Chapter: 3

Multi-objective Transportation Problem and its Solution by Grey Situation Decision Making Theory

3.0 Introduction

Multi-Objective Transportation Problem (MOTP) is a special class of Linear Programming Problem (LPP) where the constraints are of equality type and all objectives are conflicting with each other. To solve Multi-Objective Linear Programming Problem (MOLPP) the available methods generate a set of non-dominated or compromise solution. Even, many approaches have been adopted to solve MOLPP viz. Interactive Algorithm, Fuzzy Programming, Step method etc. Generally, these methods ignore some of the important aspects, like the confliction of objectives, priority of objectives, environmental constraints, unique organizational values of the firm and bureaucratic decision structures etc. which influence the decision process of transportation problems in reality. Many researchers have worked for the solution of transportation problem by adopting different approaches like as, Lee et al. [93] studied the goal programming for the optimization of multiple conflicting goals while permitting an explicit consideration of the existing decision environment. Diaz [94] and Isermann [95] have developed algorithms while, Diaz [96] also presented an alternative procedure for that. Also, in the same line multi objective design of transportation networks was reviewed by Current et al. [97] and two interactive algorithms were developed by Ringuest et al. To solve MOTP and to find optimal compromise solution of it, Li et al. [98] also presented a fuzzy approach in 1981. Leberling used hyperbolic membership function for MOTP where he found solutions are always compromise the original multi-criteria problem. Edwards [99] gave the use of an additive multi attribute utility function for the linear MOTP that provides a good initial solution. For optimum design problem, Dhinagraand Moskowitz [100] defined a nonlinear membership functions. With the fuzzy approach a modified s-curve membership function was used by Peidro and Vasant [15] [101. In 1991, Dhinra and Moskowitz defined some non-linear membership functions like exponential, quadratic and logarithmic. Bit et al. (1992) used a $k$ - objective TP which was fuzzified by fuzzy numbers and used $\alpha$-cut to obtain TP in the fuzzy sense expressed in linear programming form. Bit and Alam (1993) introduced an additive fuzzy programming
model for MOTP which aggregates the membership functions of the objectives, weights and priorities for non-equivalent objectives that gives a non-dominated compromise solution.

Verma et al. (1997) proposed a special type of non-linear membership functions to solve MOTP. He found the result obtained by this method was similar to the solution obtained by using a linear membership function. Hussien (1998) studied the complete set of possibly efficient solutions of MOTP with possibilistic coefficients of the objective functions. Kakuchi (2000) suggested that in many transportation problems the observed values of the variables are approximate, yet the variable themselves must satisfy the set of rigid relationships dictated by physical principle. To find the appropriate set of crisp numbers a simple adjustment method was proposed by them with the assumption, each observed value is an approximate number and the true value is found for membership function. The value satisfies the relationships the lowest membership grade is checked. Also, the set, whose lowest Fuzzy membership grade is the highest, is chosen as the best set of values for the problem.

In 2000 Li and Lai presented a fuzzy approach to solve MOTP where, various objectives were considered with the marginal evaluation for individual objectives and the global evaluation for all objectives. To assign weight to objective the preference of DM was considered. In 2001, Wahed and Sinna presented fuzzy approach to solve MOTP by measuring the degree of closeness of compromised solution to ideal solution. Also Sakawa et al. gave two type of two level programming problems; profit maximization problem and profitability maximization problem using interactive fuzzy programming. In 2005, Ammar and Youness examined the efficiency of solution of MOTP using fuzzy: coefficient, supply and demand quantities. In 2006, Wahed and Lee proposed interactive fuzzy goal programming to solve MOTP by considering that each objective has fuzzy goal. In the same line Zangiabadi and Maleki (2007) has also gave the solution of MOTP in which a special type of non-linear membership function is assigned to each objective function to describe each fuzzy goal. The main focus of research was to minimize the negative deviation variables from 1 to obtain a compromise solution of the MOTP.
In 2008, Surapati and Roy gave priority based fuzzy goal programming, in which the membership functions were defined for fuzzy goal. Similarly, the solution of MOTP was obtained by many researchers using different approaches described as follows. Lau et al. (2009) presented a fuzzy logic guided non-dominated sorting genetic algorithm, Lohgaonkar and Bajaj (2010) used linear and nonlinear fuzzy membership function, P.K. De and Bharti Yadav (2011) used an exponential fuzzy membership [102], J. Khan, D. K. Das (2012) used a fuzzy membership function define through the given data[103], Yousria Abo-Elnaga, Bothina El-Sobky and Hanadi Zahed (2012) used the trust-region globalization strategy [104], M. Zangiabadi and H. R. Maleki (2013) used a special type of nonlinear (hyperbolic and exponential) membership [105], Osuji, George, Okoli Cecilia, Opara, Jude (2014) used fuzzy programming algorithm [106].

