CHAPTER - 4

THREE GRIFFITH CRACKS OPENED BY THERMAL STRESS
IN AN INFINITE ISOTROPIC MEDIUM

ABSTRACT - The closed form expressions for the crack shape and the stress intensity factors for three Griffith cracks opened by thermal stress in an infinite homogeneous isotropic solid have been obtained by using Fourier transform technique. A special case of point heat source is also discussed.

1. Introduction

The experimental [99] evidences have shown that minute cracks are formed near the tip of main crack prior to fracture. The middle crack out of the three can be understood as main crack and outer ones as subsidiary. Therefore, the present research endeavour is of practical significance.

There are many problems of crack opening due to forces at crack faces, see [53]. Oleziak and Sneddon [116] have considered the penny-shaped crack opened by thermal stresses. Florence & Goodier [33] considered the same problem but used complex variable technique. Shail [126] considered the problem of penny-shaped crack in thick plate.

Kushwaha and Chandra [76] solved the problem of a Griffith crack opened by thermal stresses in an infinite solid.

The concerned problem is of the opening of three Griffith cracks occupying the space $y=0$ $0 < |x| < b$, $c < |x| < d$ while two outer cracks are symmetrically placed with respect to
Figure 4: Geometry of the problem.
inner one. The x-axis is the axis of cracks and y-axis is passing through the middle of inner crack. We assumed to medium as isotropic and homogeneous and temperature distribution is steady and the elastic properties of the medium do not change with temperature variation $T$. The heat distribution is governed by

$$\nabla^2 T = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T(x,y) = -\frac{Q(x,y)}{\chi} \quad (1.1)$$

where $Q$ is known function and $\chi$ is material constant. The boundary conditions of the problem are, see Figure 1,

$$\sigma_{xy}(x,0) = 0, \quad 0 \leq |x| < \infty \quad (1.2)$$

$$\sigma_{yy}(x,0) = 0, \quad 0 \leq |x| < b, \ c < |x| < d \quad (1.3)$$

$$U_y(x,0) = 0, \quad b \leq |x| \leq c, \ |x| \geq d \quad (1.4)$$

where $(\sigma_{xx}, \sigma_{xy}, \sigma_{yy})$ and $(U_x, U_y)$ are component of stress tensor and of displacement vector at a point of the medium. Boundary condition (1.3) means that crack is stress free.

We assumed that all physical quantities vanish as $\sqrt{x^2+y^2} \rightarrow \infty$. Medium is under plain strain conditions.

We checked throughout, [10],

$$U_y(x,0) > 0, \quad 0 \leq |x| < b, \ c < |x| < d. \quad (1.5)$$

which means that the cracks really opens out at Fourier transforms are taken as

$$F_{sc}(\xi, \zeta) = \int_0^\infty \int_0^\infty F(x,y) \sin(\xi x) \cos(\zeta y) \, dx \, dy$$
The plan of this research endeavour is as follows:

In next section we shall formulate the problem. Section 3 will reduce the above problem to quadruple integral equations. The solution of these quadruple integral equations will be given in section 4. The physical quantities will be reported in terms of solution of these quadruple integral equations. A special case will be given in section 5.

2. Formulation of the Problem

To solve above mentioned boundary value problem we are to solve the equations of equilibrium

\[ \sigma_{xx}, \sigma_{xy}, \gamma = 0, \sigma_{yx}, \sigma_{yy}, \gamma = 0 \quad (2.1) \]

with no body forces, along with the stress-strain relations

\[ \sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda (\varepsilon_{kk} - \gamma^T) \delta_{ij}; i, j = x, y \quad (2.2) \]

with \( \varepsilon_{ij} = (u_{i,j} + u_{j,i}) \), \( \gamma = \alpha_t (3\lambda + 2\mu) \quad (2.3) \)

