CHAPTER 5

TWO GRIFFITH CRACKS OPENED BY HEATED WEDGE
IN AN ORTHOTROPIC INFINITE MEDIUM

ABSTRACT:

Closed form expressions for stress intensity factors and the crack shape are obtained for two Griffith cracks opened by heated wedge in an infinite homogeneous orthotropic medium by using Fourier transforms. A special case of wedge shape with constant temperature is also reported.

1. INTRODUCTION

As Sharma has proved that if the number of layers in composite materials are more than 6, then the medium of the composite material can be treated as orthotropic medium. Therefore, the present research endeavour is of great importance in the analysis of response of composite materials.

There are numerous crack problems but only a few with temperature distribution. The thermal stresses problem set up in an isotropic body containing cracks of different configurations has attracted the attention of many mathematician, physicist and engineers. Some problems of penny-shaped cracks are done by authors of [27, 28, 31, 52, 103, 126].

Kushwaha and Gupta [9-11] has solved temperature distribution problem in an infinite isotropic medium with Griffith cracks. Kushwaha and Chandra [12-16] have solved the problems of Griffith cracks opened by heated wedge in isotropic finite and
infinite medium.

The title problem is of crack opening \( y = 0, \ b < |x| < c \), due to heated wedge of prescribed shape. The problem is to find out the stress and the displacement fields in the vicinity of crack tips. The problem at a point \((x, y)\) in the medium is divided into two parts, namely,

(A) Temperature Problem : In this we solve the temperature distribution with mixed boundary conditions and find the stress developed by temperatures.

(B) Elasticity Problem : Here we discuss the crack opening due to thermal stresses developed by temperature variation. To solve elasticity problem we follow the method of Kushwaha [74]. We assumed that the medium is under plain-strain conditions. The axes of material symmetry coincides with the co-ordinate axes. The component of physical quantities vanish as \( \sqrt{x^2+y^2} \to \infty \).

The physical problem is reduced to the following boundary value problem :

(a) \( T(x, 0) = \begin{cases} T(x), & 0 \leq |x| \leq b \\ 0, & |x| \geq c \end{cases} \) \hspace{1cm} (1.1)

\( \frac{\partial T(x, 0)}{\partial y} = -\Theta(x), \ b < |x| < c \) \hspace{1cm} (1.2)

and

(b) \( \sigma_{yy}(x, 0) = 0 \) \hspace{1cm} b < |x| < c \hspace{1cm} (1.3)

\( \sigma_{xy}(x, 0) = 0 \) \hspace{1cm} 0 \leq |x| \leq \infty \hspace{1cm} (1.4)

\( u_y^n(x, 0) + u_y^a(x, 0) = \begin{cases} f(x), & 0 \leq |x| \leq b \\ 0, & |x| \geq c \end{cases} \) \hspace{1cm} (1.5)
Figure 5: Geometry of problem.
where \((\sigma_{xx}, \sigma_{xy}, \sigma_{yy})\) and \((u_x, u_y)\) are components of stress and of displacement vector at any point \((x,y)\) of the medium.

We checked throughout that [53]

\[
u_y(x,0) > 0, \quad b < |x| < c
\]

which means that the crack faces do not meet other than the crack tips. The Fourier, sine (cosine) transforms are defined as

\[
F_{cs}(\xi, T) = \int_0^\infty \int_0^\infty F(x,y) \cos(\xi x) \sin Ty \, dx \, dy
\]

The plan of the paper is as follows: in next section we shall formulate the problem and reduce it to triple integral equations in Section 3. The solutions of these triple integral equations are presented in Section 4. The Section 5 will give the expressions for stress components \(\sigma_{yy}(x,0)\) and \(\sigma_{xy}(x,0)\) and the crack opening displacement \(u_y(x,0)\). A special case of \(T(x) = T_0 = \text{constant}\) and \(f(x) = f_0 = \text{constant}\) will be presented in Section 6.

2. FORMULATION

The equations of equilibrium

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial}{\partial y} \sigma_{xy} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial}{\partial y} \sigma_{yy} = 0
\]

are reduced to by using stress-strain relations,

\[
\nabla \phi = \frac{\partial}{\partial x} \{\nabla \cdot (T)\}
\]
where

\[ \nabla = a_{22} \frac{\partial^4}{\partial x^4} + (a_{66} + 2a_{12}) \frac{\partial^4}{\partial x^2 \partial y^2} + a_{11} \frac{\partial^4}{\partial y^4} \]  

(2.3)

\[ \nabla_1 = \frac{y_1 |A|}{a_{66}} \frac{\partial^2}{\partial x^2} + \left\{ y_1 a_{22} - \frac{y_1 |A|}{a_{66}} + y_2 a_{12} \right\} \frac{\partial^2}{\partial y^2} \]  

