CHAPTER - IV
Chapter IV

STUDY OF PRESSURE DEVELOPMENT IN TAPERED TUBE

INTRODUCTION:

The flow of non-Newtonian viscous fluid through a vessel of varying diameter is important from standpoint of blood flow in blood vessels. The taper of the tube is an important factor in the development of pressure along the tube length. The mammalians arterial tree consists of the main aortic tube which continues into two iliac and femoral arteries and number of side branches. Propagation and transformation of the pressure and flow waves in this system depends on the distribution of the characteristic impedance. In the main aortic tube the characteristic impedance increases more than 10-fold from central to the periphery. The two contributing factors are the decrease of the cross sectional area (Geometrical Tapering) and the peripheral increase of the wall stiffness (Elastic tapering). The preponderance of the latter leads to a peripheral increase of wave velocity. Average values are 400 cm/sec at the
aortic arch and about 1200 to 1500 cm/sec in the femoral artery (Gauer et al (1936)[5], Learoyd & Taylor, (1966)[11]). Kenner (1962 [6], 1963 [7]) had calculated the amplitudes based on conservation of energy and momentum equations. From this study, it was found that local step increase of the characteristic impedance leads to an increase in the transmitted pressure wave and a decrease in the amplitude of the transmitted flow wave. This agrees with what can be actually found in arteries (Attinger 1968 [1], Kroeker & Wood (1955) [10], Spencer & Denison (1963)[16]. Kenner and Wetterer (1962 a, b)[8, 9] have extensively study the problem of pressure development in the tapered tube. Later on researchers like Remington and Wetterer (1963)[13], Remington (1965)[14], Remington and O’Brien (1970)[15] and Bauer et al (1970)[2] have calculated the pressure pulse in the tapered tube. Kenner (1963) examined overall effect of continuous tapering by the two ways. In this chapter we have concentrated on the Geometrical tapering. Charm and Kurland [3] observed that when the flow is towards the converging section, theoretical and experimental results are
in closed agreement. In tubes of relatively large diameter, where the influence of a marginal gap is negligible, experimental value agrees well with the anticipated value. Oka [12] has studied non-Newtonian fluid through tapered tube and has shown that pressure gradient is not constant along the axis but increases with the decrease in the radius of the tapered tube. Chaturani and Ranganatha [4] Sahu et al [17], studied the flow of Newtonian fluid in non-uniform tubes with variable wall permeability. S. Sisavath, Jing and Zimmerman [18] have studied the creeping flow through a pipe of varying radius, and obtained the good agreement in the creeping flow regime for the pressure drop versus flow rate relationship.

**MATHEMATICAL ANALYSIS:**

To calculate pressure gradient in the flow through the tapered tube we make certain assumption to simplify the problem,

(i) The fluid is incompressible.

(ii) The motion of the fluid is laminar.

(iii) The motion has an axial symmetry.
(iv) The motion is steady.

(v) No-body force act on the fluid.

(vi) The motion is so slow that the inertial term can be neglected.

(vii) The semi-angle $\alpha$ of the converging cone is very small.

(viii) No-slip condition of the fluid at the wall is assumed.

We consider cylindrical polar co-ordinate system $(r, \theta, x)$, where $x$-axis is the axis of the tube. At the point $x$, the radius of the tube $R(x)$ is given by

$$R(x) = R_0 - \alpha x, \quad \text{--- (4.1)}$$

Where, $R_0$ is the radius of the tube at entrance point.

For non-Newtonian fluid in the tube, the shear-stress components are given by,

$$\tau_{rr} = -p + 2\eta \frac{\partial v}{\partial r}, \quad \text{--- (4.2)}$$

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Figure 4.1 Tube of Varying radius

\[ \tau_{\theta\theta} = -p + 2\eta \frac{v}{r} \quad \text{--- (4.3)} \]
\[ \tau_{xx} = -p + 2\eta \frac{\partial u}{\partial x} \quad \text{--- (4.4)} \]
\[ \tau_{xr} = \eta \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial x} \right) \quad \text{--- (4.5)} \]
\[ \tau_{r\theta} = 0 \quad \text{--- (4.6)} \]
\[ \tau_{\theta x} = 0 \quad \text{--- (4.7)} \]

Where, \( u \) is the velocity component along \( x \)-direction and \( v \) is along \( r \) direction, \( \eta \) is the coefficient of viscosity and \( p \) is the pressure.

