CHAPTER 7

A GRIFFITH CRACK OPENED BY WEDGE IN BONE (RIB)

ABSTRACT:

The closed form expressions of the stress and the displacement fields in the vicinity of crack tips are obtained by using Fourier transform method. Some special cases are discussed.

1. INTRODUCTION

The bone can be treated as orthotropic material. The dimensions of the crack are so small in comparison to rib that we can consider the rib as the infinite orthotropic medium. Thus the problem of our concern is of crack opening defined as

\[ u_y (x, 0) = \begin{cases} u_o(x), & 0 \leq |x| \leq b \\ 0, & |x| \geq c \end{cases} \]  

(1.1)

\[ \sigma_{yy} (x, 0) = -p(x), \quad b < |x| < c \]  

(1.2)

\[ \sigma_{xy} (x, 0) = 0, \quad 0 \leq |x| < \infty \]  

(1.3)

where \((u_x, u_y)\) and \((\sigma_{xy}, \sigma_{yy})\) etc are components of displacement and of stress tensor at \((x, y)\). We assumed the medium in plane strain conditions. The physical quantities vanish as \(\sqrt{x^2 + y^2} \to \infty\). We also assume that the axes of coordinate system coincides with the material symmetry. The plan of the chapter is as follows:
Figure 7: Geometry of problem
section 2 will formulate the problem and reduce it to triple integral equations. The solution of triple integral equations, along with physical quantities will be given in section 3. Some special cases will be presented in section 4.

2. FORMULATION AND REDUCTION TO TRIPLE INTEGRAL EQUATIONS

The solution of equations of equilibrium,

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \] (2.1)

in the absence of body forces, are solved by the method of Kushwaha [79], and solutions, through stress-strain relations, are given as

\[ u_y(x,y) = \frac{2}{\pi} \int_0^\infty \cos(\xi x). \xi^{-2} \left[ a_{11} \frac{\partial^3 H}{\partial y^3} - \xi^2 (a_{12} + a_{66}) H \right] d\xi \] (2.2)

\[ u_x(x,y) = \frac{2}{\pi} \int_0^\infty \xi^{-1} \sin(\xi x). \xi^{-2} \left[ a_{11} \frac{\partial^2 H}{\partial y^2} - \xi^2 a_{12} H \right] d\xi \] (2.3)

where \( H(\xi, y) = \left[ (\gamma_1 - \gamma_2) A + B \right] e^{-\gamma_1 \xi} + Be^{-\gamma_2 \xi} \) (2.4)

with \( \gamma_1 \) and \( \gamma_2 \) as roots of

\[ \gamma^4 - 2B_1 \gamma^2 + B_2 = 0 \] (2.5)

and

\[ 2B_1 = \frac{(2a_{12} + a_{66})}{a_{11}}, \quad B_2 = \frac{a_{22}}{a_{11}} \] (2.6)
The boundary condition (1.3) is satisfied identically. The
conditions (1.1)-(1.2) yield the following triple integral
equations:

\[ \int_{0}^{\infty} \cos(\xi x) A(\xi) d\xi = \begin{cases} \beta_1, & 0 \leq x < b \\ 0, & x \geq c \end{cases} \]  
(2.7)

\[ \int_{0}^{\infty} A(\xi x) \cos(\xi x) d\xi = \frac{\pi y_1}{2} p(x), \quad b < x < c \]  
(2.8)

\[ \beta_1 = \gamma_1 (\gamma_1 + \gamma_2) a_{11} \]  
(2.9)

3. SOLUTION OF TRIPLE INTEGRAL EQUATIONS

We assume the trial solution

\[ A(\xi) = \frac{2}{\xi} \left[ - \int_{0}^{b} \beta_1 u_0(t) \sin \xi t dt + \int_{b}^{c} g(t) \sin \xi t dt \right] \]  
(3.1)