In this chapter we will develop grey situation decision making theory based approach as well as grey situation decision making theory and fuzzy programming technique based approach to find the solution of multi objective transportation problem.

3.1 Mathematical Formulation of Multi-Objective Transportation Problem

In reality a transportation problem can have more than one-objects is considered as multi objective transportation problem characterized by multiple objective functions. In a MOTP a product is to be transported from \( m \) sources to \( n \) destinations with the capacities and requirements \( a_1, a_2, \ldots, a_m \) and \( b_1, b_2, \ldots, b_n \) respectively. Also the penalty \( c_{ij} \) is associated with transporting a unit of product from \( i^{th} \) source to \( j^{th} \) destination, it may be cost, delivery time or safety of delivery etc. A variable \( x_{ij} \) represents the unknown quantity to be shipped from \( i^{th} \) source to \( j^{th} \) destination. An LPP form of MOTP with \( r \) objectives, having \( m \) sources and \( n \) destinations can be written as follows [63]:

\[
\text{Minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{Subject to} \quad \sum_{j=1}^{n} x_{ij} \leq a_i, \quad i = 1, 2, \ldots, m \\
\sum_{i=1}^{m} x_{ij} \geq b_j, \quad j = 1, 2, \ldots, n \\
\]
Model 3.1

Minimize \( Z_k = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij}^{k} x_{ij}, k = 1,2,\ldots,K. \)

Subject to the constraints:
\[
\sum_{j=1}^{n} x_{ij} = a_i, \text{for } i = 1,2,\ldots,m.
\]
\[
\sum_{i=1}^{m} x_{ij} = b_j, \text{for } j = 1,2,\ldots,n.
\]
\[
x_{ij} \geq 0, \text{for } i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n. \quad (3.1)
\]

The subscript on \( Z_k \) and superscript on \( c_{ij}^{k} \) are related to the \( k^{th} \) penalty criterion. Without loss of generality, it may be assumed that \( a_i \geq 0 \) and \( b_j \geq 0 \) \( \forall \ i, j \), the equilibrium condition \( \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \) satisfied.

3.2 Grey Situation Decision Making Theory based Solution of Multi-Objective Transportation Problem

Let \( O = \{O_1, O_2,\ldots, O_m\} \) be the set of \( m \) origins having \( a_i \) \( (i = 1, 2,\ldots,m) \) units of supply respectively. \( D = \{D_1, D_2,\ldots, D_n\} \) be the set of \( n \) destinations with \( b_j \) \( (j = 1, 2,\ldots,n) \) units of requirement respectively. Let, penalty \( p_{ij} \) associated with transporting a unit of product from \( i^{th} \) source to \( j^{th} \) destination. \( x_{ij} \) represents the unknown quantity to be shipped from \( i^{th} \) source to \( j^{th} \) destination. The problem is to determine the transportation schedule when multiple objectives exist.

