CHAPTER - I
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INTRODUCTION

This work as thesis comprise the research work done by me in the pursuance of my Research Topic of the research "MATHEMATICAL ANALYSIS OF BLOOD FLOW IN CAPILLARIES" approved by C.S.J.M Kanpur University, Kanpur. This is a holy attempt to represent my comprehensive research work carried out, during the approved time by the University.

In the contemporary era, the importance of the interdisciplinary approach of studying various branches of the science and technology has been felt in many domains of human beings life. Life and related system are so complex that one requires specialized knowledge of many branches of science to understand certain phenomenon. One can't say that if he is specialized in one branch of science then he can interpret every thing about certain phenomenon without the help of any other branch. For this reason research scholars and
scientists are trying to bring closer many branches of the science and technology and as the fruit of this hard work, from so many interdisciplinary branches, a very important branch was created called as "BIO-MECHANICS". In the recent times this branch has changed the way of dealing the medical problems related to the human beings. Since this branch is related to the welfare of the mankind and existence of the life on the glove so it has attracted so many scientists from all corners of the world.

Flow of blood is the major and vital phenomenon in body of the live species and it can be well understood with the help of the branch of Mathematics "HYDRODYNAMICS" so a very different kind of interdisciplinary approach was devised to study the properties of flowing blood in the capillaries.

The desire of the research scholars to understand the microscopic and macroscopic behaviour of the blood, quantitatively, when it flows through the very fine circulatory system of vessels and other geometrics, has supplied the motivation for the appropriate investigations. As
an aspect of such research, the macroscopic rheological properties of the blood have been studied for a long time. The ultimate aim of such studies is to be able to predict from macroscopic rheological data the microscopic flow behaviour of blood in any part of the circulatory system.

Until a few years ago, most efforts were aimed at obtaining the rheological properties of blood and related fluids, and at determining how these properties varied with changes in such independent variables as red cell volume fraction (hematocrit), temperature, plasma protein concentrations, and related factors. Most of these investigations were organized with blood obtained from people and other animals, apparently in normal health.

In the last few years many scholars and scientists in this area have shifted their emphasis of their work towards attempts to relate the gross rheological behaviour to the behaviour of the constituents of the blood. Therefore, we now frequently see the mechanistic explanations of the gross properties of the blood in terms of red cell aggregation and plasma velocity in the capillary system. Same time we also
concentrate on cell membrane properties during the motion of the blood and the rheological properties of the red cell contents. We also try to see more thorough attempts to study the pathological bloods and to explain non-normal behaviour in mechanistic terms.

Blood is a liquid tissue and consists of a unique, somewhat sticky complex liquid, which is heavier than water. It is alkaline in nature. Blood, a class of fluid, is the transport medium of the body. In man and in all other vertebrate animals with the exception of two (the amphioxus and leptocephalus), it is red in colour. It is the medium by which all living tissues are related to their external environment. Blood is the medium by which growth and repaired substances are transported to the tissues and by which the various controlling glands of the body can distribute their chemical messengers. Blood is pumped from heart between tissues and the elementary canal to take up oxygen, nourishment respectively; while by its aid the products of oxidation and metabolism in the tissues are
returned to the outside of the body by the lungs, the kidneys and the skin.

Blood, the well known red fluid of the blood vessels, consists of a continuous yellowish aqueous phase (about 55 vol. %) the plasma, in which formed elements are suspended. The formed elements consist of red blood cells (Erythrocytes), white blood cells (Leukocytes) and platelets (Thrombocytes). Plasma is essentially a dilute (0.19N) electrolyte solution. It is made of mainly water (92 %) and contains traces of inorganic and organic salts. The inorganic and small organic molecules contribute little to plasma viscosity which is primarily dependent on the plasma proteins. About 7%, by weight, of plasma are plasma-proteins (mainly serum albumins, serum globulins and fibrinogens) which have molecular weight ranging from 44,000 to 1,000,000. About half of the protein mass is albumin, with a molecular weight of 69,000. Organic substances in plasma are glucose, facts urea, amino-acids, hormones and enzymes. Inorganic salts are: chlorides, sulphates, phosphates and bi-carbonates of sodium, potassium,
calcium and magnesium. The proteins and organic substances in plasma are in colloidal solution and inorganic salts are in true molecular solution. The composition of plasma is fairly constant.

**HISTORICAL LANDMARKS IN THE DEVELOPMENT OF THE HAEMODYNAMICS:**

The sudden swing from medieval 'natural philosophy' into modern science is most vividly characterized in the person of **Galileo (1564 – 1642)**. From 1598 to 1610 he was the professor of Mathematics in the Padua. During this period, **William Harvey (1578 – 1657)** was studying medicine there, and although he never mentions Galileo in his book *De Motu Cordis (1628)*, it is difficult to believe that he was not influenced by the new teaching in his approach to the problem of the circulation of the blood. **We think of him as a father of cardiovascular physiology**, for his analysis he resorts to experiments, even though they are simple observational exercises, to aid his argument. The bulk of the book is, in fact, couched in the form of old **Galenical teaching**, a world he had been brought up in and from which he was only with
difficulty freeing him. Nevertheless, his demonstration that the valves in the veins would only allow the flow of blood towards the heart is a great intellectual advance over the views of his teacher at Padua (whose classical work *De venarum ostiolis* is a magnificent first anatomical description of these structures); it was his appeal to a quantitative argument in relation to the cardiac output which entitles him clearly as the first of the moderns. The classical Galenical teaching was that the blood was constantly being created in the liver and absorbed by the tissues to which it was conveyed with an oscillatory motion mainly by the veins. At the end of a long series of observations, in a wide range of species, relating the contractions of the ventricles and the flow of blood into, and consequent dilatation of, the arteries, he finally and rather diffidently offers a quantitative argument. For example, if the ventricle ejects, let us say, half an ounce of blood with each systole, and in half an hour it makes over a thousand beats, then the ventricle will have pumped out 'over five hundred ounces' and this is far more blood than can be found in the whole body. He then
repeats it for small hypothetical stroke volumes, but this takes him into calculations in drachms and is not forceful enough to pursue in these obsolete units of weight. Had Harvey the real figures the argument could have been overwhelming, for the resting cardiac output is very close to one total blood volume per minute in nearly all mammals for which we have reliable figures.

Although Harvey's description of the circulation of the blood (1628) so clearly, to us, opens up the wide field of understanding, but was far from being immediately accepted. For example, Thomas Willis (1621 – 75) in his famous treaties on the anatomy of the brain, and more particularly, its system of anatomizing arteries which bears his name in the circle of Willis (De cerebri Anatome, published 1664), still openly rejects Harvey's idea and clings to the teachings of Galen (129 – 99) which had persisted with little question since the second century A.D. We may be more tolerant of this intellectual stubbornness when we realize that there was no descriptive evidence about the connections between the arteries and veins, the tissues which Harvey had
to postulate. The capillaries were not observed and described until the publication, in 1660–61, of the studies of the frog lung under the microscope of Malpighi (1628–94). Nevertheless, Richard Lower (1631–91), although he had been a pupil of Thomas Willis, fully believed in Harvey's views and developed his concepts on the cardiac output in his *Tractatus De Corde* (1669).

This sketches lightly the state of circulatory physiology when Stephen Hales (1677–1761) first went to the Cambridge in 1696. This was the year that Isaac Newton left the university to become warden of the Mint but it is ten years after he had presented his great work, *Principia Mathematica*, to the Royal society of publication. The concept of the viscosity of a fluid is first defined, so that, to this day, we still speak of Newtonian Viscosity. Hales did devote some of his time to physics, more particularly astronomy, but his importance to the science is not to any theoretical advances but as a great biological experimenter. Although it is his great series of investigations of circulatory function that particularly concern us here and which were
published as the volume *HAEMASTATICKS (1733)*, these were in fact, the second volume of the statistical essays, and the first subtitled vegetable Staticks (1727), is concerned with a thorough investigation of the forces necessary to get the sap to the very tall trees, his conclusions on this are amazingly complete and have formed the basis of all work to this day. In addition, he devotes a lot of attention to the force problems associated with respiration in vertebrates. Why the circulatory experimental findings were not given priority in publication is not clear, but in his biography of Hales, *Clark-Kennedy (1929, 1965)* demonstrated quite clearly that they were started in his letter years at Cambridge (1703 – 9) although they were mainly done in the years 1709 onwards, when he had just left Cambridge to take up his duties as Curate at Teddington, Middlesex.

To the most physiologists, *Stephen Hales* is only remembered as the man who measured the arterial blood pressure in the horse. This is found in the description of his experiment in *HAEMASTATICKS* and was republished by *Fulton (1966)*. Hales also made the first real advance in
computing the cardiac output since the tentative speculations of Harvey and Lower. Hales also deduced from the change in the pulsatile flow in the arteries to the steady flow in veins that this ‘smoothening’ action is largely due to the distensible properties of the arteries comprising an ‘elastic reservoir’ during ventricular systole. (In the first German translation, in 1784, of HAEMASTATICKS, this phrase is rendered as “windkessel”, thus originated the term which figures so largely in the Haemodynamics of the first half of the present century due to the large body of work by Otto Frank and his pupils). Hales was the man who introduced the concept of the peripheral resistance and realized the main site of this resistance was in the minute vessels in the tissues called capillaries. To extend this concept he then showed that various agents, e.g. water at different temperatures and in particular brandy, could change the rate of flow, presumably by altering the size of these small vessels. It was not until about 1870 when Claude Bernard did his classic experiments on the vessels of the rabbit ear that the concepts of the vasoconstriction and vasodilatation were
much advanced. Truly, Stephen Hales deserves the title “The Father of Haemodynamics”.

The eighteenth century was notable, in our present context, for the great development of the theoretical mathematical treatment of the fluid dynamics. The great name in this field is that of Leonhard Euler (1707 – 83) who is described as the most prolific Mathematician that ever lived. In our field of interest, he is important as the author of Euler’s equations of fluid motion. He enormously developed the power of the Newtonian calculus but he did not develop the Newtonian concept of viscosity, presumably because of the difficulty of introducing it into the universal laws of motion. Euler was a close friend of the Bernoulli family and particularly of Daniel Bernoulli (1700 – 82) who also devoted his life to Mathematics. His great contribution was the treatise of Hydrodynamica, from which we still use Bernoulli’s law. The treatment of the fluids in the work of both Daniel Bernoulli and Euler was of “Ideal” liquids (i.e. without viscosity), and little attention was paid to practical problems. This is particularly curious in the case of Daniel Bernoulli who
graduated initially in medicine and held the chair of Anatomy at Basel for many years. Courmand (1964) in his introductory essay to Stephen Hales facsimile reprint says, that between 1733 and 1738 he enumerated the principles of the correct calculation of the work of the heart and compared the laws of fluid flow in the tubes and in living vessels. To this day large part of theoretical hydrodynamics is concerned solely with ideal, non-viscous liquids; some of the practical results of this are discussed by Birkhoff (1960).

The work which actually led to the equations of pressure and flow was scattered rather haphazardly by a few experimentalists culminating in the classic and meticulous work of Jean Louis M. Poiseuille (1797 - 1869) which established the relationship of the flow of a viscous liquid and pressure gradient in a tube. The mathematical derivation of this flow problem was not made until nearly twenty years later. In the mean time, stokes had extended the equations of motion of Euler in 1845 and added on another set of terms to allow for viscosity to form what are now called the Navier-Stokes equations. Nevertheless the
approach here was still highly theoretical for the derivation that Hagenbach made of the Poiseuille problem makes no use of the Navier-Stokes equations whatsoever. **Thomas Young (1773 – 1829)** studied the nature of elasticity and in particular the relation between the elastic properties of arteries and velocity of propagation of the arterial pulse.