For steady viscous slow motion of the Newtonian fluid, the equation of motion becomes.
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r, \tau_{rr} \right) - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{sx}}{\partial x} = 0 \quad --- (4.8)
\]

And,
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r, \tau_{sx} \right) + \frac{\tau_{xx}}{\partial x} = 0 \quad --- (4.9)
\]

Putting the value of \( \tau_{rr}, \tau_{\theta\theta}, \tau_{sx} \) and \( \tau_{xx} \) from equation (4.2) to (4.5) into the above equations, we obtain:
\[
\frac{\partial p}{\partial r} = \frac{2}{r} \frac{\partial}{\partial r} \left( \eta_a \frac{\partial v}{\partial r} \right) - 2 \eta_a \frac{v}{r^2} + \frac{\partial}{\partial x} \left( \eta_a \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial x} \left( \eta_a \frac{\partial v}{\partial x} \right) \quad --- (4.10)
\]

And,
\[
\frac{\partial p}{\partial r} = \frac{2}{r} \frac{\partial}{\partial r} \left( \eta_a \frac{\partial u}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \eta_a \frac{\partial v}{\partial x} \right) + 2 \frac{\partial}{\partial x} \left( \eta_a \frac{\partial u}{\partial x} \right) \quad --- (4.11)
\]

Where \( \eta_a \) is the apparent viscosity of the fluid having meaning of viscosity and is a function \( \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial x} \right) \).

Since \( \alpha \) is very small, we may assume that \( u \) is of order 1 with respect to \( \alpha \) but both \( v \) and \( \partial u / \partial x \) are of the order of \( \alpha \) while \( \partial v / \partial x \) is of order of \( \alpha^2 \). Further we may assume that \( \eta_a \) is order 1 but \( \partial \eta_a / \partial x \) is of the order of \( \alpha \),

Hence equations (4.5), (4.10) are reduced to
\[
\tau_{sr} = \eta_a \frac{\partial u}{\partial r} \quad --- (4.12)
\]
\[
\frac{\partial p}{\partial r} = \frac{2}{r} \frac{\partial}{\partial r} \left( \eta a r \frac{\partial v}{\partial r} \right) - 2 \eta a \frac{v}{r^2} + \frac{\partial}{\partial x} \left( \eta a \frac{\partial u}{\partial r} \right) \quad (4.13)
\]

\[
\frac{\partial p}{\partial r} = \frac{2}{r} \frac{\partial}{\partial r} \left( \eta a r \frac{\partial u}{\partial r} \right) \quad (4.14)
\]

Eliminating \( p \) we get.

\[
\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \eta a r \frac{\partial u}{\partial r} \right) \right] = \frac{\partial}{\partial x} \left[ \frac{2}{r} \frac{\partial}{\partial r} \left( \eta a r \frac{\partial v}{\partial r} \right) - 2 \eta a \frac{v}{r^2} + \frac{\partial}{\partial x} \left( \eta a \frac{\partial u}{\partial r} \right) \right]
\]

\[
(4.15)
\]

On neglecting terms of order \( \alpha^2 \), we have

\[
\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \eta a r \frac{\partial u}{\partial r} \right) \right] = 0 \quad (4.16)
\]

This gives,

\[
0 = -\frac{\partial p}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_{x r} \right) \quad (4.17)
\]

Or,

\[
\tau_{x r} = \frac{\partial p}{\partial x} \cdot \frac{r}{2} = -\phi(x) \cdot r = \tau(x) \quad (4.18)
\]

where,

\[
\phi(x) = -\frac{1}{2} \frac{\partial p}{\partial x}
\]

If \( \tau_o(x) \) is the shear stress at the wall where \( r = R(x) \), then equation (4.18) gives,

\[
\tau(x) = \frac{\tau_o(x)}{R(x)} \cdot r \quad (4.19)
\]
If we denote strain rate by \( f(\tau) \) then,

\[
f(\tau) = \frac{du}{dr}
\]

\[
u(r) = \frac{R(x)}{\tau_{ow}(x)} \frac{\tau_{ow}(x)}{r} \int f(\tau) dz
\]

--- (4.20)

The volume of the fluid flowing per unit time across a cross section of tube is given by

\[
Q = \int_0^{\eta(x)} 2\pi r u \, dr
\]

Or,

\[
Q = \frac{\pi R^3(x)}{\tau_{ow}^3} \frac{\tau_{ow}(x)}{r^2} \int_0^{\eta(x)} \tau^2 f(\tau) \, dr
\]

--- (4.21)

Equation (4.21) gives the general formula for flow rate of non-Newtonian fluid flowing in slightly tapered tube.

