ABSTRACT: The closed form expressions of the stress and the displacement fields in the vicinity of Griffith cracks are obtained by using Fourier transform method. Two special cases are discussed.

1. INTRODUCTION

The crack opening by heated wedge has practical significance. There are many problems of crack opening due to pressure on crack faces (Sneddon and Lowengrub [33]). The penny-shaped crack opened by thermal stresses at crack faces in infinite solid have been considered by Oleziak and Sneddon [116]. Florence and Goodier [33] extended above problem and used complex variable method.


Only recently, the problem of the stress intensity factors for a Griffith crack opened by thermal stresses in an infinite solid has been discussed by Kushwaha and Chandra [121].

The concern of the present research endeavour is the problem of crack opening due to heated wedge in an infinite and isotropic medium while the cracks occupy the space $y = 0, b < |x| < c$, with the following mixed-boundary conditions:
Figure 3: Geometry of the problem.
\[ \sigma_{xy}(x,0) = 0, \quad 0 \leq |x| < \infty \]  
\[ \sigma_{yy}(x,0) = 0, \quad b < |x| < c \]  
\[ u_y(x,0) = \begin{cases} 
  u_0(x), & 0 \leq |x| \leq c \\
  0, & |x| > c 
\end{cases} \]  

where \((u_x, u_y)\) and \((\sigma_{xx}, \sigma_{xy}, \sigma_{yy})\) are the components of the displacement vector and the stress tension at a point \((x,y)\) in the medium. \(u_0(x)\) is a wedge shape function. We assume that the elastic and the thermal property of the medium do not change with heat variation. We also take that all the physical quantities vanish as \(x^2 + y^2 \to \infty\).

Since the problem is linear we first solve the heat distribution in the medium by solving the equation

\[ \nabla^2 T = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]  

under the mixed boundary conditions

\[ T(x,0) = \begin{cases} 
  T(x), & 0 \leq |x| \leq b \\
  0, & |x| > c 
\end{cases} \]  

\[ T, y(x,0) = f(x), \quad b < |x| < c \]  

where \(T(x)\) and \(f(x)\) are known functions.

The elasticity problem is solved by using the method of Sneddon [51]. It is being assumed that there are no sources/sinks in the medium. We also checked throughout that, Burniston [31]

\[ u_y(x,0) > 0, \quad b < |x| < c \]
which means that the cracks are really opened out and the crack faces do not meet each other than thrust crack tips. The Fourier transforms are taken as

\[
\hat{f}_{CS}(\xi,\eta) = \int_0^\infty \int_0^\infty f(x,y) \cos(\xi x) \sin(\eta y) \, dx \, dy
\]

(1.8)

with usual inversion formula. The paper embodies the following: next section will formulate the problem and reduce to triple integral equations. The solution of these triple integral equations will be given in Section 3. The general expressions of stress components at \( y = 0 \) and the crack opening displacement will be given in Section 4. One special case will be given in Section 5.

2. FORMULATION OF THE PROBLEM

We divide the physical problem into two problems, namely

[A] Heat Problem:

The problem of finding out the temperature at \( y = 0 \) due to heated wedge is obtained by solving the differential equation (1.4) with the conditions (1.5) - (1.6).

We take the solution of (1.4) as

\[
T(x,y) = \int_0^\infty C(\xi) e^{-\xi y} \cos(\xi x) \, d\xi
\]

(2.1)

where \( C(\xi) \) is constant to be determined.

The use of (2.1) in (1.5) - (1.6) gives
The problem of finding the stresses due to crack opening is obtained through the solution of equations of equilibrium:

\[ \sigma_{xx,x} + \sigma_{xy,y} = 0, \quad \sigma_{xy,x} + \sigma_{yy,y} = 0 \]  \hspace{1cm} (2.4)

with stress-strain relations as

\[ \sigma_{ij} = 2\mu e_{ij} + \lambda (e_{kk} - \gamma T) \delta_{ij}; \quad i, j = x, y \]  \hspace{1cm} (2.5)

where symbols have their usual meaning. We take the modulus of rigidity \( \mu \) as the unit of stress. We take the solution of (2.4) as

\[ u_x(x,y) = \int_0^\infty \xi^{-1} \left[ (1-\eta)\phi_{yy} - \eta \xi^2 \phi \right] \sin \xi x \, d\xi \]  \hspace{1cm} (2.6)

\[ u_y(x,y) = \int_0^\infty \xi^{-2} \left[ (1-\eta)\phi_{yyy} + (\eta-2) \xi^2 \phi_y \right] \cos \xi x \, d\xi \]  \hspace{1cm} (2.7)

with

\[ \phi(\xi, y) = \left[ A(\xi) + y B(\xi) \right] e^{-\xi y} \]  \hspace{1cm} (2.8)