This brings us to the threshold of our present era in Haemodynamics. It had been made more convenient by Poiseuille’s introduction of U-tube mercury manometer with the addition, by **Ludwig**, of the float and pointer which inscribed a smoked-paper **Kymograph**. The development of **manometers** which could measure pulsatile pressure had to await **Otto Frank’s (1865 – 1944)** great work of 1903, with all the theoretical developments that were to follow. With regards to flow, it has become traditional within the last twenty five years to accept the statement that pulsatile flow could not be measured in arteries. **E.H. Marey (1830 – 1904)**, in his famous text book in 1881, gave full chapter on velocity of blood in arteries.
The early 20th century was completely dominated by Otto Frank and his pupils. In 1939 paper of Wezler and Boger was devoted to the distinction between the elastic and nonelastic arteries. McDonald and Taylor (1959) give a fuller account in their sketches of pressure flow in elastic arteries. Aperia (1940) even had shown considerable interest in windkessel formulations.

The main developments since 1950 have been in terms of treating the whole arterial system as being in a steady state oscillation produced by regularly repeated beat of the heart. In 1952 Womersley and McDonald had formulated on the basis of Fourier series. After coming of digital computer in 1960 and beyond, the Blood flow in arteries and Capillaries gained the momentum as academic interest.

**BRIEF DESCRIPTION OF THE BLOOD:**

Blood is a very highly concentrated suspension (about 40 – 45% in volume fraction) of a variety of cells, red cells (RBC), White cells (WBC), Platelets (alternatively called Erythrocytes, Leucocytes & Thrombocytes respectively) are
suspended in a continuous phase called plasma. Plasma is an aqueous solution of electrolytes and organic substances mainly proteins. Relative proportions of cell elements and concentrations of plasma contents in blood are shown in table no. 1.1 given below.

Table No. 1.1 – Blood constituents (5x10^6 particles/mm³)

<table>
<thead>
<tr>
<th>Cell Elements</th>
<th>Relative proportions</th>
</tr>
</thead>
<tbody>
<tr>
<td>White cells (all kinds)</td>
<td>1</td>
</tr>
<tr>
<td>Platelets</td>
<td>30</td>
</tr>
<tr>
<td>Red Cells</td>
<td>600</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Plasma</th>
<th>Weight Fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>0.91</td>
</tr>
<tr>
<td>Inorganic Salt</td>
<td>0.01</td>
</tr>
<tr>
<td>Proteins</td>
<td>0.07</td>
</tr>
<tr>
<td>Other organic substances</td>
<td>0.01</td>
</tr>
</tbody>
</table>
PLASMA PROTEINS AND COAGULATION

Plasma proteins (table no - 1.1) are mainly albumins and globulins which control water exchanges between blood and tissues by their action on osmotic balance. The remainders are lipoproteins and fibrinogen. The latter plays a very important role in the clotting mechanism of the blood. However, under the normal conditions, and in association with β-globulins, fibrinogen is closely involved in the reversible aggregation of RBC which forms Rouleaux. We will see later on that it is one of the fundamental determinants of the blood rheology.

Whole blood solidifies in the presence of air oxygen, or coming into contact with extracorporeal surfaces. The formation of the clot would be observed after an early increase in viscosity (Dintenfass, 1971)[24]. In vitro, this clot mainly consists of a complex structure of RBC and fibrin which derives from the plasma fibrinogen by a chain reaction. Platelets play a predominant role in clotting under in vivo conditions. After complete formation, a contraction of the clot (Copley, 1960)[21] begins, which resembles syneresis,
a similar retraction observed during gelation process in colloids. After the clotting is achieved, the extra-fluid is called serum, which is roughly plasma without fibrinogen.

**NATURE OF PLASMA PROTEINS:**

If blood is allowed to clot and the solid material be removed, the remaining fluid (serum) has essentially the same composition as plasma except that fibrinogen and some of clotting factors have been removed. The multiform nature of the plasma proteins are described below:

(a) Plasma proteins produce a definite Viscosity of plasma which is important in maintaining blood pressure

(b) They are responsible for osmotic pressure, the level of which is important for regulating water exchange between blood and tissues.

(c) They promote stabilization of the blood by providing conditions that prevent sedimentation of erythrocytes.

(d) They posses buffer properties and maintain the acid base Equilibrium of blood.

(e) They are important factors in immunity.

(f) Plasma proteins play an important role in coagulation.
Human blood which is slightly alkaline (pH $\approx 7.4$) is a suspension of particles in a complex aqueous continuous phase. In normal blood cellular components other than red cells compose only about 3% of the total cell volume and usually have only a small effect by viscosity. The red cells outnumber the white cells and platelets by approximately 5000:1 and 12:1, respectively. A suspension of leukocytes shows higher viscosity than that for the same volume fraction of the red cell suspension (Steinberg and Charm) [84]. Further white cells may significantly increase blood viscosity in diseases where there is large increase in leukocyte concentration (Dintezfass)[22].

Blood flows in the form of different laminar containing different types of cells. Considering these factors, the rheological properties of blood might be expected to be rather complex. When a test tube filled with the blood (to which an anticoagulant has been added) is centrifuged, the corpuscles tend to settle at the bottom. Therefore the blood is divided into two layers: namely a transparent slightly yellowish or colour less upper layer consisting of plasma and
a red lower layer consisting of the farmed elements. The white blood cells (Leukocytes) and the plasma have specific gravity is less than that of the erythrocytes.

The erythrocytes are about 95% of the total cell volume in the blood. Therefore, the removal of white blood cells and platelets from blood does not measurably modify the experimentally determined flow properties of this suspension. Because of this, and because platelets specially can cause experimental difficulties in the viscometers, most blood rheological studies are performed with suspension which have most of the white cells and platelets removed.

The concentration of most red blood cells (RBCs) in suspension is generally reported as the suspension volume fraction occupied by the red blood cells, which is called Hematocrit. The hematocrit is normally about 42-45% by volume at the sea level.

**THE RED BLOOD CORPUSCLES (RBC)**

The red cells, also called Erythrocytes, consists of a thin, flexible, but essentially unstretchable envelope or membrane, and an interior filled with a complex aqueous
solution, which is nearly a saturated Hemoglobin solution (Fig. 1.1), (viscosity 6.0 centipoises). The red cell is not in the spherical shape. In blood, which is not flowing the red cell has a biconcave discoid shape, with a major diameter of 8.1 microns and a maximum thickness of about 2 microns, so they may be deformed forever into a bullet shaped entity during passage through small capillaries. The membrane is specialized structure which serves as a barrier to the transfer of the macromolecules, is the site of metabolism-linked transfer mechanisms for particular chemical species, and it permits some small molecules, such as water, to transfer very rapidly. Thus, the erythrocytes are a complicated type of Osmometer. In addition, the red cells (as well as the protein molecules) all carry a net Negative charge. The red colour of the red blood corpuscles is due to the presence of hemoglobin, which is an iron-containing protein compound. Hemoglobin has the characteristics property of combining with the oxygen to form an unstable compound, called oxy-hemoglobin. This compound decomposes and gives oxygen to the tissues of the body. The red blood
corpuscles are the re-oxygen-carrier of the blood. These have elastic walls and can squeeze through narrower passage. The membrane is highly deformable, if the changes in surface are small, but becomes very much stiffer if the deformation produces a large change in the area of the cells membrane. Prime function of the RBCs is the transport of $O_2$ and $CO_2$ through the body and buffering the blood so as to regulate pH. The RBCs are heavier than plasma. In non flow blood, the RBCs therefore tend to settle. The red blood cell has a large surface area relative to its volume. In the normal flowing blood the red cells aggregates and as a result of these aggregation **6-10 red cells in a stack**, such a primary aggregation is called "**Rouleau**". Secondary aggregates and "**Rouleau**" break up, and at sufficiently high shear rate, the cells exists as individuals. If the shear rate again becomes about zero, these aggregate structures reform very rapidly.
WHITE BLOOD CORPUSCLES

White Blood Corpuscles are fewer in number. These are irregular in shape and nucleated. Their number also varies as they can pass out from blood capillaries and enter the lymph. Leukocytes have short life and disintegrate very quickly. The protoplasm of WBCs form pseudopodium and each behaves like an independent and complete
organism. They show all the properties of life and respond to the external stimuli. White blood corpuscles also take food from surrounding old and dead cells or any other foreign matter in the blood. They also serve to ingest fast globules. Leukocytes are of two types: granulocytes and agranulocytes.

THE BIRTH AND DEVELOPMENT OF THE RED BLOOD CELL:

The birth place of the erythrocyte and all other formed elements of the blood, except the lymphocytes, is the bone marrow and that all the protein synthesis necessary of its future existence, takes place before it is mature. This includes the manufacture of the hemoglobin. Its shape and deformability allow it to travel fast through the larger vessels or slip easily through the narrowest sinuses or finest capillaries. Nevertheless it survives an average 120 days and normally fulfils its function during this time with perfect efficiency. As the red blood cell ages, the enzyme systems start to bread down. It becomes more rigid and can no longer sustain its journey through the vessels and organs of the blood system.
It fails at last in the demanding environment of the spleen and entrapped by the very sinus pores through which it passes so easily at the start of its existence. Finally it is consumed by macro-phases. But even as the last traces of the red blood cell disappear, its successor, is already taking its place in the bone marrow. Within the bone marrow a complex network of venus sinuses radiates out from a large central sinus. Nutrient arteries also send out finer vessels and these are believed to drain into the sinus network after passing into the surrounding bone. A discontinuous layer of adventitial cells line the sinuses and within this structural matrix lie all the blood cell precursors including the primitive stem cells.

It is with these undifferential stem cells that the life history of the erythrocyte begins. Stem cells can be divided into two classes. The first forms are reserves which are not committed to the production of any specific blood cell type. The second group derived from the first is already programmed and gives rise to the cell lines from which platelets, leucocytes and the red blood cells are formed.
Erythroproteins secreted in the kidney under conditions of relative hypoxia initiates red cell production. The proerythroblast is the first recognizable erythroid cell and this undergoes three to four mitotic diversions. In this process there is a stepwise reduction in the size and the nuclear protein condenses to give a dense pyknotic nucleus incapable of further DNA synthesis. These morphological changes are accompanied by intense biochemical activity. Ribosomes linked by the messenger mRNA, manufacture those proteins necessary for the development of the red cell. These include membrane proteins which act as receptor sites. Some of these are specific for iron. Iron provided from iron, changed transferring gains entry through receptors and transformed to the mitochondria. Haeme is synthesized by combing iron with protoporphyrin. The presence of haeme has an effect on mRNA transcription of the nucleus. Ribosome linked by this mRNA manufacture alpha and beta globin chains. These link up to from the hemoglobin molecule which provides the reason for the red blood cell's existence.
Each mature red cell contains approximately 300 million hemoglobin molecules.

It is now a mature red cell, concerned only with the protection of its vital cargo and with; its bio-concave shape which maximizes its surface area/volume ratio is perfectly adopted to perform its primary function, the delivery of oxygen and the removal of carbon-di-oxide.