where \( \alpha_t \) is the coefficient of linear expansion of the medium, \( \gamma \) and \( \mu \) are Lame’s constants and symbol (,) at suffix over functions represents differentiation with respect to symbol following it. In the following lines we took the unit of stress as \( \mu \). Substituting (2.2) into (2.1) and solving for \( U_x \) we get

\[ (\nabla^2)^2 U_x = \frac{b'}{\beta^2} \left( \frac{\partial}{\partial x} \nabla^2 \right) T \quad (2.4) \]

with

\[ \beta^2 = \frac{\lambda + 2\mu}{\kappa}, \quad b' = (2\beta^2 - 4) \alpha_t \quad (2.5) \]
Since the problem is linear, we assume that

\[ U_x(x,y) = U_e(x,y) + U_t(x,y) \]  \hspace{0.5cm} (2.6)

where means \( U_x \) displacement due to crack opening and \( U_t \) displacement due to heat distribution. Similarly there will be two types of components of stress also. Thus we divide the problem into two, namely.

[A] **Heat distribution problem** - The problem of this type is solved through the equation (2.4) by taking Fourier transform.

The solution of heat problem is given as

\[ U_t(x,y) = \frac{4b'}{\pi^2 \beta^2 \chi} \int_0^\infty \int_0^\infty \frac{\xi \sin(\xi x) Q_{cc} \cos(\xi y) d\xi d\xi}{(\xi^2 + \xi^2)^2} \]  \hspace{0.5cm} (2.7)

where \( Q_{cc} \) is through \( T_{cc} \). Thus

\[ \sigma_{yy}(x,y) = \frac{4b'}{\pi^2 \beta^2 \chi} \int_0^\infty \int_0^\infty \frac{\xi^2 Q_{cc} \cos(\xi x) \cos(\xi y) d\xi d\xi}{(\xi^2 + \xi^2)^2} \]  \hspace{0.5cm} (2.8)

\[ U_x(x,y) = \int_0^\infty \xi^{-1} \left[ (1-\eta) \phi_{yy} - \eta \xi^2 \phi \right] \sin \xi x d\xi \]  \hspace{0.5cm} (2.9)

[B] **Elasticity Problem** - This problem is solved by the method of Sneddon [134] where crack is opened by thermal stress \( \sigma_{yy}^t(x,0) \) whose solution is given in section A. The solution is assumed as

\[ U_y(x,y) = \int_0^\infty \xi^{-2} \left[ (1-\eta) \phi_{yyy} + (\eta-2) \xi^2 \phi, y \right] \cos \xi x d\xi \]  \hspace{0.5cm} (2.10)
where $\eta$ is Poisson ratio of the medium with

$$\phi(\xi, \gamma) = [A(\xi)+\gamma B(\xi)] e^{-\xi \gamma} \quad (2.11)$$

3. Reduction and Solution of Quadruple Integral Equations

The boundary condition (1.2) is satisfied identically if

$$\xi A(\xi) = -B(\xi)/(\beta^2-1) \quad (3.1)$$

while from the conditions (1.4) and (1.3) along with (2.7) - (2.11) and stress-strain relations, we get

$$\int_0^\infty A(\xi) \cos (\xi x) d\xi = 0, \quad b \leq x \leq c, \quad x \geq d \quad (3.2)$$

$$\int_0^\infty \xi A(\xi) \cos(\xi x) d\xi = \frac{1}{2(\beta^2+1)} \phi^t(x,0), \quad 0 \leq x < b, \quad c < x < d. \quad (3.3)$$

We assume the trial solution of (3.2) - (3.3) as

$$\xi A(\xi) = 2 \int_0^b g_1(t) \sin \xi t \, dt + 2 \int_c^d g_2(t) \sin(\xi t) \, dt \quad (3.4)$$