Here, Grey situation decision making theory is used to minimize the total transportation penalty according to the problem which satisfying supply and demand conditions where the penalty \( p_{ij} \) as the situation set denotes by
\[
P = \{p_{ij} = (O_i, D_j) / O_i \in O, D_j \in D\} \quad (3.2)
\]

Here, it is necessary to confirm the decision making goals (objectives) and seek the corresponding effect measure matrix \( U^{(k)} \) for each situation as
Here, the data of decision making goals for transporting a product is the effect value \( u^{(k)} \) of situation \( p_i \in P \) with objective \( k = 1, 2, \ldots, K \), which is nothing but costs \( C^{(k)} \) of the MOTP.

Now, the upper and lower effects are measured by the formula:

Upper effect measure
\[
\bar{r}_{ij}^{(k)} = u^{(k)} / \max_{i} \max_{j} \{ u^{(k)} \},
\]
(3.4)

Lower effect measure
\[
\underline{r}_{ij}^{(k)} = \min_{i} \min_{j} \{ u^{(k)} \} / u^{(k)}. \tag{3.5}
\]

By using either (3.4) or (3.5) achieve the consistent matrix of effect measure \( R^{(k)} \)

\[
R^{(k)} = \begin{bmatrix}
\bar{r}_{11}^{(k)} & \bar{r}_{12}^{(k)} & \cdots & \bar{r}_{1m}^{(k)} \\
\bar{r}_{21}^{(k)} & \bar{r}_{22}^{(k)} & \cdots & \bar{r}_{2m}^{(k)} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{r}_{n1}^{(k)} & \bar{r}_{n2}^{(k)} & \cdots & \bar{r}_{nm}^{(k)} 
\end{bmatrix}. \tag{3.6}
\]

To convert multi objective optimization problem in single objective optimization problem, decide weight of each objective.

There are two different ways to obtain objective weights \( \eta_1, \eta_2, \ldots, \eta_s \).

(1) Decision maker gives weight to the objectives according to their first preference.

(2) Decision maker applies some methods to obtain the objective weights.

From the objective weights and consistent matrices of effect measure, acquire the comprehensive effect measure of situation \( s_{\bar{y}} \)

\[
r_{\bar{y}} = \sum_{k=1}^{s} \bar{r}_{\bar{y}}^{(k)} \eta_k, \tag{3.7}
\]
and achieve the comprehensive matrix \( R \) of effect measure.
\[ R = \begin{bmatrix}
    r_{11} & r_{12} & \cdots & r_{1m} \\
    r_{21} & r_{22} & \cdots & r_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{n1} & r_{n2} & \cdots & r_{nm}
\end{bmatrix}, \quad (3.8) \]

Subtract each data of comprehensive matrix \( R \) of effect measure from value 1 because comprehensive matrix \( R \) of effect measure contain maximum value so it is necessary to convert combine maximum objective in minimum form and then find solution of this grey theory based combined single objective problem by using given supply and demand as like LPP problem.

### 3.3 Algorithm for Finding Solution of MOTP using MGSD Theory

**Input**

Multi-objective transportation problem with effect measure matrix

\[ U^{(k)} = (U^{(1)}, U^{(2)}, \ldots, U^{(s)}; n \times m) \]

**Output**

Solution of MOTP

Compute the efficient solution of MOTP using the optimization model of objective weight.

**Begin**

**Step-1:** Read: problem

while problem = MOTP do,

for \( k = 1 \) to \( K \) do,

enter effect measure matrix \( U^{(k)} \),

end

**Step-2:** Find the lower effect measure \( r_{ij}^{(k)} \) and upper effect measure \( r_{ij}^{(k)} \) and accomplish the consistent matrix of effect measure \( R^{(k)} = [r_{ij}^{(k)}] \).

for \( k = 1 \) to \( K \) do,

\[ r_{ij}^{(k)} = \min_i \min_j \{ u_{ij}^{(k)} \} / u_{ij}^{(k)} , \]

\[ r_{ij}^{(k)} = u_{ij}^{(k)} / \max_i \max_j \{ u_{ij}^{(k)} \} , \]