**BEHAVIOUR OF RBC:**

Human RBC can be considered as a partially inflated balloon (Brochard, 1977), thus having a very high level of deformability, which decreases with the age of the cell. Internal fluid is a hemoglobin solution (35 gr / 100 ml), the viscosity of which is about 6 to 8 CP (figure 1.1 and figure 1.2 below). RBC membrane consists of biomolecular leaflet (a lipid bilayer, with inclusion of anchord proteins) supported by a more rigid skeletal structure (the actin – spectrin network). RBC membrane carries a negative electrical charge, equivalent to about 6,000 electronic charges per cell. The RBC biconcave discoid shape (in the case of mammals) results from the ejection by the cell of its nucleus on entering
the circulation. This leads to a volume reduction of the cell, its area remaining constant. RBC dimensions are greater than the critical ones for which Brownian motion becomes important. Therefore, under steady conditions, blood settling occurs. After one hour, a plasma column above the settling RBC is formed, although these sedimentations effects can be neglected when blood flows very rapidly, they are more and more important as flow rate is decreasing. Change in the osmotic conditions (Normal ones corresponds to those of an aqueous solution containing about 0.9% by weight of NaCl) induces drastic change in the shape of RBC. Indeed, RBC becomes spherical (and then is called Spherocyte) after swelling in the solution with lower NaCl concentrations than normal ones. Further swelling leads to hemolysis of the cell which eliminates hemoglobin towards the plasma, the result being ghost, i.e. a RBC reduces to its membrane only. Under higher NaCl concentrations, on the contrary, the RBC volume decreases, this leads to the formation of Echinocytes, the membrane structure of which likely involves the actin-spectrum network.
Rheological Properties of the Erythrocyte Membrane:

One of the consequences of the red cell’s rheological properties is the existence of perpetual rotating movement of the membrane round the hemoglobin. Sugihara and Seki (1996) [87] have clearly observed that the movement, which allows stresses to be transmitted through the membrane, consequently encouraging transfers (mainly oxygen).
THE ELASTIC BEHAVIOR OF RBC:

The elastic behavior is generally observed in red cells. The amount of deformation a cell undergo, depends on the outside forces consist of the stresses produced on the membrane by the outside flow. In vivo, this will in fact consist of plasma viscosity and flow conditions. Intrinsic deformability is dependent on the membrane structure and on the protein network on the inner face of the membrane. The deformation of red cells allows the blood to remain fluid up to hematocrit of 98%. Rigid cell without deformation will cease to flow at cell concentration of about 60%. The fall in viscosity with increasing shear rate is also manifestation of deformation of erythrocytes (Murata [41]). The hematocrit also influences the deformation of the erythrocytes i.e. raising the cell concentration produce an increase in cell deformation ([Goldsmith] [35]) and therefore fall in viscosity of blood occurs.

At low flow rates the presence of red cell aggregates strongly influences the viscosity of the blood. The size of the rouleaux and aggregates progressively decreases.
as the shear rate increases and this produce the typical shear thinning, seen in blood at low shear rates. In normal blood, the disaggregation is probably complete at shear rate about 50 sec\(^{-1}\) (Chein et. al. [15]). The build up of a three dimensional structure of interacting aggregates suggests that blood may show a yield stress. Also the cell flexibility influences blood viscosity, particularly at high hematocrit values. At high shear rates of about 50 sec\(^{-1}\), hardened cells show a shear thickening behavior (Singh and Gupta [79], Chein et al. [15] and Dintenfass [23]). Apart from temperature viscosity relationship, it is observed that the osmotic pressure difference also influences the blood viscosity. This determines the shape, size and elasticity of the red cells a large extent.

**GENERAL PROPERTIES OF BLOOD:**

Since RBCs compose about 97% of the total cell volume, therefore whole blood properties are (Rheologically) dominated by the RBC properties except some cases of the diseases like Leukaemia, in which very large quantity of white cells increases both viscosity and shear rate dependence of the blood (Dintenfass, 1965 [23]).
(i). COAGULATION OF THE BLOOD:

Blood maintains its fluidity in the blood vessels throughout its life but rapidly becomes solid when shed. These characteristics of blood are essential for the preservation of life. The maintenance of fluidity is necessary for the circulation of blood, whilst the solidification of the shed blood provides an indispensable defense against excessive bleeding from wounds. The coagulation of the blood is due to formation of a jelly in plasma by the deposition of protein material called 'fibrin' and it is the formation of this body that the fundamental change in blood clotting occurs. Coagulation of the blood is due to the Plasma. The injured tissue produces a substance, called thromboplastin or thrombokinase. It is in the presence of calcium salts, which are also present in the plasma, which changes prothrombin in the blood plasma into the active enzyme, called thrombin or thrombase. Thrombin change fibrinosin into fibrin, which become a mass of fine fibers. These fibers entangle the red blood corpuscles, so as to form
the clot. The clot latter on shrink and squeezes out of blood-serum

Considering these factors, the rheological properties of blood might be expected to be rather complex. This is the case, and the complexity grows when one examines pathological bloods. Fortunately, there is no experimental evidence showing that the rheological properties of blood are history or time dependent and no normal stress effects have been demonstrated.

It must be noted that when blood is withdrawn from its natural environment, certain irreversible chemical and mechanical reactions occur, the cumulative effect being called “Clotting”. To prevent this from occurring various Anticoagulants are added to the blood as it is drawn from the animals. The effect of an Anticoagulant on the rheological properties of the blood are difficult to ascertain because of the difficulty in making reliable rheological measurements in the short time available between the blood drawing and the onset of the blood clotting. The best study of this problem is probably that of Frasher, et al (1968) [34].
who found that the natural anticoagulant **Heparin** had no effect on the flow properties of the blood of the dog. There is some evidence (*Meiselman effect, 1965*) that calcium—chelating anticoagulants slightly affect high shear rate behaviour, but that low shear rate properties are unaffected. The effect of anticoagulants on blood viscosity appears to be small (*Cokolet et al [17, 18], Charm et. al. and Kurland [14]*).

Hematocrit is the volume fraction of red cells plus trapped plasma obtained by the centrifuge of a volume of the whole blood prevented from clotting by addition of the anticoagulants. Blood rheological properties have been found free from any effect that would be induced by these anticoagulants. The Hematocrit value ‘H’ is normally a little greater than the true volume fraction $\Phi$, since a small volume of plasma is trapped between the cells and it is advisable to use a corrected hematocrit $\Phi = 0.96H$ in the normal conditions (*Chien and Usami, 1971 [15]*). Never the less, this correction strongly depends on RBC deformability and becomes very important in the case of rigid particles for
which packing volume fraction is about 0.60. Such a value for correction factor was found in the case of hardened cell suspension (Chien and Usami, 1971 [15]).

(ii). THE VISCOSITY OF BLOOD:

Viscosity is defined, for fluid, as the ratio of shear stress to shear rate (or velocity gradient). Viscosity is the property of a fluid that determines its resistance to flow. An understanding of the viscosity of blood and of the factors that influence blood viscosity is important due to various reasons. The clinical importance of blood viscosity as a parameter lies in its sensitivity to small variations in composition. One can often diagnose pathological states by detecting a change in blood viscosity. Blood viscosity is an important factor in determining the local pressure variation through the cardiovascular system which in turn influences the local flow rates through each section of the vascular network.

Blood viscosity depends on several rheological parameters (e.g. hematocrit, blood cells, plasma etc.). Plasma viscosity depends on its protein concentration and
viscosity increases with increase in protein concentration. The different types of plasma proteins have different influence on plasma viscosity according to their shape and size. The largest in the plasma protein is the fibrinogen. The influence of Fibrinogen on plasma viscosity can be seen in the difference between plasma and serum viscosities. Serum usually has a viscosity which is 20% less than the plasma. Influence of the globulins on viscosity is illustrated in the disease macroglobulinemia (Somer, T. [81]). The smallest in the plasma proteins is albumin, which contributes in the largest concentration. Variation in the albumin structures has the least effect of the three proteins on plasma viscosity, but the substances make an important contribution to plasma viscosity through its high concentration. In plasma many relationships have been suggested to express blood viscosity as a function of cell concentrations, plasma viscosity and shear rate. As the temperature increases, the viscosity of plasma and blood (like other liquids) decreases. Observations showed are made at constant temperature 37°C (the deep body temperature). Blood viscosity has
been examined by several authors like Bugliarello et al [10], Rand et al [71] and Sugihara [86].

When apparent blood viscosity is represented as a function of shear rate, high viscosity at low shear rates is observed caused primarily by the formation of rouleaux of red cell aggregates. Viscosity decreases rapidly until it becomes practically constant at high shear rates. The high viscosity observed at low shear rates is particularly important in pathologies with low blood flow rates (e.g. venous flow with stagnation).

**Apparent blood viscosity depends primarily on the following factors:**

1. The cell volume concentration (parameter similar to the classical hematocrit value)
2. The reversible aggregation of the red cells.
3. The mechanical properties of red cells.
4. Plasma viscosity (Fibrinogen and Albumin).

**HYPER VISCOSITY SYNDROMES:**

Pathological changes are one of the factors controlling blood viscosity and the resulting clinical symptoms...
constitute the hyper viscosity syndromes. The field of hyper
viscosity syndromes concerns the situations when the
increased blood viscosity and the accompanying
modifications in flow resistance must be considered as being
the result of the overall rheological behavior of blood. In this
general context, hyper viscosity syndromes can be divided
into 4 main groups:

(a) Increase of number of blood cells (mainly RBC)
(b) Increase in the plasma proteins concentrations or
the appearance of high amounts of a
monoclonal protein
(c) Increase in the internal RBC viscosity or a change
in the rheological properties of the RBC
membrane.
(d) Increases in erythrocyte aggregation

Considered from a hemodynamic view point
appearance of a hyper viscosity syndrome could (by "feed
back" mechanisms) enhance the phenomenon and
decrease the local blood flow or even stop flow completely,
thus making ischemia and thrombosis easier.
(iv). THE UPTAKE AND RELEASE OF OXYGEN IN THE HEMOGLOBIN MOLECULES:

The newly mature red blood cells move along the major veins towards the heart and then to the lungs. In the thin walled pulmonary capillaries this encounter the high oxygen tension necessary to charge the hemoglobin molecule. The uptake of oxygen has an effect on the molecular configuration of hemoglobin.

Before the first oxygen molecule is accepted the hemoglobin molecule must be exposed to a relatively high $O_2$ tension. When this point is reached the binding of $O_2$ haeme initiates a process of molecular rearrangement that molds it progressively easier for hemoglobin to accept $O_2$ molecules. The deoxygenated and the oxygenated forms have been described as the "tense" and "relaxed" states of hemoglobin. The lighter the partial pressure of $O_2$ encountered the greater the proportion of hemoglobin which will exist in the relaxed state.

In the tissue, hemoglobin encounters relatively low levels of $O_2$ and relative high concentration of the organic
phosphate, Diphosphoglycerate under these circumstances
O₂ is released and hemoglobin tends to revert to the "tense"
structure. Diphosphoglycerate has a stabilizing effect on the
deoxygynated form and enhances the ability of hemoglobin
to lose O₂. The sigmoid shape of the O₂ dissociation, hence
effects these sequential changes. The configuration of the
hemoglobin molecule accounts for the case with which O₂ is
taken up in the lungs and unloaded in the tissues.

During vigorous muscular effect there is a rise in
the blood temperature, an increase in blood acidity and a
rise in the concentration of certain organic phosphates.

Under these circumstances the dissociation of O₂
from hemoglobin is enhanced, shifting the curve to the right.
This response makes O₂ more readily available in the region
of rapidly metabolizing tissue.

**(v). THE CIRCULATORY SYSTEM:**

Blood circulatory system is responsible for many
complex functions (respiratory, excretory, nutritive,
protective, and regulatory) in the human body. The blood
vessels comprise of arteries, veins, capillaries and sinusoids.
The heart that provides the energy for the circulation has four components: namely left - right atrium and left - right ventricles which are interconnected to each other by one way valves. Blood coming from body tissues enters the right atrium through the venacavae. Contraction of the right atrium forces blood past the tricuspid valve into the right ventricles. From this point there begins two sub-divisions, viz., the pulmonary and the systemic circulations. As the term imply the former services the lung and the latter various systems of the body.

The food, which is indigested by the human and oxygen should always be available for metabolism to all the parts of the body and waste should be constantly removed. This is very essential for the life. The digested food materials reach to the different parts of the body by the transport of substances, commonly called circulatory system. **Circulatory system includes blood, heart and blood vessels.** The heart is provided with muscular walls that contracts periodically to pump the blood through the body and blood vessels are tubular and are responsible for the movement of the fluid.
Heart beats rhythmically and automatically. However, it can be influenced by several factors, such as temperature, nervous-control, presence of certain salts and hormones in the blood. In heart, there is a special constricting tissue, which is responsible for the instigation and regulation of the heart beats. This specialized tissue is known as nodal tissue. This nodal tissue conducts impulses to the ventricles. Therefore, all the parts of the heart contract simultaneously. The nervous control of the heart-beat is due to the sympathetic and parasympathetic sets of fibers. The beating of heart includes contraction or systole and relaxation or diastoles. The normal rate of the heart beat is about 70 to 80 beats per minute. However it depends upon many factors, such as age, sex, fever, metabolic rates etc.