Then (3.2) will be satisfied identically if

$$\int_b^c g_2(t) \, dt = 0 \quad (3.5)$$

The substitution of (3.4) into (3.3) and repeated use of method due to Srivastava and Lowengrub [149] we get the solution as

$$g_1(t) = 2t \Delta_0(t) \left\{ \pi^2 \theta(t) \right\}^{-1}, \quad 0 \leq t < b \quad (3.6)$$
\[ g_2(t) = -2t \Delta_0(t) \left\{ \pi^2 \theta(t) \right\}^{-1}, \quad c < t < d \]  

\[ \theta(t) = \left\{ \left| \left( b^2-t^2 \right) \left( c^2-t^2 \right) \left( d^2-t^2 \right) \right| \right\}^{1/2} \]  

and

\[ \Delta_0(t) = \left\{ \int_0^b \int_0^d \frac{p(x) \theta(x) \, dx}{t^2-x^2} \right\} + D. \]  

where \( D \) is an arbitrary constant, and

\[ p(x) = \sigma_{yy}(x,0) / \left[ \pi \left( \beta^2-1 \right) \right] \]  

4. Physical Quantities and Special Case

The quantities of physical interest are crack opening displacement and normal stress component at \( y = 0 \). We find the value of integral \( (3.2) \) for \( 0 \leq x \leq b, \quad c \leq x \leq d \) and then evaluating the integrals we get

\[ U_y(x,0) = \frac{4(1-\eta)^2}{E} \left[ \int_x^b g_1(t) \, dt + \int_x^d g_2(t) \, dt \right] \]  

where \( g_1 \) & \( g_2 \) are given by \( (3.6) - (3.8) \). Similarly, the value of integral \( (3.3) \) for \( b < x < c, \quad x > d \) is given as

\[ \sigma_{yy}(x,0) = \pm \frac{4 \pi^2}{E} \Delta_0(x) / \left\{ \theta(x) \right\} - \sigma_{yy}^t(x,0) \]  

where \((+)\) and \((-)\) sign are to be taken for \( b \leq x \leq c \) & \( x \geq d \) respectively. The stress-intensity factors at crack tips are defined as
\[ K_b = \lim_{x \to b^+} \sqrt{x-b} \sigma_{yy}^e (x, 0) \quad K_c = \lim_{x \to c^-} \sqrt{c-x} \sigma_{yy}^e (x, 0) \quad K_d = \lim_{x \to d^+} \sqrt{x-d} \sigma_{yy}^e (x, 0) \quad (4.3) \]

there is no singularity in \( \sigma_{yy}^e (x, 0) \).

5. Special Case

Now we consider one special case of thermal distribution. A heat source of intensity \( \chi p \) is put at the points \((0, \pm h)\) given as

\[ Q(x, y) = \chi p \delta(x) \left[ \delta(y-h) - \delta(y+h) \right] \quad (4.4) \]

where \( \delta \) is Dirac delta function, see figure 2.

Thus using (4.4) we get

\[ Q_{cc} = \chi p \cos (\xi h) \quad (4.5) \]

Thus

\[ p(x) = \frac{b_p}{4(\beta^2+1)} \left[ \frac{h}{h^2+x^2} - \log \frac{h^2+x^2}{h^2} \right] \quad (4.6) \]

Evaluating the stress \( \sigma_{yy}^e (x, 0) \), we get as

\[ \sigma_{yy}^e (x, 0) = \pm 2pb' [\pi \theta(x)]^{-1} \left[ \pi \left( <h^2+x^2> - N_0 <h^2+d^2>^{-1} \right) \right. \]

\[ \left\{ \delta' (h) \pm \theta (h) \frac{h}{h^2+x^2} \right\} + \int \left\{ \left( <h^2+x^2> \pi - \frac{N_0}{h^2+d^2} \right) dh \right\} \quad (4.7) \]
\[ K_0^2 = \frac{d^2 - c^2}{d^2 - b^2} \]  \hspace{1cm} (4.9)

where \( \Pi \) & \( F \) are complete elliptic integrals of third and first type respectively, and

\[ \delta_2(h) = \left[ (h^2 + b^2) \left( h^2 + c^2 \right) \left( h^2 + d^2 \right) \right]^{1/2} \]  \hspace{1cm} (4.10)