(2.4)

with

\[ |A| y_1 = \alpha_2 (a_{11} - a_{12}) a_{22}, \quad |A| y_2 = \alpha_2 a_{11} (a_{22} - a_{12}) \]  

(2.5)

and

\[ |A| = a_{11} a_{22} - a_{12}^2 \]  

(2.6)

The function \( \phi \) is related to displacement components as

\[ u_x(x, y) = \int_0^\infty \xi^{-1} \left[ a_{11} \phi', xy - \xi^2 a_{12} \phi' \right] \sin \xi x d\xi \]  

(2.7)

\[ u_y(x, y) = \int_0^\infty \xi^{-2} \left[ a_{11} \phi'', yyy - \xi^2 (a_{12} + a_{66}) \phi' y \right] \cos \xi x d\xi \]  

(2.8)

with the following stress-strain relations

\[ \sigma_{xx}(x, y) = \left( a_{22} u_x, x - a_{12} u_y, y \right) / |A| - y_1 T \]
\[ \sigma_{yy}(x, y) = \left( -a_{22} u_x, x + a_{11} u_y, y \right) / |A| - y_2 T \]
\[ \sigma_{xy}(x, y) = \left( u_x, y + u_y, x \right) / a_{66} \]  

(2.9)

where \( a_{11}, a_{66} \) are elastic constants. The symbol \( (,) \) in subscripts refer to differentiation with respect to variable written on the right of symbol.

The heat distribution problem is solved through the equation

\[ \nabla_1 T = 0. \]  

(2.10)
then (2.2) gives

$$\nabla \phi = 0 \quad (2.11)$$

if we make substitution

$$a_{11} = a_{22} = (1-\eta^2)/E, \quad a_{12} = -\eta(1+\eta)/E, \quad a_{66} = 2(1+\eta)/E$$

then the problem is reduced to isotropic medium and the equation (2.2) reduces to the same as Kushwaha and Chandra [121] did for isotropic case. In the above \( \eta \) and \( E \) are Poisson's ratio and Young's modulus of isotropic medium.

3. REDUCTION TO TRIPLE INTEGRAL EQUATIONS

The problem formulated in previous section will be reduced to following triple integral equations

$$T(x,y) = \int_{0}^{\infty} B(\xi) e^{-\xi \beta_1 y} \cos(\xi x) d\xi \quad (3.1)$$

where

$$\beta_1 = \frac{\chi_1 |A|}{\chi_1 a_{22} a_{66} - \chi_2 (|A| - a_{12} a_{66})} \quad (3.2)$$

Thus using (1.1) - (1.2), we get

$$\int_{0}^{\infty} B(\xi) \cos(\xi x) d\xi = \begin{cases} T(x), & 0 \leq x < b \\ 0, & x > c \end{cases} \quad (3.3)$$

$$\int_{0}^{\infty} \xi_1 B(\xi) \cos(\xi x) d\xi = \Theta(x), \quad b \leq x \leq c \quad (3.4)$$
Elasticity problem: For elasticity problem we solve (2.1) which is to be substituted in (2.8). The solution $\phi'$ is assumed as

$$(r_1 - r_2)\phi'(\xi, \gamma) = \left\{ (r_1 - r_2) A(\xi) - B_1(\xi) \right\} e^{-r_1 \xi x}$$

$$+ B_1(\xi) e^{-r_2 \xi x}$$

where $r_1$ and $r_2$ are solutions of

$$r^4 - 2D_1 r^2 + D_2 = 0$$

with

$$2D_1 = (2a_{12} + 6a_{66})/a_{11}, \quad D_2 = a_{22}/a_{11}$$

The boundary conditions (1.3) - (1.5) along with (2.7) and (3.5) - (3.7) reduces to the following triple integral equations

$$\int_0^\infty B_2(\xi) \cos(\xi x) \, d\xi = \beta_2 \left\{ \begin{array}{ll} f(x) & 0 \leq x \leq b \\ 0 & x \geq c \end{array} \right. \quad (3.8)$$

and

$$\int_0^\infty \xi B_2(\xi) \cos(\xi x) \, d\xi = \frac{\pi}{2} r_1 T(x, 0), \quad b \leq x \leq c \quad (3.9)$$

with

$$\beta_2 = \frac{2}{\pi} a_{11} r_1 (r_1 + r_2), \quad B_2 = \xi B_1$$

The solution of above triple integral equations will be given in the next section.