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\[ R^{(k)} = \begin{bmatrix} r_{11}^{(k)} & r_{12}^{(k)} & \cdots & r_{1m}^{(k)} \\ r_{21}^{(k)} & r_{22}^{(k)} & \cdots & r_{2m}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1}^{(k)} & r_{n2}^{(k)} & \cdots & r_{nm}^{(k)} \end{bmatrix}, \]

end

Step-3: Get the comprehensive matrix of effect measure for situation \( s_{ij} \) is \( R = [r_{ij}] \).

for \( k = 1 \) to \( s \) do

\[ R = [r_{ij}] = \sum_{k=1}^{s} r_{ij}^{(k)} \eta_k, \]

end

Step-4: Convert comprehensive matrix \( R \) in minimization form.

Step-5: Find the solutions of converted single objective optimization problem by given demand and supply as like TP.

### 3.4 Numerical Illustrations

This section discussed several multi objective transportation problem solution with developed approach as well as its comparison with other approaches.

**Numerical illustration: 1**

A company has three manufacture accommodations \( A_1, A_2 \) and \( A_3 \) with production capacity of 8, 19 and 17 units of a product respectively. These units are to be shipped to four store rooms \( B_1, B_2, B_3 \) and \( B_4 \) with requirement of 11, 3, 14 and 16 units respectively. The transportation cost and transportation time between accommodations to store rooms are given below [107]:

\[
\begin{align*}
\text{Min } z_1 &= x_{11} + 2x_{12} + 7x_{13} + 7x_{14} + 1x_{21} + 9x_{22} + 3x_{23} + 4x_{24} + 8x_{31} + 9x_{32} + 4x_{33} + 6x_{34}, \\
\text{Min } z_2 &= 4x_{11} + 4x_{12} + 3x_{13} + 4x_{14} + 5x_{21} + 8x_{22} + 9x_{23} + 10x_{24} + 6x_{31} + 2x_{32} + 5x_{33} + x_{34},
\end{align*}
\]

subject to the constraints:

\[
\begin{align*}
x_{11} + x_{12} + x_{13} + x_{14} &= 8, \\
x_{21} + x_{22} + x_{23} + x_{24} &= 19, \\
x_{31} + x_{32} + x_{33} + x_{34} &= 17, \\
x_{11} + x_{21} + x_{31} &= 11, \\
x_{12} + x_{22} + x_{32} &= 3, \\
x_{13} + x_{23} + x_{33} &= 14, \\
x_{14} + x_{24} + x_{34} &= 16, \\
x_{ij} &\geq 0, i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4
\end{align*}
\]
Solution:

By using Grey situation decision making theory the solution obtained with different weights are as follows:

Consider case set, counter set and situation set. Production facilities of company are the cases. i.e. \( A=\{A_1, A_2, A_3\} \) is the case set. Destination is the counter. i.e. \( B=\{B_1, B_2, B_3, B_4\} \) is the counter set. Situation set \( C=\{c_{ij} = (A_i, B_j) \mid A_i \in A, B_j \in B\} \) is structured by A and B.

(1) Effect measure matrices under goals are given below.

\[
U^{(1)} = \begin{bmatrix}
1 & 2 & 7 & 7 \\
1 & 9 & 3 & 4 \\
8 & 9 & 4 & 6
\end{bmatrix}, \quad U^{(2)} = \begin{bmatrix}
4 & 4 & 3 & 4 \\
5 & 8 & 9 & 10 \\
6 & 2 & 5 & 1
\end{bmatrix}
\]

(2) For transporting a product, goals are less than its better, hence lower effect measure is utilized at here. The lower effect measure for first data is

\[
i_{11}^{(1)} = \min \min \{u_{11}\} = \frac{1}{1}.
\]

Similarly, obtain lower effect measure for each data. Therefore, the consistent matrices of effect measure are given below.