Then the substitution of (3.1) into (2.7) yields

\[ \int_{b}^{c} g(t) dt = \beta_1 u_0(b) \]  
(3.2)

and (3.1) will be satisfied there. The equations (2.8) and (3.1)
give, after using the method of Srivastava and Löwengrub (62)

\[ g(t) = \pi^2 \frac{\Delta(t)}{\delta(t)} \]  
(3.3)

\[ \Delta(t) = \left[ \int_{b}^{c} \frac{f(x) \delta(x) \cdot x \cdot dx}{x^2 - t^2} + L \right] \]  
(3.4)
where $L$ is constant to be determined through (3.3) and (3.2).

\[
\delta(t) = \left\{ \frac{1}{(t^2 - b^2)} \frac{1}{(c^2 - t^2)} \right\}^{1/2} \tag{3.5}
\]

\[
f(x) = \frac{\lambda}{2} p(x) - \frac{2}{\pi} \int_0^b u'_0(t) \frac{t \, dt}{t^2 - x^2} \tag{3.6}
\]

**PHYSICAL QUANTITIES**

The quantities of interest in fracture mechanics are crack opening displacement and normal stress component in the vicinity of crack tips.

**Crack Shape**

The value of the integral (2.7) for $b < x < c$ will give $u_y(x,0)$. Hence

\[
u_y(x,0) = \int_x^C g(t) \, dt \tag{3.7}
\]

The value of $\sigma_{yy}(x,0)$ is evaluated from the value of integral (2.8) for $0 \leq x \leq b$ and $x \geq C$. The value of stress for $0 \leq x \leq b$ is called the stress underneath the stamp.

\[
\sigma_{yy}(X,0) = \pm \frac{\Delta(x)}{\pi \delta(x)} \tag{3.8}
\]

where (+) sign is to be taken for $0 \leq x \leq b$ and (-) sign for $x \geq C$. $\Delta(x)$ is defined in (3.4).
Stress Intensity Factor

It is defined as

\[ K_b = \lim_{x \to b} \sqrt{b-x} \sigma_{yy}(x,0) \] (3.9)

\[ K_c = \lim_{x \to C} \sqrt{x-C} \sigma_{yy}(x,0) \] (3.10)

Using (3.9) - (3.1) in to (3.8) we get

\[ K_b = \frac{\Delta(b)}{\pi \sqrt{b(b^2 - b^3)}} , \quad K_c = \frac{\Delta(b)}{\pi \sqrt{2c(c^2 - b^2)}} \] (3.11)

4. SPECIAL CASE

To make our analysis clear we consider the case

\[ p(x) = p_0 = \text{constant} \] (4.1)

\[ u_o(x) = u_0 = \text{constant} \] (4.2)

Therefore the condition (3.2) is changed to

\[ \int_b^C g(t) \, dt = u_0 \] (4.3)

and

\[ f(x) = \frac{\pi}{2} p_0 \] (4.4)
Then substitution of (4.4) into (3.4) and evaluation of integrals give

\[ \Delta(t) = \frac{\lambda^2 \rho_0}{\xi} \left[ 2t^2 - \left(b^2 + c^2\right) + L \right], \quad b \leq t \leq c \]  

(4.5)

Thus using (4.5) into (3.11) we get

\[ K_b = \frac{\rho_0}{\pi \sqrt{2} b (c^2 - b^2)} \left[ b^2 - c^2 + L \right] \]  

(4.6)

\[ K_c = \frac{\rho_0}{\pi \sqrt{2} c (c^2 - b^2)} \left[ c^2 - b^2 + L \right] \]  

(4.7)

with

\[ L = \pi^2 \beta_1 \frac{C_{0_1}^0(b)}{F} = \pi \rho_o \left\{ \frac{2c^2 E}{F} - \left(b^2 + c^2\right) \right\} \]  

(4.8)

where \( F \) and \( E \) complete elliptic integrals defined as

\[ F = F\left(\frac{\pi}{2}, k\right), \quad E = E\left(\frac{\pi}{2}, k\right), = k^2 \frac{c^2 - b^2}{c^2} \]

The expressions for stress intensity factors (4.6) - (4.7) show that it is not depending upon the properties of medium. The crack opening displacement is constant multiple of that of isotropic case while this constant depends upon the properties of the medium.