While blood perfuses all body, most of the circulation of a person at rest is delivered through the major organs. The liver, kidneys, brain and heart are supplied with as much as 60% to 65% of the blood circulation at rest. The internal diameter of blood vessels in human circulatory system lies in the range 2.5cm in aorta to about 4 microns in
Blood flows in a large number of interconnected tubes of varying diameters and lengths in the circulatory system thus blood is pumped from the heart into the aorta from where it goes to the circulatory system consisting of about 40 large arteries, 1600 main artery branches, 1,800 terminal branches, 4,00,00,000 arterioles, 1,20,00,00,000 capillaries, 8,00,00,000 venules, 1,800 terminal veins, 1,600 main venous branches, 40 large venous branches 40 large veins and them return to the heart through the venacavae.

Blood flows from the high pressure area (arterial blood) to the low pressure area (venous blood) after flowing through the thousand of vessels. It is estimated that the heart muscles itself consume about 10% of the energy required to sustain life. Only a part of this energy goes into the mechanical work of pumping blood. Blood flow is pulsatile in larger blood vessels, but in smaller blood vessels and capillaries it approaches steady flow condition. The pressure in the aorta rises rapidly to its maximum (systolic) value of about 120mm Hg. As the left heart relaxes, the pressure in the
left ventricle falls below the aortic pressure to a low value (diastolic) of around 80mm Hg. The requirement of the circulation is the supply of O$_2$ required for metabolic processes. In an average man, 200ml/min O$_2$ is required. Under physical stresses the need may rise to 5lit/min. Blood has a capacity of 200ml of oxygen per litre. The efficiency of transfer of O$_2$ from blood to the cells is such that for an average man it is necessary for the heart to circulate 5 to 6 litres of blood per minute or about five times the metabolic requirement of oxygen. The metabolic requirements are the energy requirements for biological functions. Some main features of circulatory system are as described below:

(a) Blood flows in the cardiovascular system through a large number of interconnected tubes of varying diameters and lengths.

(b) Blood flow is pulsatile because of the periodic nature of pumping by the heart.

(c) Tubes are not rigid, but are elastic and the walls are made of many types of elastic materials.
(d) Tubes are not of uniform diameters, but the diameters of tube may change along its length.

(e) Tubes are not straight, but are curved.

(f) A tube branches off into two or more smaller tubes at many junctions.

(g) Blood is not homogeneous fluid, because it consists of plasma with suspensions of RBC, WBC and platelets. Some of the tubes in which it flows have diameters less than the size of a RBC, so that the cell gets deformed during the motion.

(h) Blood is neither a Newtonian fluid nor can its stress strain-rate relation be expressed by a simple formula.

From fluid mechanical point of view the circulatory system can be broadly classified into two parts, viz., macro-circulation and microcirculation defined as:

**(vi). MACROCIRCULATION:**

The flow in the vessels, whose diameter is greater than 500 \(\mu m\) \((1\mu m - 10^{-6})\), is characterized by macro-circulation. Generally such type of flow occurs in aorta's
main arteries and terminal branches. The flow is mainly characterized by high Reynolds number. The Reynolds number is defined by the ratio of inertial to viscous forces. The governing equations, to analyze such flow conditions, should include inertial effects, effect of curvature of blood vessel, pulsatile flow and the distensibility of the vessel wall. Turbulent flow can occur in blood vessels for the Reynold's number greater than 2300. Blood can be considered as a homogeneous and continuous fluid if the vessel diameter is large compared to the pulsatile flow. Effects are the most widely studied as problems in macro-circulation by Fung [33], Jones [42], Skalak [80], Taylor [90]. The flow distribution at bifurcation bends etc. and their effects on pathological states have been studied by Bergel [7], Patel et al [67].

(vii). MICROCIRCULATION:

Microcirculation can be said to be the flow of blood at the micro-circular level. The blood flow in the vessels of diameter less than 500µm (such flow occurs in arterioles, capillaries and venues) is called microcirculation. Microcirculation may be in pulmonary capillaries or in system
capillaries. In the latter case, the flow is unsteady, the number of red cells in a capillary varies widely and the red cells are substantially deformed in passing through a capillary. Blood flow in microcirculation is very important because, it is responsible for about 80% of the total pressure drop in circulatory system. Also, the transfer of nutrients and removal of wastes from the living cells of the body occurs in the capillary bed, which is a part of microcirculation (Chien et al.) [16].

(viii). BLOOD FLOW IN CAPILLARIES:

Blood flow in capillaries is of great interest to physiologists involved in micro-vascular research. The fluid mechanical and biomechanical processes which occur during the movement of an erythrocyte through the human of a capillary are very complex. In the capillaries, the total vascular cross-section is 700 to 800 times greater than the cross-section of the aorta; even through the area of a single capillary is tiny. In the microcirculation regions which are of particular interest to the angiologists, blood can no longer be considered as a continuous fluid and here the intrinsic
rheological properties of the cell become all important. Study of microcirculation give us the vital information about the health of the tissue under study. Each organ has got a typical micro-vascular structure. The basic flow pathway is common to all organs. There is the difference lying in the branching pattern and the type of the capillaries. Capillaries are the vessels made of endothelial cells surrounded by a basement membrane. These vessels are devoid of smooth muscle cells and hence are incapable of active vasomotion. Study of microcirculation is also important in delivery of drugs to the specific sites and plays a key role in the successful healing of wounds (Puniyani et al [70]). Blood flow in capillaries is characterized by very low Reynold's number \((10^{-3})\) due to smallness in tube size \((3.5-10\mu m)\) and flow velocity \((= 1mm/sec)\). The treatment of flow as homogeneous does not seem to be reasonable as the red cells are large in their bodies when compared to the capillary dimensions, being larger in their under-formed state than the internal diameter of capillary.
In recent years many research works have been done for better understanding of the various aspects of microcirculation (Brabee [5]), Bugliarello and Sevilla [10], Cokelet [18], Fitz Gerald [31], Goldsmith and Skalak [36]). However, as the size of red blood cell is not negligible compared to vessels diameter, it becomes necessary to treat the flow as two phase non-homogeneous flow. This two phase nature of blood is responsible for anomalous behaviour of blood through narrow tubes (capillaries). The problem is generally analysed by either high speed photography or by model studies (Branemark [9], Gross & Araesty [37] and Sutera and Hochmuth [88]). In microcirculation, viscous effects are dominant over inertial forces. Consequently, the average velocity of blood is very small (0.01 – 0.07 cm/sec). In this case, the effect of non-Newtonian nature of blood becomes very important (Oka [62]).
(ix). FACTORS AFFECTING BLOOD FLOW IN THE MICROCIRCULATION:

An examination of the microcirculation reveals that the normal blood flow is extensively rapid in the arterioles, and somewhat slower in the collecting venules. Reunifying between these vessels are the capillaries, which are so fine that their caliber is less than that of red cells which must travel through them.

Significant changes of shape are needed as the cell traverse these pathways. Yet under normal circumstances the considerable deformability of the red blood cell allows them to make passage without difficulty. **Red cell deformability** is dependent of the integrity of the cell membrane and it maintains a favorable ionic and osmotic balance. This requires energy.

(x). EXPERIMENTAL EVIDENCES IN THE BLOOD-CIRCULATION:

The table given below provides the main flow characteristics of the human microcirculation (adapted from Sutera, 1977 [89]):
Table No. 1.2 – Blood flow in human microcirculation

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Vessel Diameter (µm)</th>
<th>Diameter ratio $\xi = 2R/2a$</th>
<th>Blood(°) velocity (cm/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.</td>
<td>Small Arteries</td>
<td>70 – 500</td>
<td>10 – 60</td>
</tr>
<tr>
<td>3.</td>
<td>Arteriole</td>
<td>10 – 70</td>
<td>1 – 10</td>
</tr>
<tr>
<td>4.</td>
<td>Capilla</td>
<td>4 – 10</td>
<td>2 – 1</td>
</tr>
<tr>
<td>5.</td>
<td>Venules</td>
<td>10 – 110</td>
<td>1 – 14</td>
</tr>
<tr>
<td>6.</td>
<td>Small veins</td>
<td>110 – 500</td>
<td>15 – 60</td>
</tr>
<tr>
<td>7.</td>
<td>Terminal Veins</td>
<td>500 – 5000</td>
<td>60 – 600</td>
</tr>
</tbody>
</table>

(°) – Average peak values

Experiments on steady blood flow exhibit four "anomalous" features, namely the blunting of velocity profile, the formation of the plasma layer, the Fahraeus Effect and lastly the Fahraeus-Lindquist effect. These are discussed as,
(a) A blunted velocity profile is observed near the axis of the vessel, leading to a plug flow either at high hematocrit or at low value of the vessel to cell diameter ratio, $\xi = 2R/2a$. Such a blunting is shown in the fig no. 1.4 of human blood flowing through a narrow slit (Dufaux et al., 1980) $H = 20$ and $40$ (the data has been obtained by laser Doppler velocimetry). Such results, accounting for the fact that higher flow rates $Q$ values are obtained under the same pressure gradient $P$, are comparable with similar results on the flow of rigid sphere suspensions through tubes (Karnis et al., 1966 [46]). Nevertheless, the independence of velocity profile against flow rate (i.e. linearity of the pressure flow rate relation) observed by these authors seems to result from a too short range of flow rate variations. Conversely figure no. 1.5 shows this non linearity in the $P$-$Q$ relation, i.e. non-Newtonian effects in the apparent viscosity.
Fig No. 1.4

Velocity profiles in a narrow slit (1 x e = 200 µm x 1.1 cm)
Normal blood at H = 0.20 and 0.40 (from DUFAUX et al., 1980)
Fig. no. 1.5

Pressure – flow rate relation of blood flow in a narrow slit \((l \times e = 1.1 \text{cm} \times 350 \mu\text{m})\) at various Hematocrit. \(\sigma_w = Pe/2\) and \(\dot{\gamma}_0 = 4Q/le^3\) (\(P = \text{pressure gradient} = P/10 \text{ cm}; Q = \text{flow rate}\)).

(From DUFAUX et al., 1980)

(b) In very narrow tubes \((3 < \xi < 50)\), the existence of the marginal layer near the wall is currently accepted, leading to consider the flow as two phase, one with a particle rich axial core surrounded by a particle depleted (or a particle less) with layer. Such existence is now well established after...
Bloch's photographic records (Bloch, 1962 [8]) of blood flow in various animals. Copley and Staple (1962) [21] observed similar features. Bugliarello et al., (1965) [11] carried out measurements in the glass capillaries, using anticoagulated normal human blood at different hematocrit, changing the pressure gradient. They found that the relative layer thickness, $\delta / R$, increases, increasing the hematocrit $H$ and / or decreasing the wall shear stress $\sigma_w$, whatever the radius may be expected at low hematocrit ($H = 28\%$) for which $\delta / R$ appears as very sensitive to the $R$ value. Values of $\delta / R$ are found of about $0.05 - 0.1$ at normal hematocrit.

(c) A mean (tube averaged) hematocrit $H_t$ is found lesser than the feed (reservoir) hematocrit $H_r$. It decreases as the tube diameter diminishes (Fahraeus effect) (Fahraeus, 1929).