\[
R^{(1)} = \begin{bmatrix}
1 & 0.5 & 0.14286 & 0.14286 \\
1 & 0.11111 & 0.33333 & 0.25 \\
0.125 & 0.22222 & 0.75 & 0.66667
\end{bmatrix},
\]

\[
R^{(2)} = \begin{bmatrix}
0.75 & 0.5 & 1 & 0.25 \\
0.8 & 0.25 & 0.33333 & 0.1 \\
0.16667 & 0.5 & 0.2 & 1
\end{bmatrix}
\]

(3) Combine \( R^{(1)} \) and \( R^{(2)} \) by taking equal weight \( \eta_k = \frac{1}{2} \) (k=1,2) we get,

\[
R = \begin{bmatrix}
0.875 & 0.5 & 0.57143 & 0.19643 \\
0.9 & 0.18056 & 0.33333 & 0.175 \\
0.14583 & 0.36111 & 0.475 & 0.83333
\end{bmatrix}.
\]
(4) Subtract each data of comprehensive matrix R of effect measure from 1 to convert combine maximization comprehensive matrix R in minimization form.

\[
R = \begin{bmatrix}
0.125 & 0.5 & 0.42857 & 0.80357 \\
0.1 & 0.81944 & 0.66667 & 0.825 \\
0.85417 & 0.63889 & 0.525 & 0.16667
\end{bmatrix},
\]

(5) By using this comprehensive matrix \( R = [r_{ij}] \) of effect measure as well as demand and supply the formed single objective optimization problem (LPP) solution is;

\[
x_{12} = 3, x_{13} = 5, x_{21} = 11, x_{23} = 8, x_{31} = 1, x_{34} = 16
\]

and its objectives values are \( z_1 = 176, z_2 = 175 \).

Similarly, for different values of \( \eta_k (k = 1, 2) \) (weights) we get different solution, which are noted in Table 3.1.

<table>
<thead>
<tr>
<th>( \eta_1 )</th>
<th>( \eta_2 )</th>
<th>( z_1 )</th>
<th>( z_2 )</th>
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<td>1</td>
<td>208</td>
<td>167</td>
</tr>
</tbody>
</table>

Table 3.1: Solution of numerical illustration-1 with different weights

Table 3.1 contains solution of multi-objective transportation problems with different weights. It also shows that as per the value of objective weight there is some change in objective value. This weight is decided by decision makers and according to their requirements solutions are achieved.

**Numerical illustration: 2**

A company has four manufacture accommodations \( A_1, A_2, A_3 \) and \( A_4 \) with production capacity of 6, 8, 7 and 7 units of a product respectively. These units are to be shipped to six store rooms \( B_1, B_2, B_3, B_4, B_5 \) and \( B_6 \) with requirement of 3, 5, 5, 6, 4 and 5 units
respectively. The transportation cost and transportation time between accommodations to store rooms are given below [108]:

\[
\begin{align*}
\text{Min } z_1 &= 2x_{11} + 3x_{12} + 5x_{13} + 11x_{14} + 4x_{15} + 2x_{16} + 4x_{21} + 7x_{22} + 9x_{23} + 5x_{24} + 10x_{25} + 4x_{26} + 12x_{31} + 25x_{32} + 9x_{33} + 6x_{34} + 26x_{35} + 12x_{36} + 8x_{41} + 7x_{42} + 9x_{43} + 24x_{44} + 10x_{45} + 8x_{46}, \\
\text{Min } z_2 &= x_{11} + 2x_{12} + 7x_{13} + 7x_{14} + 4x_{15} + 4x_{16} + 1x_{21} + 9x_{22} + 3x_{23} + 4x_{24} + 5x_{25} + 8x_{26} + 8x_{31} + 9x_{32} + 4x_{33} + 6x_{34} + 6x_{35} + 2x_{36} + 3x_{41} + 4x_{42} + 9x_{43} + 10x_{44} + 5x_{45} + 1x_{46},
\end{align*}
\]