From equation (4.18), we obtain,

\[
\tau_{ow}(x) = -\phi(x) R(x)
\]

--- (4.22)

From equation (4.21) we calculate pressure gradient as a function of \( x \) provided the flow curve of non-Newtonian fluid is given.
DETERMINATION OF PRESSURE GRADIENT FOR FLUID OBEYING
CASSON CONSTITUTION EQUATION:

It has been generally accepted that blood flow in small vessels obeys Casson's equation. For this model, flow chart is given by –

\[
f(\tau) = \frac{1}{\eta_c} \left( \sqrt{\tau - \tau_y} \right)^2 ; \text{ where } \tau > \tau_y \quad \text{--- (4.23)}
\]

And,

\[
f(\tau) = 0 \quad ; \text{ where } \tau \leq \tau_y
\]

Where, \( \tau_y \) is the Casson yield stress and \( \tau \) is the viscosity of the fluid. We consider the flow for which \( \tau > \tau_y \) because flow rate \( Q \) vanishes when \( \tau \leq \tau_y \).

From equations (4.21) and (4.23) we have,

\[
Q = \frac{\pi R^3(x)}{\eta_c} \frac{\tau_o(x)}{\tau_o^3(x) \eta_c} \int_0^x \tau^2 \left[ \tau - 2\sqrt{\tau_y \frac{1}{\tau^2} + \tau_y} \right] dz \quad \text{--- (4.24)}
\]

\[
Q = \frac{\pi R^3(x)}{4\tau_o^3(x) \eta_c} \left[ \tau_o^4(x) - \frac{16}{7} \frac{1}{\tau_y^2} \frac{7}{2} + \frac{4}{3} \frac{\tau_y \tau_o^3(x) - \frac{1}{21} \tau_y^4} \right]
\]

Or,

\[
Q = \frac{\pi R_{(3)}^3 \tau_o(x)}{4 \eta_c} \left[ 1 - \frac{16}{7} \left( \frac{\tau_y}{\tau_o(x)} \right)^2 + \frac{4}{3} \left( \frac{\tau_y}{\tau_o(x)} \right) - \frac{1}{21} \left( \frac{\tau_y}{\tau_o(x)} \right)^4 \right] \quad \text{--- (4.25)}
\]
For very small yield stress, we may assume \( \tau_y / \tau_\omega (x) << 1 \ldots \)

So, neglecting small terms like \( \tau_y / \tau_\omega (x) \) and \( [\tau_y / \tau_\omega (x)]^4 \) we have,

\[
Q = \frac{\pi R^3(x)}{4 \eta_c} \left[ \tau_\omega (x) - \frac{16}{7} \frac{1}{\tau_y^2 (x)} \right] \quad \text{--- (4.26)}
\]

This in view of equation (4.18) becomes,

\[
Q = \frac{\pi R^3(x)}{4 \eta_c} \left[ -R(x) \phi (x) - \frac{16}{7} \frac{1}{\tau_y^2 (-R(x) \phi(x))^2} \right] \quad \text{--- (4.27)}
\]

Or,

\[
R^3(x) \left[ -R(x) \phi (x) - \frac{16}{7} \frac{1}{\tau_y^2 (-R(x) \phi(x))^2} \right] = R^3_0 \left[ -R_0 \phi_0 - \frac{16}{7} \frac{1}{\tau_y^2 (-R \phi)^2} \right] \quad \text{--- (4.28)}
\]

Where, the subscript 0 denotes the value of the respective quantity at \( x = 0 \). The above expression shows that the pressure gradient \( \phi(x) \) decreases with the decrease in the tube radius \( R(x) \).