As Cokelet (1976) [20] showed, the discharge hematocrit $H_d$ (i.e., the average hematocrit of the outflowing blood from the tube) practically equals the feed hematocrit $H_t$ if $\xi \geq 3$. Below this value, $H_d < H_t$, that can be thought as resulting from some particle redistribution at the entrance of the tube. For $3 \leq \xi \leq 50$, Barbee and Cokelet (1971) [6] found that the tube relative hematocrit, $H_r = H_t / H_t$, varies linearly with $H_r$, the slope increasing as $R$ decreases (fig no. 1.6). Similar variations but with markedly different slopes
from the ones expected were obtained with very narrow tubes (down to $2R = 8.1 \, \mu m$) (Cokelet, 1976) [20]. Thus the following approximate relation is obtained,

$$H_T = \alpha_1 H_f^2 + \alpha_2 H_f, \quad \text{with } \alpha_{1,2} = \alpha_{1,2}(R) \quad ---- \ (1.1)$$

![Graph showing Relative Hematocrit $H_r = H_t / H_f$ versus $H_f$ for different tube diameters](image)

**Fig. no 1.6**

*Relative Hematocrit $H_r = H_t / H_f$ versus $H_f$ for different tube Diameter $D$. (From BARBEE and COKELET, 1971)*

(d). The apparent viscosity lowers as the tube diameter decreases: (FAHRAEUS – LINDQVIST Effect) (FAHRAEUS & LINDQVIST, 1931).

Numbers of studies of this effect have been carried out. We shall limit here to display the results of
Haynes and Burton (1959) in long tubes (to avoid any entrance effects \((L/R = 90)\)) with \(67 \mu m \leq R \leq 750 \mu m\). They observed that the higher the hematocrit, the stronger the \(\eta_a\) lowering. (Shown in the Fig. no. 1.7 given below).

![Fig. No. - 1.7](image)

**Fig. No. - 1.7**

Fahraeus-Lindqvist effect for human erythrocyte suspensions of various hematocrit at 25.5°C. Smooth curve versus fitted to the points by the method of least squares. Asymptotic values for a tube of infinite radius are indicated by broken lines to gather with their standard errors of estimate.

(from HAYNES and BURTON, 1959)
Fig. no. 1.8 shows the data obtained,

(i) With a large tube (where no phase separation occurs, hence

\[ H_t = H_f. \]

(ii) With a narrow tube, 29 \( \mu m \) in diameter, where \( H_t \neq H_f \) varies as a function of \( H_f \), as indicated. Comparing large and small tubes data at \( H_f = 0.559 \), for instance, shows Fahraeus–Lindqvist effect at the lowering in \( \sigma_w \) at any \( \bar{u} \) value. All these authors from their analysis found an important empirical fact from their data: The relation \( \sigma_w \) versus \( \bar{u} \) for large tubes fits the small tube data at a given feed hematocrit \( H_f \), provided the tube hematocrit \( H_t \) was used in the relation \( \sigma_w (\bar{u}, H_f, R) \) instead of \( H_f \) as in the large tube that is,

\[ \sigma_w (\bar{U}, H_f, R) \approx \sigma_w (\bar{U}, H_t, \infty) \]

(1.2)

Where \( H_t = H_f \) (\( H_t, R \)) is given in the equation \( H_f = \alpha_1 H_f^2 + \alpha_2 H_f \), with \( \alpha_{1,2} = \alpha_{1,2} (R) \).
Therefore one can assert that, empirically the main part of the Fahraeus–Lindqvist effect results from the Fahraeus

Fig. No. – 1.8

Relationship between wall shear stress and average blood velocity divided by the tube diameter. The points are experimental data obtained with 29 μm. Similar results, expressed as wall shear stress \( \sigma_w \) versus \( \bar{U} = Q / \pi R^2 = \dot{\gamma}_a / 4 \) (where \( \dot{\gamma}_a \) is the apparent shear rate) have been obtained by BARBEE and COKELET (1971). The curve represents data obtained with an 811 μm tube. (from BARBEE and COKELET, 1971)
effect, i.e. the wall shear stress, as function of $\bar{U}, H_f$ and $R$, must (at least approximately) verify the equation

$\sigma_w(\bar{U}, H_f, R) \approx \sigma_w(\bar{U}, H_f, \infty)$. Further similar results hold in the smaller tube, down to $2R = 8.1 \, \mu m$ (Cokelet 1976) [20].

**SOME RECENT DEVELOPMENTS IN THE FIELD OF HAEMODYNAMICS:**

An understanding of cardiac health and disease requires knowledge of the various factors that control coronary capillary blood flow. So, a work by Ghassan S. Kassab, Kha N. le and Y.C. Fung [34a] was presented in the year 1999. In their study they obtained distensibility data on the coronary capillary blood vessels on the epicardial surface in the form of a pressure-diameter relationship using intravital microscopy. A mathematical model of the coronary capillary blood flow was then constructed on the basis of measured anatomic and elasticity data of the coronary capillary network, rheology of blood, physical laws governing blood flow and appropriate boundary conditions.

In the same year 1999, J. David Briers, Glenn Richards and Xiao Wei He [43], studied the capillary blood flow through
the LASCA (laser speckle contrast analysis) technique and described some recent developments in their research for a real time, noninvasive, full-field technique for visualizing capillary blood flow. In 1999, Sridhar Gorti, Hiroshi Tone and Genji Imokawa [83] presented their work based on Triangulation method for determining capillary blood flow and physical characteristics of the skin.

In tissue, medical or dental engineering, when blood comes into contact with a new artificial material, the flow may be influenced by surface tension between the blood and the surface of the material. Therefore a study was presented by the Wei Huang, Raghbir Bhullar and Yuan Cheng Fung [93] in 2001, in this direction showing the effect of surface tension on the flow of blood is significant, especially at the micro-scale. The leading edge of the flowing blood is the triple point where the blood, the material surface and a stationary gas or fluids meet. The movement of the triple point i.e. the advancing front of the flow, is driven by surface tension, resisted by viscous shear stress and balanced by the inertial force (mass x acceleration). In their
article the dynamics was illustrated in detail in the case of blood flowing into a capillary tube by contact.

With the increasing use of the computers in the field of biological sciences, a digital blood flow analysis was presented by the M. Manjunath & Megha Singh [55] in 2002. In their paper the blood flow through frog mesenteric micro-vessel with multiple branching, at Reynolds number 0.022, was analyzed. After pre-processing the images the velocity and erythrocyte distribution profiles by image velocimetry and axial tomography were obtained, respectively. The vascular parameters and shape descriptors were obtained by image processing techniques. The axial velocity shown the variation around branching areas associated with an increase in radial velocity, which was up to 10% of the axial velocity. A paper was presented by J. C. Misra and B. K. Kar [44] in 2002 which deals with a mathematical analysis of branching in the microcirculatory system. The specific problem considered is that of blood entering from a feeding artery into a branch capillary. Flow of blood is investigated by considering blood to be plastic of Herschel-Bulkley type.
In their model red cells in blood are taken to be
concentrated in the core region, where the concentration
profile is specified by piecewise linear function. A
mathematical model which describes the production and
diffusion of vasoactive chemical factors involved in oxygen-
dependent cerebral blood flow (CBF) regulation in the rat
was presented by Ursino, M. Di Giammarco and P. Belardinell
[91] in the year 2002. In their study they have used the partial
differential equations describing the relations between input
and output variables, replacing with similar ordinary
differential equations by using mathematical approximations
of the hyperbolic functions in the Laplace transform domain.

The biological zero (BZ) problem is a critical issue
inherent in laser Doppler flowmetry (LDF). It causes confusion
when measuring low tissue blood flows. Many experimental
studies have been done on the question of whether the BZ
flux should be subtracted from the normally measured flux in
various situations. However this problem can only be solved
after a proper mathematical analysis. Only then one can
clearly define and formulate what flux is truly meaningful in
blood perfusion measurement and what movement generates the BZ flux and how can one correctly remove it. Therefore in this area a work was presented by Jicun Zhong Seifalian, A. M. Salerud, G. E. Nilsson [45] in the year 2002 at the Stockholm, Sweden. In their paper, the movement of moving blood cells (MBCs) is decomposed into a net translation and a random wandering based on in vivo observations. This important step leads to a clear definition of the BZ and net perfusion flux and reveals that subtraction of BZ flux from the normal flux will certainly cause an underestimation of the net flux.

Erik F Hauck, Sebastian Apostel, Julie F Hoffmann, Axel Heimann and Oliver Kempski [26] presented a study on the capillary flow and diameter changes in 2003. In their study they tried to observe by intra-vital video microscopy, the diameter changes in capillary flow during reperfusion after global cerebral ischemia.

In 2004 a study was undertaken by K. P. Ivanov, M.K. Kalinina and Yu. I. Levkovich [49] on the Blood flow velocity in capillaries of brain and muscles. Intra-vital
microfilming by means of a dark-field contact epiobjective was used for measuring capillary blood flow velocity in the brain and skeletal muscles of the rat. The linear flow rate in capillaries was determined by measuring the rate of motion of plasma-filled "gaps" in the continuous erythrocyte flow. The mean linear red cell velocity for 100 cerebral capillaries 2 - 5µm in diameter was found to be 0.79±0.03 mm/sec. In the temporalis muscle the velocity was equal to 1.14±0.04 mm/sec in 123 capillaries and 2.43±0.08 mm/sec in 34 arterioles and precapillaries not more than 5µm in luminal diameter. The experimentally obtained average values of blood flow velocities in cerebral capillaries indicate that these velocities vary mainly from 0.5 to 1.5 mm/sec. This agrees with previously performed calculations based on the mathematical model which suggests that this range of velocities is optimal to adequately supply neurons with oxygen. In the same year 2004 a study was presented by the Elshahed M. [27] for studying the blood flow in capillaries under the starling hypothesis.
A mathematical model was presented by Eric P Salathe [28] in 2005 for analyzing oxygen transport in the skeletal muscle that includes both the interaction of very large number of capillaries and the effect of axial diffusion in the tissue. The analysis takes into account differences in the blood flow velocity among the capillaries and in the oxygen concentration and hematocrit of the blood entering the capillaries at the arterial end.

Microcirculation is common to every organ and nurtures various tissues by providing oxygen and nutrients and removing waste products. To understand this phenomenon a mathematical model was suggested by the Rekha Bali, Swati Mishra & Shraddhya Dubey [73] for the red cell motion in narrow capillary surrounded by tissue in 2007. In this paper they developed a model of capillary-tissue fluid exchange through squeezing flow of plasma in the gap between cells moving through a capillary of smaller diameter than that of the cell introducing hydrodynamic lubrication theory [A. Cameron, The principle of lubrication, Willey, New York, 1966]
Significant achievements have been made in the area of lab-on-a-chip devices for biomedical analysis. The recent developments were observed of lab-on-a-chip devices by enabling researchers to use numerical methods to improve device performance. In this area a numerical study was presented by Liu & Ming [52] in the year 2008. In this they have used the volume of fluid (VOF) model, in which they studied the non-Newtonian shear-thinning viscosity of blood capillary flow in micro-channels with diameters under 100µm. In order to understand the normal and pathologic behavior of the human vascular system, detailed knowledge of blood flow and the response of the blood vessels is required. For this, a study was presented by A. Jafari, P. Zamankhan, S.M. Mousavi and P. Kolari [1] in the year 2008. In this study they simulated the behaviour of blood flow in micro-vessels using computational fluid dynamics (CFD). Numerical analysis was performed using a commercially available CFD package fluent 6.2 which was based on the finite volume method. A continuum approach was proposed in which fluid structure interaction had been taken into
Based on the limitations imposed by the computational resources, a more simplified model based on the volume of fluid (VOF) model approach was suggested to simulate movements of RBCs in capillaries and also to predict RBCs deformation.

In Dec 2009, a study was presented by the Andrew Moulden [2] for mass blood flow under anoxia spectrum syndromes. In 2009, from the Dept of chemical Engineering, University of Massachusetts, Amherst (USA), a numerical method is implemented for computing blood flow through a branching micro-vascular capillary network was devised by the C. Pozrikidis [12]. The simulations follow the motion of individual red cells as they enter the network from an arterial entrance point with a specified tube hematocrit, while simultaneously updating the nodal capillary pressures.

**SOME PRELIMINARIES OF FLUID DYNAMICS:**

Since we are discussing here the flow of blood in capillaries, so direct involvement of the principles of fluid dynamics is inevitable. According to the basic definitions
and fundamental equations for various important flows, the fluids can be divided into two categories:

(i) Ideal fluids

(ii) Real fluids

**IDEAL FLUIDS:**

The fluids which are non-viscous and incompressible are called ideal fluids. In ideal fluids there is no tangential force between adjoining layers of the fluids, but there is only the normal force called pressure.