subject to the constraints:

\[
\begin{align*}
x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} &= 6, \\
x_{11} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} &= 8, \\
x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} &= 7, \\
x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} &= 7, \\
x_{11} + x_{21} + x_{31} + x_{41} &= 3, \\
x_{12} + x_{22} + x_{32} + x_{42} &= 5, \\
x_{13} + x_{23} + x_{33} + x_{43} &= 5, \\
x_{14} + x_{24} + x_{34} + x_{44} &= 6, \\
x_{15} + x_{25} + x_{35} + x_{45} &= 4, \\
x_{16} + x_{26} + x_{36} + x_{46} &= 5, \\
x_{ij} &\geq 0, i = 1,2,3,4 \text{ and } j = 1,2,3,4,5,6.
\end{align*}
\]

By using Grey situation decision making theory the solution obtained with different weights are as follows: For different values of \( \eta_k(k = 1,2) \) we get different solution which are noted in Table 3.2.

<table>
<thead>
<tr>
<th>( \eta_1 )</th>
<th>( \eta_2 )</th>
<th>( z_1 )</th>
<th>( z_2 )</th>
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<td>84</td>
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</table>

Table 3.2: Solution of numerical illustration-2 with different weights

Table 3.2 contains solution of multi-objective transportation problems with different weights. It also shows that as per the value of objective weight there is some change in
objective value. This weight is decided by decision makers and according to their requirements solutions are achieved.

**Numerical illustration: 3**

A company has three manufacture accommodations $A_1, A_2$ and $A_3$ with production capacity of 15, 20 and 10 units of a product respectively. These units are to be shipped to three storerooms $B_1, B_2$, and $B_3$ with requirement of 10, 4 and 31 units respectively. The transportation cost and transportation time between accommodations to storerooms are given below [109]:

\[
\begin{array}{ccccccccccc}
11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\
11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\
11 & 21 & 31 & 12 & 22 & & & & \\
11 & 21 & 31 & 12 & 22 & & & & \\
\end{array}
\]

subject to the constraints:

\[
\begin{align*}
x_{11} + x_{12} + x_{13} &= 15 \\
x_{21} + x_{22} + x_{23} &= 20 \\
x_{31} + x_{32} + x_{33} &= 10 \\
x_{11} + x_{21} + x_{31} &= 10 \\
x_{12} + x_{22} + x_{32} &= 4 \\
x_{13} + x_{23} + x_{33} &= 31 \\
x_{ij} &\geq 0, i = 1, 2, 3 \text{ and } j = 1, 2, 3.
\end{align*}
\]

**Solution:**

By using Grey situation decision making theory the solution obtained with different weights are as follows: For different values of $\eta_k (k = 1, 2)$ we get different solution which are noted in Table 3.3.

<table>
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<td>0</td>
<td>1</td>
<td>221</td>
<td>159</td>
</tr>
</tbody>
</table>

Table 3.3: Solution of numerical illustration-3 with different weights
Table 3.3 contains solution of multi-objective transportation problems with different weights. It also shows that as per the value of objective weight there is some change in objective value. This weight is deciding by decision makers and according to their requirements solutions are achieved.

3.5 Comparison of Grey Situation Decision Making Theory approach with other approach

This section discusses comparison of grey situation decision making theory based approach with other developed approaches. The comparison in Table 3.4 shows that the grey situation decision making theory based approach provides very efficient alternative approach to find solution of multi-objective transportation problem.