\( \eta \) is Poisson ratio of the medium. The substitution of (2.6) - (2.8), through relations (2.5), into boundary conditions (1.1) - (1.3) and using the symmetry of geometry we get the following
triple integral equations:

\[
\int_0^\infty A(\xi) \cos(\xi x) \, d\xi = \beta^{-2} \begin{cases} 
& u_0(x), \quad 0 \leq x \leq b \\
& 0, \quad x \geq c
\end{cases} 
\]  \hspace{1cm} (2.9)

\[
\int_0^\infty \xi A(\xi) \cos(\xi x) \, d\xi = \frac{(3\beta^2-4)\alpha t}{2(\beta^2-1)} T(x,0), \quad b < x < c 
\]  \hspace{1cm} (2.10)

where

\[
\beta^2 = \frac{\lambda + 2\mu}{\mu} 
\]  \hspace{1cm} (2.11)

The solutions of triple integral equations (2.2) - (2.3) and (2.9) - (2.10) will be given in next section.

3. SOLUTION OF TRIPLE INTEGRAL EQUATIONS

To solve these triple integral equations, we shall follow the method of Srivastava and Lowengrub [95]. We assume the trial solutions as

\[
C(\xi) = 2 \pi \frac{1}{\xi} \left[ \int_b^c g_1(t) \sin(\xi t) \, dt - \int_0^b T(t) \sin \xi t \, dt \right] 
\]  \hspace{1cm} (3.1)

and

\[
A(\xi) = 2 \pi \frac{1}{\xi} \left[ \int_b^c g_2(t) \sin(\xi t) \, dt - \beta^{-2} \int_0^b u_0(t) \sin \xi t \, dt \right] 
\]  \hspace{1cm} (3.2)

then the equations (2.2) and (2.9) are satisfied identically if

\[
\int_b^c g_1(t) \, dt = T(b), \quad \int_b^c g_2(t) \, dt = \beta^{-1} u_0(b) 
\]  \hspace{1cm} (3.3)

and the substitution of (3.1) - (3.2) into (2.2) and (2.10) respectively and then inverting, we get
\[ g_1(t) = \pi^{-2} \left[ -\int_{b}^{c} \frac{x \delta(x) p(x) \, dx}{x^2 - t^2} + D_1 \right] \]  

(3.3)

\[ \delta(t) = \left[ \left( t^2 - b^2 \right) \left( c^2 - t^2 \right) \right]^{1/2} \]  

(3.4)

\[ g_2(t) = A_0(t) / \delta(t) \]  

(3.5)

with

\[ p(x) = \frac{2}{\pi} \int_{0}^{b} t' (t) \frac{tdt}{t^2 - x^2} - f(x) \]  

(3.6)

\[ A_0(t) = \pi^{-2} \left( \int_{b}^{c} \frac{x \delta(x) p_1(x) \, dx}{t^2 - x^2} + D_2 \right) \]  

(3.7)

\[ p_1(x) = \nu T(x, 0) + \frac{2\pi^{-2}}{\pi} \int_{0}^{b} \frac{tu_0(t) \, dt}{t^2 - x^2} \]  

(3.8)

\[ \nu = \frac{(3\beta^2 - 4)}{2(\beta^2 - 1)} \alpha \]

where \( D_1 \) and \( D_2 \) are two arbitrary constants which will be determined through conditions (3.3).

4. PHYSICAL QUANTITIES

The temperature \( T(x, 0) \) is important because it causes the cracks to open. Evaluating the value of integral (2.1) for \( b \leq x \leq c \) we get

\[ T(x, 0) = \int_{x}^{c} g_1(t) \, dt \]  

(4.1)

where \( g_1(t) \) will be given by (3.3).
Crack Shape

The crack opening displacement is another quantity which is important in fracture design criteria. It is evaluated through the value of integral (2.9) for \( b < x < c \) and given as

\[
u_y(x,0) = \beta^2 \int_x^c g_2(t) dt \quad b < x < c
\]  