**REAL FLUIDS:**

The fluids which are viscous and compressible are called real or actual fluids.

**FLUID DENSITY:**

Fluid density is defined as the mass per unit volume. It generally varies with pressure and temperature.

**VISCOSITY:**

Fluids may be defined as materials that continue to deform in the presence of any shearing stress. We refer to continuous deformation of a fluid as 'flow' and the property by virtue of which the fluid resists any deformation is called
viscosity. Viscosity is thus a measure of the reluctance of fluid to yield to shear when the fluid is in motion. Further we can define viscosity as the property of real fluids as a result of which they offer some resistance to shearing i.e. sliding moment of one particle past of near another particle. Viscosity is also known as internal friction of fluid. All well known fluids have this property in varying degrees. This is very small for some fluids, such as water and gasses etc. whereas it is large in some cases like oil, glycerin etc. In viscous fluids both the tangential and normal forces exist.

**SHEARING STRESS:**

Two types of forces act on a fluid element, one of them is body force and the other is surface force. The body force is proportional to the mass of the body on which it acts while the other force acts on the boundary of the body so it is proportional to the surface area. Surface F is the surface force acting on elementary surface area ds at point P of the surface. S. Let F₁ and F₂ are the resolved parts of the force in the directions of tangent and the normal at P. The normal force per unit area is called normal stress and is also called
pressure. The tangential force per unit area is called shearing stress. Hence $F_1$ is a kind of shearing stress and $F_2$ is the normal stress.

Shear Tensor and Strain rate tensor:

The shearing stress $\tau$ between any thin sheet of a fluid is defined as, $\tau = \frac{\text{Force}}{\text{Area}}$

Now $\tau \propto \frac{U}{y_o}$, where $y_o$ is the distance between two planes such that one is at rest, while other is moving with uniform velocity $v$ parallel to itself. Therefore,

$$\tau = \mu \frac{U}{y_o}$$
Where, $\mu$ is the constant of proportionality and is defined as viscosity.

For all practical purpose, fluids are divided into two categories,

(A) Newtonian fluids

(B) Non-Newtonian fluids

(A). NEWTONIAN FLUIDS:

The physical property that characterizes the flow resistance of simple fluids is the viscosity. The Newtonian hypothesis states that the stress depends linearly on the rate of strain and is independent of the strain. Thus, in the Newtonian law of viscosity for two dimensional flows, the shear stress $\tau$ and the rate of the strain $du/dy$ are related by,

$$\tau = \mu \frac{du}{dy} \quad \text{(1.3)}$$

Where, $\tau$ is the viscosity, a proportionality constant which is independent of the shear rate (rate of strain) but is affected only by the temperature and the pressure for a given fluid system. The graph between $\tau$ and the velocity gradient $\frac{du}{dy}$ becomes straight line and is called "flow curve". Gasses and
low molecular weight (i.e. non-polymeric) liquids are almost Newtonian under conventional rate of shear (say 1 to $10^{-5}$ sec$^{-1}$). At the extremes of very low and very high shear rate polymer solutions, polymers melts and most slurries of limited concentration are found to be Newtonian.

The generalized constitutive equation for such isotropic fluid flows i.e. the linear relation between the components of stress-tensor $\tau_{ij}$ and the rate of strain tensor $e_{ij}$ is given by,

$$\tau_{ij} = -p\delta_{ij} + 2\eta_0 e_{ij} \quad ---- (1.4)$$

Where, $p$ is the pressure, $\delta_{ij}$ is the kronecker's delta, $\eta_0$ is the material constant called the coefficient of viscosity and

$$e_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right], \text{is the rate of deformation tensor} \quad ---- (1.5)$$

In the Newtonian fluids, boundary conditions are based on the experimental fact and slip occurs at the surface of a solid body.

The theory based on this linear relationship has achieved remarkable success in explaining the phenomena
of lift, skin-friction, drag, separation and secondary flows etc. But this theory has been inadequate to explain the flow behavior of many materials of industrial importance like plastics of all kinds, colloids, high polymers, suspensions etc. In fact, we can say safely that all real fluids show deviations from this hypothesis when subjected to sufficiently precise measurements.

(B). NON-NEWTONIAN FLUIDS:

The fluids, whose flow behavior shows deviations from Newtonian relationship, are called Non-Newtonian fluids. The field of non-Newtonian fluid mechanics has gained new inputs both in engineering techniques as well as applications in the mathematical and physical foundations with the recent growing industrial and technical importance of the non-Newtonian fluids cited above. The study of non-Newtonian fluids has thrown a good deal of light on non-linear continuum mechanics. They are broadly classified into six main categories according to their observed flow behaviour (Kapur et. al. [47]):
(i) **VISCO-INELASTIC FLUIDS:**

Power-law fluids, Reiner-Rivlin fluids, Bingham plastics, Ellis fluids, reiner Philippoff fluids, Prandtl fluids, Erying fluids, Powello-Erying fluids, Willianson fluids, Rabinwtich type fluids, Meter fluids, Casson fluids are generally known as visco-inelastic fluids.

(ii) **VISCO-ELASTIC FLUIDS:**

It includes Oldroyd fluids, Rivlin-Eriksen second order fluids, walter's fluids and other visco-elastic fluids.

(iii) **POLAR FLUIDS:**

A Dipolar fluid, micropolar fluids, model of condiff and Dahler, stokes fluids with couple stresses are known as polar fluids.

(iv) **HEAT CONDUCTING NEOMATIC LIQUID CRYSTALS:**

Though this does not give an exhaustive list, but the most important of these are power-law fluids, Bingham plastic, Oldroyd fluids, second order fluids, walter's fluids, micropolar fluids and anisotropic fluids (Kumar [51])
Two main categories of non-Newtonian fluids are as follows:

(a) TIME INDEPENDENT FLUIDS OR VISCO-INELASTIC FLUIDS:

The majority of the non-Newtonian fluids that we encounter probably fall into this category. A common feature of this class of fluids is that when at rest, they are isotropic and homogenous and when they are subjected to shear, the resultant stress depends only on the rate of shear. However this sub-class shows diverse behaviour in response to applied stress. A number of rheological models have been proposed to explain such a diverse behaviour. Some of the important models which have attracted the researchers are:

(i) POWER LAW FLUIDS:

These fluids do not have the yield stress. The rheological relation between the stress tensor $\tau_{ij}$ and the stress rate $e_{ij}$ is given by the Bird et. al [3] as,

$$\tau_{ij} = k \left[ \left( \sum_{m=1}^{3} \sum_{l=1}^{3} \frac{e_{ml} e_{lm}}{2} \right)^{1/2} \right]^{n-1} e_{ij}$$

(1.6)

Where, 'k' and 'n' are the constant for a particular fluid, $k$ is a measure of consistency of the fluid and $n$, the exponent, is
a measure of how the fluid deviates from a Newtonian fluid.

The apparent viscosity for such fluids is given by,

\[ \mu_a = k \left[ \sum_{m=1}^{3} \sum_{l=1}^{3} \frac{e_{ml} e_{lm}}{2} \right]^{1/2} e_{ij} \]  \quad (1.7)

The power-law fluids is further characterized as pseudo-plastic fluid or dilatants fluid, according as \( n < 1 \) or \( n > 1 \) respectively and it reduces to Newtonian fluids if \( n = 1 \), then ‘\( k \)’ represents the Newtonian viscosity. The apparent viscosity decreases or increases with the increasing shear rate according as \( n < 1 \) or \( n > 1 \) respectively.

Equation (1.6) for \( \tau_{ij} \) is found to be remarkably versatile and useful for explaining the flow behaviour of majority of non-Newtonian fluids. Pseudo-plastic fluids are polymer solutions or melts, grease, adhesives, rubber solutions, starch suspensions, cellulose acetate, soaps, detergent slurries, solutions used in rayon manufacturing, mayonnaise, paper pulp, napalm, paints, dispersion media in certain pharmaceutical, biological fluids etc.

Examples of dilatants fluids are aqueous suspensions of titanium dioxide, some gum Arabic / borax
solutions, starch potassium silicate, quicksand, same corn flour / sugar solutions etc.

Although, there are a number of empirical relations that have been used to describe power law fluids, the simple of these is the power law due to Ostwald [66]. It may be written as,

\[ \tau = k\dot{\gamma}^n \] ---- (1.8)

Taking log on both sides, equation (1.8) becomes,

\[ \log \tau = \log k + n \log \dot{\gamma} \] ---- (1.9)

Which is the equation of the straight line where the slope is ‘n’ and the intercept of the curve with \( \log \dot{\gamma} = 0 \) or \( \log \dot{\gamma} = 1 \), is just a special case of a power law fluid.

(ii) REINER-RIVLIN FLUIDS:

Reiner [72] and Rivlin [74] have given the most general equations for isotropic, incompressible, non-Newtonian viscoelastic fluid connecting the stress tensor and the rate of deformation tensor as,

\[ \tau_{ij} = 2\eta_0 e_{ij} + 4\mu e_i e_j \] ---- (1.10)

\[ t_{ij} = p\delta_{ij} + 2\eta_0 e_{ij} + 4\mu e_i e_j \] ---- (1.11)
Where, the coefficient of viscosity $\eta_0$ and the coefficient of 'cross-viscosity' $\mu_e$ depend on the invariants $I_1, I_2, I_3$

Where,

$$I_1 = e_y, \quad I_2 = \frac{1}{2} e_\theta e_\theta, \quad I_3 = \frac{1}{3} e_\alpha e_\alpha e_\mu$$

---- (1.12)

For an isotropic fluid, assuming that the components of stress tensor can be expressed as polynomials in the components of rate of shear tensor and using Cayley-Hamilton theorem, it can be shown that this is the most general relation between the stress and the strain-rate tensor.

(iii) BINGHAM PLASTICS:

These fluids possess yield stress. The plastic flow of an isotropic fluid is described by the following system of equations [Bingham [4]]

$$\frac{2\eta_0 e_\theta}{\sqrt{1/2 \tau_\theta \tau_\theta}} = \frac{\tau_\theta}{\sqrt{1/2 \tau_\theta \tau_\theta}}$$

---- (1.13)

$$\tau_\theta = 2\eta_0 e_\theta \quad \text{and} \quad 1/2 \tau_\theta \tau_\theta > \tau_0^2$$

---- (1.14)

$$e_\theta = 0 \quad \text{if} \quad \tau_\theta \tau_\theta \leq \tau_0^2$$

---- (1.15)

$$\tau_\theta = -p \delta_\theta + \tau_\theta$$

---- (1.16)
Where, $\tau_{ij}$ are the derivative stress tensor components, $\eta$ is the constant reciprocal mobility and $\tau_0$ is the yield value of the material and $\eta_o$ is the apparent viscosity of the material.

For these materials, the apparent viscosity decreases with increasing shear rate. Examples of such fluids are certain plastic melts, oil well drilling moods, aces, cement, rock and chalk slurries, tooth paste, paper pulp etc. Some other models which are proposed for this type of fluids are given by Herschel-Bulkey and Crowley-Kitzes [39].

These fluids are governed by the empirical relation,

$$e = \phi_0 \tau + \phi_1 \tau^n$$

---- (1.17)

Where $\phi_0, \phi_1 \& n$ the positive parameters, 'e' is the rate of strain and $\tau$ is the stress. The possible generalized form of the equations given above, is

$$e_{ij} = \phi_0 \tau_1 + \phi_1 \left| \sum_{q=1}^{3} \sum_{p=1}^{3} \frac{\tau_{pq} \cdot \tau_{qp}}{2} \right|^{n-1/2} \tau_{lk}$$

---- (1.18)

This model included the Newtonian ($\phi_1 = 1$) and the Power-law ($\phi_0 = 0$) models as the special cases.
(iv) REINER-PHILIPPOFF FLUIDS:

The consecutive equation for such fluids is given by,

$$\tau_{ik} = \mu_0 + \frac{\mu_n - \mu_0}{1 + \frac{1}{2\tau_0} \left( \sum_{p=1}^{3} \sum_{q=1}^{3} \tau_{pq} \tau_{qp} \right)}$$  \hspace{1cm} (1.19)

Where, $\mu_0, \mu_n$ & $\tau_0$ are three adjustable positive parameters.