To get the approximation value of \( \phi(x) \), we put

\[
\phi (x) = \phi_0 \left[ 1 + \alpha \psi (x) \right] \quad \text{--- (4.29)}
\]
In equation (4.29) and neglect the small quantities of order $a^2$, thus,

$$
\psi(x) = \frac{4x}{R_0} \left[ 1 + \frac{6}{7} \left( \frac{\tau_y}{-R_0 \phi_0} \right)^{\frac{1}{2}} \right]
$$

--- (4.30)

Hence,

$$
\frac{-\phi(x)}{-\phi_0} = 1 + \frac{4 \alpha x}{R_0} \left[ 1 + \frac{6}{7} \left( \frac{\tau_y}{-R_0 \phi_0} \right)^{\frac{1}{2}} \right]
$$

--- (4.31)

The graph of equation (4.31) shows that pressure gradient again increases linearly with the distance $x$.

---Figure 4.2 Variation of pressure gradient with respect to axial distance $x$.---

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Determination of Pressure Gradient for Fluid Obeysing

Herschel-Bulkley Constitutive Equation:

Herschel-Bulkley constitution equation regarding shear stress and strain rate is written as,

\[
f(\tau) = \begin{cases} 
\frac{1}{K} (\tau - f_H)^n & \text{if } \tau > f_H \\
0 & \text{if } \tau \leq f_H 
\end{cases}
\]  

--- (4.32)

Where, \( f_H \) denotes the yield stress and \( K \) denotes the viscosity of the fluid.

From the equation (4.21) and (4.31), we obtain flow rate \( Q \) as

\[
Q = \frac{\pi R^3(x)}{K \tau^3_0(x)} \left[ \tau^2 \left( \frac{(\tau - f_H)_{n+1}}{n+1} \right) \right]_0^\tau^\omega(x) - \frac{2}{n+1} \left[ \tau^\omega(x) - f_H \right] \int_0^\tau (\tau - f_H) \, d\tau \\
\left\{ \tau^\omega(x) - \frac{2\tau^\omega(x) [\tau^\omega(x) - f_H]}{n+2} + \frac{2[\tau^\omega(x) - f_H]^2}{(n+2)(n+3)} \right\}
\]  

--- (4.34)

Or,

\[
Q = \frac{\pi R^3(x)}{K \tau^3_0(x)} \frac{(\tau^\omega(x) - f_H)_{n+1}}{(n+1)(n+2)(n+3)} \\
\left[ (n+1)(n+2) \tau^2(x) + 2(n+1) \tau^\omega(x) f_H + 2f_H^2 \right]
\]  

--- (4.35)
Or, 
\[ Q = \frac{\pi R^3(x)\left(\tau_\omega(x) - f_H\right)^{n+1}}{K \tau_\omega^3(x)} \]

\[ \left[ (n+1)(n+2) + 2(n+1)\left(\frac{f_H}{\tau_\omega(x)}\right) + 2\left(\frac{f_H}{\tau_\omega(x)}\right)^2 \right] \]

(4.36)

When, \( n = 1 \), the constitutive equation (4.32) reduces to Bingham Plastic fluid model and hence for this fluid \( Q \) becomes,

\[ Q = \frac{\pi R^3(x)}{24 K_H} \left(\frac{f_H}{\tau_\omega(x)}\right)^2 \tau_\omega(x) \left[ 6 + 4 \frac{f_B}{\tau_\omega(x)} + 2\left(\frac{f_B}{\tau_\omega(x)}\right)^2 \right] \]

(4.37)

On simplifying equation (4.37), we obtain.

\[ Q = \frac{\pi R^3(x)\tau_\omega(x)}{4 K_H} \left[ 1 - \frac{4}{3} \left(\frac{f_H}{\tau_\omega(x)}\right) + \frac{1}{3} \left(\frac{f_H}{\tau_\omega(x)}\right)^4 \right] \]

(4.38)

We assume that the yield value \( f_H \) is so small that condition is satisfied. After neglecting the small term we have equation (4.8) in the form,

\[ Q = \frac{\pi R^3(x)\phi(x)}{4 K_B} \left[ -R(x)\phi(x) - \frac{4}{3} f_B \right] \]

(4.39)

Or,

\[ R^3(x) \left[ -R(x)\phi(x) - \frac{4}{3} f_B \right] = R_0^3 \left[ -R_0\phi_0 - \frac{4}{3} f_B \right] \]

(4.40)

From the equation (4.40) we observe that pressure gradient \( \phi(x) \) increases with decrease in the tube radius \( R(x) \).
REFERENCES


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