<table>
<thead>
<tr>
<th>Sr.no.</th>
<th>Multi-objective Grey situation decision (MGSD) approach (using weights)</th>
<th>other approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical Illustration:1</td>
<td>Using weights $\eta_1 = \eta_2 = 0.5$ we have $z_1 = 176, z_2 = 175$.</td>
<td>By Waiel et.al. [107]: $Z_1 = 186, Z_2 = 173$.</td>
</tr>
<tr>
<td>Numerical Illustration:2</td>
<td>Using weights $\eta_1 = \eta_2 = 0.5$ we have $z_1 = 186, z_2 = 90$.</td>
<td>By W.Ritha et.al. [108]: $Z_1 = 171, Z_2 = 101$.</td>
</tr>
<tr>
<td>Numerical Illustration:3</td>
<td>Using weights $\eta_1 = \eta_2 = 0.5$ we have $z_1 = 221, z_2 = 159$.</td>
<td>By Abbas et.al.[109]: $Z_1 = 161, Z_2 = 199$.</td>
</tr>
</tbody>
</table>

**Table 3.4: Comparison of numerical illustrations with other approach:**

Table 3.4 contains comparison of multi-objective transportation problems with different approaches. In this table MGSD theory based approach having same objective weight. As per the Table-3.1, Table-3.2, and Table-3.3, some change in objective weight, there is some change in objective value. This weight is decided by decision makers and according to their requirements, solutions are achieved.
3.6 Grey Situation Decision Making Theory and Fuzzy Programming Technique based Approach to find Solution of MOTP

In fuzzy programming technique, it is necessary to find the lower bound as \( L_k \) and the upper bound as \( U_k \) for the \( K \)th objective function \( Z_k, k = 1, 2, \ldots, K \) where \( U_k \) is the highest acceptable level of achievement for objective \( k \), \( L_k \) the aspired level of achievement for objective \( k \) and \( d_k = U_k - L_k \) the degradation allowance for objective \( k \).

Here, in this developed approach we first utilized Grey situation decision making theory to find the lower effect measure \( r_{ij}^{(k)} \) or upper effect measure \( r_{ij}^{(k)} \) and accomplish the consistent matrix of effect measure \( R^{(k)} = [r_{ij}^{(k)}] \) for each objective \( k \). That is here Grey situation decision making theory is utilized for normalization of data. The solution of MOTP can be obtained by the following steps:

**Step-1:** Find the lower effect measure \( r_{ij}^{(k)} \) or upper effect measure \( r_{ij}^{(k)} \) and accomplish the consistent matrix of effect measure \( R^{(k)} = [r_{ij}^{(k)}] \) for each objective \( k \) as per section 3.2.

**Step-2:** Solve the single-objective transportation problem \( K \) times with consistent matrix of effect measure \( R^{(k)} = [r_{ij}^{(k)}] \) as a cost by taking one objectives at a time.

**Step-3:** Determine the corresponding values for every objective at each solution. According to each solution and value for every objective, we can find a pay-off matrix as follows:

\[
\begin{array}{cccc}
z_1(x) & z_2(x) & \cdots & z_K(x) \\
X^{(1)} & z_{11} & z_{12} & \cdots & z_{1k} \\
X^{(2)} & z_{21} & z_{22} & \cdots & z_{2k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X^{(k)} & z_{k1} & z_{k2} & \cdots & z_{kk}
\end{array}
\]

**Table 3.5: Pay-off matrix**

Where, \( X^{(1)}, X^{(2)}, \ldots, X^{(k)} \) are the isolated optimal solutions of the \( K \) different transportation problems for \( K \) different objective function.
**Step-4:** Define a membership function $\mu(z_k)$ for the $k^{th}$ objective function.

**Step-5:** Convert the MOTP of the problem, obtained in step 3, into the following crisp model;

**Model 3.2:**

Maximize $\lambda$,

$$\lambda \leq \mu(z_k),$$

Subject to the constraints

$$\sum_{j=1}^{m} X_{ij} = a_i, \quad i = 1, 2, \ldots, m.$$  
$$\sum_{i=1}^{n} X_{ij} = b_j, \quad j = 1, 2, \ldots, n.$$  
$$X_{ij} \geq 0, \forall i, j. \quad \lambda \geq 0.$$  

(3.9)

If we use a linear membership function, the crisp model can be simplified as:

