lambda_n} \) defines a compact operator on \( \Omega^2 \) but

\[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}a_k}{\alpha_n} \right|^2 = \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \infty. \]

Thus we see that the matrix \( \{\xi_{nk}\} \) defines a bounded linear operator \( A \) on the Hilbert space \( \Omega^2 \) provided \( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\xi_{nk}a_k}{\alpha_n} \right|^2 < \infty. \)

Note that

\[ \langle A\varphi, \varphi_n \rangle = \frac{\xi_{nk}a_k}{\alpha_n} \] hold for all \( n, k = 1, 2, 3, \ldots \ldots \)
Further if the bounded linear operator $A$ is defined by the matrix $\{\xi_{nk}\}$ on $\Omega_u^2$, then its adjoint $A^*$ which is also a bounded linear operator on $\Omega_u^2$ is given by;

$$\langle A^* \phi_k, \phi_n \rangle = \langle \phi_k, A \phi_n \rangle = \langle A \phi_n, \phi_k \rangle = \left( \frac{\xi_{kn} \alpha_n}{\alpha_k} \right)$$

Thus $A$ will be a self-adjoint operator i.e. $(A = A^*)$ iff

$$\xi_{nk} = \left( \frac{\xi_{kn} \alpha_n}{\alpha_k} \right) \text{ for each } n, k = 1, 2, 3, \ldots \ldots \ldots$$

or

$$\xi_{nk} \overline{\alpha_k} = \overline{\xi_{kn} \alpha_n} \text{ for each } n, k = 1, 2, 3, \ldots \ldots \ldots$$

$$\Rightarrow \quad \frac{\xi_{nk}}{\xi_{kn}} = \frac{\overline{\alpha_n}}{\alpha_k} \text{ for each } n, k = 1, 2, 3, \ldots \ldots \ldots$$

Many important properties and application of this space has been studied in the next chapter also see [129].

### 3.5:- Multiplier on the space $\Omega_u^2$

**Multipliers** - An element $f, f(s) = \sum_{k=1}^{\infty} a_k e^{s \lambda_k}$ from a class of function $f$ represented by a Dirichlet series will be called an multiplier for the Hilbert space $\Omega_u^2$ if $(f, g) \in \Omega_u^2$ for all $g, g(s) = \sum_{k=1}^{\infty} b_k e^{s \lambda_k}$ belong to $\Omega_u^2$

where

$$(f, g)(s) = \sum_{k=1}^{\infty} (a_k b_k) e^{s \lambda_k} \quad \ldots \ldots \ldots \quad (3.5.1)$$

We characterize multiplier class of $\Omega_u^2$ in the following theorem.
Note that the space $\Omega^2_u$ is particularly denoted as $\Omega$ (see Remark 2.3.3) here if $\{\alpha_k\}$ is taken as $\{1\}$ i.e. when the fixed Dirichlet series, as described in 2.2 chapter 2 becomes like $u(s) = \sum_{k=1}^{\infty} e^{s\lambda_k}$.

**Theorem - 3.5.1:-** The space $\Omega$ is the multipliers class for $\Omega^2_u$.

**Proof:** Let $f \in \Omega$ and $g \in \Omega^2_u$

where $f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k}$ and $g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k}$. Then $|a_k| < M_1$ ($M_1 > 0$)

for each $k$ and $\sum_{k=1}^{\infty} \frac{|b_k|}{|a_k|} < \infty$. Point wise product as defined above in (3.5.1)

$$(f \cdot g)(s) = \sum_{k=1}^{\infty} a_k b_k e^{s\lambda_k}$$

is meaningful and $(f \cdot g) \in \Omega^2_u$ because

$$\sum_{k=1}^{\infty} \left| \frac{a_k b_k}{a_k} \right|^2 = \sum |a_k|^2 \left| \frac{b_k}{a_k} \right|^2 \leq M_1^2 \sum \left| \frac{b_k}{a_k} \right|^2 < \infty \quad \text{...........................}$$

(3.5.2)

Conversely also if for any $f$ where

$$f(s) = \sum_{k=1}^{\infty} a_k e^{s\lambda_k} \quad \text{and} \quad g, g(s) = \sum_{k=1}^{\infty} b_k e^{s\lambda_k} \text{ belongs to } \Omega^2_u$$

such that $(f \cdot g) \in \Omega^2_u$ then we must have

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\[ \sum_{k=1}^{\infty} \left| \frac{a_k b_k}{\alpha_k} \right|^2 < \infty \]

or

\[ \sum |a_k|^2 \left| \frac{b_k}{\alpha_k} \right|^2 < \infty \] \hspace{1cm} \text{(3.5.3)}

We claim that \( f \in \Omega_u \), in order that (3.5.3) holds.

Suppose contrary that \( \{a_k\} \) is not bounded then we can find a subsequence \( \{k_j\} \) of \( \{k\} \) such that \( |a_{k_p}| > p^2 \)

Therefore

\[ \sum |a_{k_p}|^2 \left| \frac{b_{k_p}}{\alpha_{k_p}} \right|^2 > p^2 \sum \left| \frac{b_{k_p}}{\alpha_{k_p}} \right|^2 \rightarrow \infty \]

which is a contradiction to (3.5.3).

Hence the class \( \Omega \) is the multipliers class from \( \Omega_u^2 \) to itself i.e. \((\Omega_u^2, \Omega_u^2) = \Omega\).