Normal stress component at \( y = 0 \), \( \sigma_{yy}(x,0) \) is the value of integral (2.10) for \( 0 \leq x \leq b \) and \( x > c \), where

\[
\sigma_{yy}(x,0) = \frac{2}{\pi} \left[ \pm \frac{A_0(x)}{\delta_1(x)} - \beta^2 \int_0^b \frac{tu_0(t)dt}{t^2-x^2} \right]
\]

while (±) signs are taken for \( 0 \leq x < b \) and \( x > c \) respectively and \( \delta_1(x) \) are given as

\[
\delta_1(x) = \left( (x^2-c^2)(x^2-b^2) \right)^{1/2}, \quad c < x < \infty
\]

\[
\delta_1(x) = \left( (b^2-x^2)(c^2-x^2) \right)^{1/2}, \quad 0 < x < \infty
\]

Stress intensity factors are defined as

\[
K_b = \lim_\frac{\sqrt{b-x}}{x-b} \sigma_{yy}(x,0)
\]

\[
K_c = \lim_\frac{\sqrt{c-x}}{x-c} \sigma_{yy}(x,0)
\]

Thus using (4.5) - (4.6) and (4.3) we get

\[
K_b = \frac{2}{\pi} \left[ n(b) A_0(b) - t_1(b) \right]
\]
\[ K_c = -\frac{2}{\pi} \left[ n(c)\Delta_0(c) \right] \]  
(4.8)

where

\[ n(x) = \left[ 2x(c^2-b^2) \right]^{-1/2} \]

\[ t_1(b) = \lim_{x \to b} \left( \frac{1}{b-x} \right) \int_{0}^{b} \frac{tu'(t)dt}{t^2-x^2} \]  
(4.9)

Thus the physical problem, namely, the crack shape, stress-intensity factors are reduced in terms of solution of triple integral equations.

In next section we shall solve some special cases so that the validity of calculations is established.

5. SPECIAL CASE

We take the wedge shape functions \( u_0(x) \), temperature distribution \( T(x) \) and flux \( f(x) \) as

\[ u_0(x) = u_0 = \text{constant} \]

\[ T(x) = T_0 = \text{constant} \]

\[ f(x) = f_0 = \text{constant} \]  
(5.1)

Substituting the values from (5.1) into (3.6) we get

\[ p(x) = -f_0 = \text{constant} \]  
(5.2)

Then

\[ g_1(t) = \frac{1}{\pi^28(t)} \left[ \frac{\pi}{4} f_0 \left( 2t^2-b^2-c^2 \right) + D_1 \right] \]  
(5.3)

with

\[ D_1 = \frac{\pi}{4F} \left[ 4\pi cT_0 + f_0 \left( \left( b^2+b^2 \right) F - 2c^2E \right) \right] \]  
(5.4)

Where \( F \) and \( E \) are complete elliptic integrals of the first and the
second type given as
\[ F = (\pi/2, \mu_0), \quad E = E(\pi/2, \mu_0), \quad \mu_0 = \sqrt{\frac{c^2-b^2}{c^2}} \]  
(5.5)

Substituting the value of \( g_1(t) \) from (5.3) into (4.1), we get
\[ T(x,0) = \left[ f_0 \left( 2c^2E(\phi, \mu_0) - (b^2+c^2)F(\phi, \mu_0) \right) + DF(\phi, \mu_0) \right] / 4\pi c \]  
(5.6)

where
\[ \phi = \sin^{-1} \sqrt{\frac{c^2-x^2}{c^2-b^2}} \quad b < x < c \]  
(5.7)

(5.6), first of (5.1) and (2.8), we get
\[ p_1(x) = \nu^2 T(x,0) \]  
(5.8)
Thus,
\[ g_2(t) = \frac{1}{\pi^2 \delta(t)} \left[ \nu \int_b^c \frac{x\delta(x) T(x,0)}{t^2-x^2} \, dx + D_2 \right] \]  
(5.9)

where,
\[ D_2 = \frac{1}{\pi^2} \left\{ \frac{c^2}{\pi^2} \right\} \int_b^c \frac{x\delta(x) T(x,0)}{c^2-x^2} \, dx \]
\[ \Pi \left\{ \frac{\pi}{2}, \mu_0, K(x) \right\} \]  
(5.10)

\( \Pi \) is complete elliptic integral of third type.
\[ K(x) = \frac{c^2-x^2}{c^2-b^2} \]  
(5.11)

Using (5.1) in (4.7) – (4.9), we get
\[ t_1(b) = 0 \]  
(5.12)
Thus,
\[ \Delta_0(t) = \pi^2 \left[ \nu \int_b^c \frac{x\delta(x) T(x,0)}{t^2-x^2} \, dx + D_2 \right] \]  
(5.13)

where \( T(x,0) \) and \( D_2 \) are given by (5.6) (5.10) respectively.