For small and large values of $\tau_0$, the behaviour of these fluids is nearly Newtonian but for intermediate values of $\tau_0$, the behaviour is markably non-Newtonian and lies between two extremes of non-Newtonian behaviour.

(v) PRANDTL FLUIDS:

These fluids are characterized by the relation (Williamson [94]),

$$\tau = A \sin^{-1} \left( \frac{\dot{y}}{B} \right)$$  \hspace{1cm} (1.20)

Where A and B are the material constants of the fluid. The relation given above can be generalized as,

$$\tau_{ik} = \frac{A \sin^{-1} \left( \sum_{p=1}^{3} \sum_{q=1}^{3} e_{pq} e_{qp} \right)}{\left( \sum_{q=1}^{3} \sum_{p=1}^{3} e_{pq} e_{qp} \right) \left( \sum_{p=1}^{3} \sum_{q=1}^{3} e_{pq} e_{qp} \right)}$$  \hspace{1cm} (1.21)
(vi) ERYING FLUIDS:

These fluids are characterized by the rheological relation,

\[ \tau = \frac{e}{B} + C \sin^{-1}\left( \frac{\tau}{B} \right) \]  ---- (1.22)

Where A, B and C are the constants, which are typical of the particular fluid. A possible generalization (Williamson [94]) of the above relation can be given as,

\[ \tau_{ik} = \frac{1}{B} e_{ik} + \frac{C \sin^{-1} \left( \sum_{p=1}^{3} \sum_{q=1}^{3} \frac{\tau_{pq} \tau_{qp}}{2A^2} \right)^{1/2} \left( \sum_{p=1}^{3} \sum_{q=1}^{3} \frac{\tau_{pq} \tau_{qp}}{2} \right)}{\sum_{p=1}^{3} \sum_{q=1}^{3} \frac{\tau_{pq} \tau_{qp}}{2}} \]  ---- (1.23)

(vii) POWELL-ERYING FLUIDS:

The empirical relation for such fluids is given as,

\[ \tau = Ae + B \sin^{-1}(Ce) \]  ---- (1.24)

This can be generalized as,

\[ \tau_0 = Ae_{ij} + B \sinh^{-1} \left[ \frac{1}{\sqrt{2} I_2} \right] \frac{e_{ij}}{\sqrt{2} I_2} \]  ---- (1.25)

Where, A, B and C are the fluid parameters.

(viii) WILLIAMSON FLUIDS:

These fluids are characterized by the following relation (Williamson [94]).
\[ \tau = \frac{Ae}{B + e} + \mu e \] \hspace{1cm} (1.26)

The above equation may also be put as,

\[ e = \alpha \tau + \beta + \sqrt{(\alpha E + \beta) + \gamma} \] \hspace{1cm} (1.27)

Where \( \alpha, \beta \) & \( \gamma \) are the parameters.

A possible generalization is,

\[ \tau_{ij} = \frac{Ae_{ij}}{B + \left( \sum_{p=1}^{3} \sum_{q=1}^{3} e_{pq} e_{qp} \right) / 2} + \mu e_{ij} \] \hspace{1cm} (1.28)

(ix) RABINOWITSCH TYPE FLUIDS:

These fluids are governed by the following relation,

\[ e = \frac{1}{\mu_0} \tau + \sum B_q \tau^{q+1} \] \hspace{1cm} (1.29)

This is generalized to,

\[ e_0 = \frac{1}{\mu_0} \tau_0 + \sum B_q J_{ij} \tau_{ij} \] \hspace{1cm} (1.30)

(x) METER \- MODEL FLUIDS:

The constitutive equation for this class of fluids (Meter [58]) is,

\[ \tau = -\eta e_{ij} \quad \text{And,} \quad \eta = \eta_0 + \frac{\eta_0 - \eta_{\infty}}{1 + |\tau / t_m|^{a-1}} \] \hspace{1cm} (1.31)
Where, $\eta_0$, $\eta_\infty$ and $\tau_m$ are the fluid parameters whose values have been tabulated by Meter and Bird [57].

(b) TIME DEPENDENT FLUIDS or VISCO-ELASTIC FLUIDS:

These fluids are characterized by the fact that the shear rate depends not only on the applied stress, but in addition, on the duration of the stress. Fluids which show a decrease in viscosity with time under isothermal conditions and steady shear are called Thixotropic while those fluids which show increase in viscosity are called Rheoplectic. The thixotropic fluids are more common than rheoplectic fluids and are of great importance in industries.

There are materials for which a suddenly applied and maintained state of uniform stress induces an instantaneous deformation, followed by a flow process which may or may not be limited to magnitude, as time grows. A material which responds in this manner, is said and to exhibit both instantaneous elastic effect and creep characteristics. This behaviour is clearly not described either by elasticity of viscosity theory but by combined features of both. Hence they are called viscoelastic materials. When a
viscoelastic fluid is in motion, a certain amount of energy is stored up in the material as strain energy, while some energy is lost due to viscous dissipation. In this class of fluids, unlike the case of purely viscous fluids, the strain, however, small, can not be neglected, as it is responsible for the recovery of the original state and for the reverse flow that may flow for the removal of the stress.

During the flow, the natural state of the fluid changes constantly and tries to attain the instantaneous state of deformation but never succeeds completely. This leg is measure of elasticity or so called 'memory' or elastic response of the fluid. In the formation of the rheological models or the consecutive equations, an account of this memory is taken by introducing 'stress relaxation times' and 'strain-rate retardation times'. The problem of allowing for memory in rheological model has results in several of viscoelasticity. A number of researchers like Oldroyd [64, 65], Dewitt [25], Pao [68], Green and Rivlin [38], Noll [59] and Walters [92] have developed the consecutive equations of visco-elastic fluids.
For these fluids, the shear rate $\mu'$ depends on the imposed stress $\tau$ and the extent of deformation $\gamma$ so that,

$$\mu' = f(\tau, \gamma) \quad \text{---- (1.32)}$$

The Weissenberg 'climbing' effect was the pioneering work in this field. Weissenberg [95, 96] showed that the elastic fluid 'climb' onto a shaft rotator within opposition to centrifugal forces. The Navier-Stokes equations had, therefore, to be modified. The consecutive equations were derived from considerations of invariance and general assumptions regarding the variables occurring in these equations.

We give now some of the well known models for this VISCO-ELASTIC FLUIDS category:

(i) **THE MAXWELL MODEL:**

One of the simplest models which can be made is one spring and one dash-pot series and is known as the Maxwell Model. The governing equation for this model are given as,

$$\gamma_s = \frac{\tau}{G} \quad \text{---- (1.33)}$$
And, \[ \gamma_d = \frac{r}{\mu} \] \hspace{1cm} \text{--- (1.34)}

This is a fluid model in which a constant applied stress will result in a continuous deformation due to unrestrained extension of the dash-pot. Thus

\[ \gamma = \gamma_s + \gamma_d \] \hspace{1cm} \text{--- (1.35)}

Differentiating equation (1.35) with respect to the time and with the help of the equations (1.33) and (1.34) we get,

\[ \tau + \lambda t = \mu \gamma \] \hspace{1cm} \text{--- (1.36)}

Where, \( \lambda = \frac{\mu}{G} \) is the relaxation time.

If \( G \to \infty \), that is \( \lambda \to 0 \), the spring becomes a rigid connection and the model reduces to that of a Newtonian fluids.

Conversely, if \( \mu \to \infty \), that is \( \lambda \to \infty \), the dash pot becomes rigid and model becomes that of Hookean solid.

To get a feel for the behaviour of the Maxwell fluid, let us consider its response to a stress-relaxation test in simple shear, in which a shear strain of magnitude \( \gamma_0 \) is suddenly applied at the time \( t = 0 \) and is held constant. If the
initial extension were infinitely fast, the dash pot would remain rigid, since as infinite strain rate would require an infinite stress to expand it. Hence, only the spring will extend initially. If the extension is then maintained constant, however, the spring contraction force will react on the dash pot causing it to extend as the spring relaxes.

(ii) THE RIVLIN-ERICKSEN MODEL:

This is the most general model put forward so far from purely phenomenological considerations.

Let a visco-elastic isotropic material undergo deformation and let the components in the system of velocity, acceleration, second acceleration, ... n\textsuperscript{th} acceleration at the instant 't' be defined by \( v_i(t), v_{i(1)}(t), v_{i(2)}(t), \ldots \ldots v_{i(n)}(t) \) respectively.

If \( D/Dt \) denotes the material time derivative, then

\[ v_{i(\alpha-1)}(t) = \frac{D}{Dt} v_{i(\alpha)}(t) \]

or,

\[ v_{i(\alpha-1)}(t) = \frac{\partial v_{i(\alpha)}(t)}{\partial t} + v_m(t) = \frac{\partial v_{i(\alpha)}(t)}{\partial x_m(t)} \] ---- (1.37)

Where, \( \alpha = 0, 1, 2, \ldots \ldots (n - 1) \) and \( v_i(t) = \frac{Dx_i(t)}{Dt} \)
Where a generic particle $x_i$ in the rectangular Cartesian frame 'x' at some reference time $t = 0$ has coordinate $x_i(t)$ at generic time $t$.

**Rivlin and Ericksen [75]** assumed that the stress components $\tau_{ij}$ at a point $x_i$ at the time 't' are single valued functions of gradients in the system 'x' of velocity, acceleration, nth acceleration at the point $x_i$ at the time 't', so that

$$\tau_{ij} = f_{ij}(v_{p,q}, v^1_{p,q}, v^2_{p,q}, \ldots, v^n_{p,q}) \quad \text{--- (1.38)}$$

Where 'p' denotes the differentiation with respect to $x_p$ that is $x_p(t)$.

(iii) **THE OLDROYD FLUIDS:**

**Oldroyd [63],** after considering the problem in a general perspective, proposed the consecutive equation,

$$P'_{ik} + \lambda_i \frac{D}{Dt} P'_{ik} = 2\eta_0 \left( e_{ik} + \lambda_2 \frac{D}{Dt} e_{ik} \right) \quad \text{--- (1.39)}$$

$$\frac{D}{Dt} e_{ik} = \frac{\partial}{\partial t} e_{ik} + v_j \frac{\partial e_{ik}}{\partial x_j} + \omega_y e_{jk} + \omega_j e_{ik} \quad \text{--- (1.40)}$$

$$\frac{D}{Dt} P'_{ik} = \frac{\partial}{\partial t} P'_{ik} + v_j \frac{\partial P'_{ij}}{\partial x_j} + \omega_y P'_{jk} + \omega_j P'_{ik} \quad \text{--- (1.41)}$$
Where, \( P_{ik}' = P_{ik}' - p\delta_{ik} \), is the stress tensor, and \( \delta_{ik} \) is the kronecker delta,

Also, \( \omega_{ik} = \frac{1}{2}(v_{i,k} - v_{k,i}) \) and \( e_{ik} = \frac{1}{2}(v_{i,k} + v_{k,i}) \)

And \( \eta_0 \) is the coefficient of viscosity.

These gives rise to non-linear equations, the simple generalized form of which is,

\[
P_{ik}' + \lambda_i \frac{D}{Dt} P_{ik}' + \mu_0 P_{jk}' e_{jk} - \mu_1 (P' e_{jk} + P_{jk}' e_\theta) - v_1 P_{jk}' e_{j} \delta_{ik} = \]

\[
2\eta_0 \left\{ e_{ik} + \lambda_2 \frac{D}{Dt} e_{ik} - 2\mu_2 e_{ij} e_{jk} + v_2 e_{ij} e_{j} \delta_{ik} \right\} \quad \text{--- (1.42)}
\]

Where, \( \mu_0, \mu_1, \mu_2, v_1 \) and \( v_2 \) are scalar physical quantities, having dimensions of time. He emphasized that the above equations with proper choice of constant can explain the behaviour of many real fluids including bio-fluids.