4. SOLUTION OF TRIPLE INTEGRAL EQUATIONS

We take trial solution of $B(\xi)$ as
\[ B(\xi) = \frac{2}{\pi} \int_{b}^{c} g(t) \frac{\sin \xi t}{\xi} \, dt - \frac{2}{\pi} \int_{0}^{b} T'(t) \sin \xi t \, dt \]  \hspace{1cm} (4.1)

Then (4.1) will satisfy (3.3) identically if
\[ \int_{b}^{c} g(t) \, dt = T(b) \]  \hspace{1cm} (4.2)

and then the substitution of \( B(\xi) \) into (3.4) which after inversion, using the method of Srivstava and Lowengrub [89], gives
\[ g(t) = \frac{1}{\pi^2} \delta(t) \left[ \int_{b}^{c} \frac{x\theta(x) \delta(x) \, dx}{x^2 - t^2} + D \right] \]  \hspace{1cm} (4.3)

where \( D \) is an arbitrary constant which will be determined through (4.2) - (4.3) and
\[ \delta(t) = \begin{cases} \left\{ |(c^2 - t^2)(t^2 - b^2)| \right\}^{1/2} & b < t < c \end{cases} \]  \hspace{1cm} (4.4)

we take
\[ B_2(\xi) = \frac{2}{\pi} \left[ \int_{b}^{c} k(t) \frac{\sin \xi t \, dt}{\xi} - \beta_2 \int_{0}^{b} f'(t) \sin \xi t \, dt \right] \]  \hspace{1cm} (4.5)

which satisfies (3.8) identically if
\[ \int_{b}^{c} h(t) \, dt = \beta_2 f(b) \]  \hspace{1cm} (4.6)

and substitution of (4.5) into (3.9) which gives after inversion
\[ h(t) = \frac{1}{\beta_2 \pi^2 \delta(t)} \left[ \int_{b}^{c} \frac{x\delta(x) P(x,0) \, dx}{x^2 - t^2} + H \right] \]  \hspace{1cm} (4.7)
where $H$ is constant of integration and

$$P(x,0) = r_1 T(x,0) - \beta \int_0^b \frac{f'(t)tdt}{x^2-t^2} \quad (4.8)$$

where $T(x,0)$ i.e. the value of integral (3.3) in the interval $b < x < c$.

$$T(x,0) = \int_x^c g(t)dt \quad (5.1)$$

where $g(t)$ is the solution given by (4.3). The heat given by (5.1) will open crack $y=0$, $b < |x| < c$.

6. PHYSICAL QUANTITIES

Now we are interested in evaluating $\sigma_{yy}^e(x,0)$ for $0 \leq x \leq b$ and $x > c$. We find the value of integral (3.9) for above $x$. Using (4.5) and evaluating the integral then using (4.7) we get

$$\sigma_{yy}^e(x,0) = \frac{1}{\pi \beta_2} \begin{cases} \frac{M(x)}{\delta(x)} & 0 \leq x \leq b \\ -\frac{M(x)}{\delta(x)} & x > c \end{cases} \quad (5.2)$$

where

$$M(x) = \int_b^c \frac{\delta(y) P(y,0)dy}{y^2-x^2} + H \quad (5.3)$$

Now we evaluated crack opening displacement $u_y(x,0)$, which is given through (3.8), (4.5) as

$$u_y(x,0) = \frac{1}{\beta_2} \int_x^c h(t)dt \quad (5.4)$$
where $\beta_2$ and $h$ are given by (3.10) and (4.7).

6. SPECIAL CASE

We take

\[ T(x) = T_0 = \text{constant}, \quad (6.1) \]
\[ \theta(x) = 0 = \text{(NO FLUX)} \quad (6.2) \]

and \[ f(x) = d = \text{constant} \quad (6.3) \]

Using (6.2) in (4.3), we get

\[ g(t) = \frac{D}{\pi^2 \delta(t)} , \quad b < t < c \quad (6.4) \]

where

\[ D = E \cdot T_0 \frac{n^2}{c^2} F(n/2, K) \quad (6.5) \]

and

\[ K = \frac{c^2 - b^2}{c^2} , \quad E = E(n/2, u) \]

$F$ and $E$ are called complete elliptic integrals of first and of second type.

Now using (6.4) in (5.1), we get,

\[ T(x,0) = \frac{D}{\pi^2} \int_x^c \frac{dt}{\delta(t)} \quad (6.6) \]

Using (6.3) in (4.8), we get

\[ P(x,0) = r_1 T(x,0) \quad (6.7) \]

which gives
\[ h(t) = \frac{r_1}{\beta_1 n^2 \delta(t)} \left[ \frac{D}{n^2} \int_0^c x \delta(x) \frac{dx}{x^2-t^2} \int_0^c \delta(x) + \frac{H}{r_1} \right] \quad (6.8) \]

Equation (6.8) and (4.6) will determine \( H \). Using (6.7) in (5.3) we get for \( 0 \leq x \leq b \) and \( x \geq c \)

\[ M(x) = \int_0^c \frac{y \delta(y) T(y,0) dy}{y^2-x^2} + H \quad (6.9) \]

Then we see that the normal stress component possesses the square root singularity only.