(iv) WalterS Fluids Model 'B':

The consecutive equations for the Walters [92]

fluids 'B' are given by \( S_{ik} = -p g_{ik} + \tau_{ik} \) \quad \text{--- (1.43)}

And,

\[
\tau_{ik}(x,t) = 2 \int_{-\infty}^{t} \Psi(t-t') \frac{dx^i}{dx^m} \frac{dx^j}{dx^n} \theta^{(1)mr}(x',t') dt' \quad \text{--- (1.44)}
\]

Where \( S_{ik} \) is the stress tensor, \( p \) an arbitrary isotropic pressure, \( g_{ik} \) the metric tensor of fixed coordinate system \( x_i, x'^i \) the
position at the time $t'$ of the element which is instantaneously at the point $x_i$ at the time $t$, $e^{(i)jk}$ the rate of strain tensor and,

$$\Psi(t-t') = \int_0^\tau \frac{n(\tau)}{\tau} e^{-(t-t')/\tau} d\tau$$  

Where $n(\tau)$ being the distributive function of relaxation time $\tau$.

**MICRO POLAR FLUIDS:**

Eringen [29] proposed the theory of micro-polar fluids which exhibits the micro-rotational effect and micro-rotational inertia and studied the flow of micropolar fluids in channels and pipes. The consecutive equations of micropolar fluid, according to Eringen [29] are,

$$\tau_{ij} = (-p + \lambda d_{kk})\delta_{ij} + (2\mu + k)d_{ij} + Ke_{ijk}(W_k - v_k)$$  

$$M_{ij} = \alpha v_{pp}\delta_{ij} + \beta v_{ij}$$

Where $\tau_{ij}$ is the stress tensor, $M_{ij}$ is the couple stress tensor, $\delta_{ij}$ is the Kronecker delta, $e_{ijk}$ is the alternating tensor and represents micro rotation.
Some Important Flows:

(i) MAGNETO-HYDRODYNAMIC (MHD) FLOW:

In the later stage of the development of hydrodynamics, much interest has been awakened in the response of conducting fluids to the application of electric and magnetic fields, for, it touches the problem of harnessing fusion energy etc. The ultimate dynamical behaviour of the fluids determined by the mechanical fluid force as well as the magnetic forces exerted on the particles as a result of the magnetic field induced by the motion of the charged fluid particles a corollary of the property of the conductivity.

The effect is described mathematically by Maxwell's electromagnetic equations as,

\[ \nabla \cdot \vec{E} = \frac{4\pi}{\varepsilon} q; \quad \nabla \cdot \vec{H} = 0; \]

\[ \nabla \times \vec{H} = 4\pi \vec{J} \]

\[ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}; \quad \vec{J} = \sigma (\vec{E} \times \vec{\mu} \times \vec{H}) \]

Where, \( \vec{E} \), is the electric field intensity

\( \varepsilon \), is the dielectric constant of the fluid
\( \overline{H} \), is the magnetic field intensity
\( \mu \), is the permeability of the medium
\( \overline{J} \), is the conduction current density vector
\( q \), is the charge per unit volume
\( \sigma \), is the conductivity of the medium

Hence, the three consecutive equations in the case of a viscous fluid are,

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0, \quad k = 1, 2, 3
\]

\[
\rho \left[ \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} \right] = -\nabla p + \overline{J} \times \overline{B} \quad ---- (1.48)
\]

\[
\rho \left[ \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} \right] = -\nabla \left( \rho v + \overline{Q} \right) + \left( \overline{J} \cdot \overline{E} \right) \quad ---- (1.49)
\]

Where, the pressure tensor \( P \) has components \( P_{ij} \) is given by,

\[
P_{ij} = \left( \frac{2}{3} \rho \frac{\partial u_k}{\partial x_k} \right) \delta_{ij} - \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad ---- (1.50)
\]

Where, \( p \) is the internal energy per unit mass, \( \eta \), is the coefficient of viscosity, \( \overline{Q} \), is the heat flow rate \( \overline{B} = \mu \overline{H} \) and \( \delta_{ij} \) is the kronecker delta,

Further,
\[
\left( n.P \right) = \sum_k \frac{\partial}{\partial x_k} P_{ik}
\]

\[
\left( P.n \right) = P_{ik} v_k
\]

(93)
And the energy equations for the incompressible fluid becomes,

$$\rho \frac{d\varepsilon}{dt} = kV^2 + \eta \phi + \frac{j^2}{\sigma} \quad \text{(1.51)}$$

The terms on the right hand side represent heat conduction, viscous dissipation and joule heating effect respectively and $\phi$ being a positive definite quadratic form of the velocity derivative $\frac{\partial u_i}{\partial x_i}$.

The Hartmann number $M = \frac{\mu H_0}{\rho \nu} \left(\frac{\sigma}{\rho \nu}\right)^{1/2}$ gives the relative effects of the magnetic and viscous drag, where $\nu = \eta / \rho$.

(ii) TWO-PHASE FLOW:

Multi phase system consists of a fluid medium (media) and particulate phase of any number of chemical components. These fluids media can be gas liquid or plasma and the particulate phase may consist of solid particles, liquid droplets, as bubbles, etc. Since most common types of multi-phase flows consist of two phase of some substance only, these may be classified as:  
(i) Liquid gas flow  
(ii) liquid solid flow  
(iii) gas solid flow  
(iv) liquid plasma flow  
(v) plasma
solid flow (vi) gas plasma flow. These six classes of two phase flow arise in different fields of physical and technological interest and hence their basic features may not be the same, implying thereby that each class and phase should be treated in a manner most suitable for it. A few examples of practical applications of such systems are,

(a) **Gas-Solid particles**: Dust collectors, fluidized beds, metalized propellants rockets, cosmic dusts, nuclear fall-out problems.

(b) **Gas-Solid droplets**: Atomizers, rocket engine injectors, clouds, factory stalk effluents and evaporators.

In particular, for the motion of the gas containing solid particles or liquid droplets, the force of interaction depends in general upon the particle Reynolds number, Mach number, Knudsen number as well as on the details of the local flow field and the interaction between the particles. This simply indicates the extent of difficulties that has to be encountered in **multi phase flows** when compared to single phase flows. However, the dynamics of such fluid particle system has been extensively investigated for a long
time with the help of considerable idealizations had been rather ad hoc and empirical; systematically analytical treatments have been developed only in the last few years. (Marble [53, 54]).

(iii) DUSTY FLOWS:

Another interesting subject worked on, is that of dusty fluids. Sproull [82] observed that dusty air in turbulent flow through a pipe show a reduced resistance coefficient. The same had been observed by Kazakevich and Krapivins [48] in the case of aerodynamic resistance of a dusty gas. Sproull suggested that the viscosity of a dusty gas is appreciably reduced but this was in direct contradiction to Einstein formula for the viscosity of suspension.

Saffman [78] then suggested that the higher inertia of the dust particles make them to resist the turbulent motion causing a relative motion of dust and air which extracted energy from the turbulent fluctuations. As is certainly possible, if the turbulent intensity is reduced, then the Reynold's stress will be reduced and the force required maintaining a given flow rate will, likewise, be reduced. Since the problem of
turbulence is related to stability of the laminar flow. Saffman [78] discussed the problem of stability of laminar flow by making the following assumptions:

(a) Dust particles are uniform in the shape and size.

(b) Their velocity and number density can be described by fields \( \nu(x,t) \) and \( N(x,t) \) respectively.

(c) Bulk concentrations of the dust is small, so that the net effect of the dust on the air is equivalent to an extra form \( KN(v - u) \) per unit volume, where \( u \) is the velocity of the gas, \( K \) is a constant (called Stoke's resistance coefficient) and Reynold's number of the relative motion of dust and gas is proportional to relative velocity.

Accordingly, the governing equations of motion for a dusty viscous incompressible fluid are given by,

\[
\frac{\partial \tilde{u}}{\partial t} + (u - \nabla \cdot \nu) \tilde{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \tilde{u} + \frac{KN}{\rho} (u - \tilde{v}) \quad \text{(1.52)}
\]

\[
m \left( \frac{\partial \tilde{v}}{\partial t} + \tilde{v} \nabla \tilde{v} \right) = K(u - \tilde{v}) + mg \quad \text{(1.53)}
\]
(iv) CONVECTIVE FLOWS:

Free and forced convective flows have aroused the interest of engineers and Mathematicians. Motions, which are caused by the density gradients created by temperature differences, are termed 'natural' as distinct from those 'forced' on the scream by external reasons.

(v) FLOWS THROUGH POROUS MEDIA:

The study of the flows through porous media has been motivated by its immense importance and continuing interest of scientists and Mathematicians in many engineering and technological fields. Such diversified fields illustrated as petroleum technology, soil mechanics, water filtration and purification, groundwater hydrology, civil engineering, textile and ceramic engineering and many other rely heavily upon it as fundamental to their individual problems.

(vi) DARCY LAW:

An important factor which provides the Mathematical foundation for the fluid through porous media is called "Darcy law". This was established by Darcy in
empirical relationship based on his experimental study of the flow of water through filter beds of sand which established a linear relationship between hydraulic gradient ‘J’ and filtration velocity ‘v’. Mathematically, Darcy law is given by,

$$V = k' \frac{dh}{ds} \quad \text{--- (1.54)}$$

Where, $k'$ is the constant of proportionality and called the coefficient of filtration which depends on the properties of both fluid and the porous media. The coefficient $k'$ has the dimensionality of a vector and is equal to the filtration velocity for a piezometric slope equal to one.

The coefficient of proportionality $k'$ appearing in the “Darcy law” is also called hydraulic conductivity. It is scalar quantity with dimension [LT$^{-1}$] and depends on the properties of both porous medium and the following fluid. According to Nutting [61a], $k'$ expressed as,

$$k' = \frac{k \rho g}{\mu} \quad \text{--- (1.55)}$$

Where, $\mu$ is the viscosity of the fluid $\rho$ is the density of the fluid and ‘$k$’ is a coefficient depending upon the properties
of porous medium and is called the permeability of the porous medium.

Many Rheologists \{Feincher and Lewis [30], Cerman [13], Nissan [61], Ruth [77], Fox [32], Niclsen [60], Hudson and Robertz [40], Plain and Morrison [69], Romita [76] and Steward and Owens [85] etc.\} have investigated a great discrepancy regarding the ‘Universal’ Reynold’s numbers above which ‘Darcy law’ would no longer be valid, is well evident. The Reynolds numbers range between 0 and 1 (Niclson [60]) and plain & Morrison [69]). Bear [3a] analyze the flow through porous media by three regimes as follows:

(a) At low Reynold’s numbers, there exist regions of laminar flow, in which viscous forces are predominant and the linear “Darcy law” is valid.

(b) As ‘Re’ number increases, there exist a transition zone from laminar to turbulent.

(c) At high ‘Re’ numbers, there exists the turbulent flow regime, in which, inertial force govern the flow.
Keeping in view of the above discussion we have divided our work into six chapters. The chapter-wise discussion is given in the next part of this THESIS.
REFERENCES


[34a]. Ghassan S. Kassab, Kha N. Le and Y. C. Fung (1999):
Chapter - I  
Thesis "MATHEMATICAL ANALYSIS OF BLOOD FLOW IN CAPILLARIES" – By Ravindra Saxena


(106)

[44]. J. C. Misra and B. K. Kar (2002): A mathematical analysis of blood flow from a feeding artery into a branch capillary; Dept of Mathematics IIT Kharagpur-India


(108)
[60]. Niclsen R. F. (1951): World Oil, 132, 6, 188
   Amer. Ass. Petrol Geol., 14 : 1337 - 1349


[91]. Ursino, M. Di Giammarco and P. Belardinell in the year 2002


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