**Model 3.2.1:**

Maximize $\lambda$,

$$z_k + \lambda(U_k - L_k) \leq U_k, \quad k = 1, 2, \ldots, K.$$  

Subject to the constraints

$$\sum_{j=1}^{m} X_{ij} = a_i, \quad i = 1, 2, \ldots, m.$$  
$$\sum_{i=1}^{n} X_{ij} = b_j, \quad j = 1, 2, \ldots, n.$$  
$$X_{ij} \geq 0, \forall i, j. \quad \lambda \geq 0.$$  

(3.10)

If we will use the hyperbolic membership function, then an equivalent crisp model for the fuzzy model can be formulated as:
Model 3.2.2:

Maximize \( \lambda \),

\[
\lambda \leq \frac{1}{2} e^{\left[ \frac{(U_k+L_k)}{2} - z_k(X) \right]} a_k - e^{\left[ \frac{(U_k+L_k)}{2} - z_k(X) \right]} a_k + \frac{1}{2}, \text{ if } k = 1, 2, \ldots, K
\]

where \( a_k = \frac{6}{U_k - L_k} \).

Subject to the constraints

\[
\sum_{j=1}^{n} X_{ij} = a_i, \quad i = 1, 2, \ldots, m.
\]

\[
\sum_{i=1}^{m} X_{ij} = b_j, \quad j = 1, 2, \ldots, n.
\]  \hspace{1cm} (3.11)

\( X_{ij} \geq 0, \forall i, j. \quad \lambda \geq 0 \)

Model constraints can further be simplified as:

\[
\lambda \leq \frac{1}{2} \tan h \left[ \left( \frac{U_k + L_k}{2} \right) - z_k(X) \right] a_k + \frac{1}{2},
\]

\[
2\lambda \leq \tan h \left[ \left( \frac{U_k + L_k}{2} \right) - z_k(X) \right] a_k + 1, \hspace{1cm} \text{tanh}^{-1} (2\lambda - 1) \leq \left[ \left( \frac{U_k + L_k}{2} \right) - z_k(X) \right] a_k,
\]

\[
a_k z_k + \text{tanh}^{-1} (2\lambda - 1) \leq \frac{a_k(U_k + L_k)}{2},
\]

Now, putting \( \text{tanh}^{-1} (2\lambda - 1) = X_{mn+1} \) it is converted to

\[
a_k z_k (X) + X_{mn+1} \leq \frac{a_k(U_k + L_k)}{2}.
\]

Hence, the given model is simplified as:
Model 3.2.2.1

Maximize \( X_{mn+1} \),

Subject to the constraints
\[
a_k z_k (X) + X_{mn+1} \leq \frac{a_k (U_k + L_k)}{2}, \quad k = 1, 2, \ldots, K.
\]
\[
\sum_{j=1}^{n} X_{ij} = a_i, \quad i = 1, 2, \ldots, m.
\]
\[
\sum_{i=1}^{m} X_{ij} = b_j, \quad j = 1, 2, \ldots, n.
\]
\[
X_{ij} \geq 0, \quad \forall i, j. \quad X_{mn+1} \geq 0 \text{ Where } \tanh^{-1} \left( 2 \lambda - 1 \right) = X_{mn+1} \quad (3.12)
\]

**Step-6:** Solve the crisp model by LINGO software.

**Step-7:** The solution obtained in step 6 will be the compromise solution of the MOTP.

### 3.7 Algorithm for Finding Solution of MOTP with Membership Function using MGSD Theory:

**Input**

Multi objective Transportation Problem with Effect measure matrix

\[
U^{(k)} = \left( U^{(1)}, U^{(2)}, \ldots, U^{(s)}; n \times m \right)
\]

**Output**

Solution of MOTP

Compute the efficient solution of MOTP using the optimization model of objective weight.

**Begin**

**Step-1:** Read: problem

while problem= MOTP do

for k=1 to s do

enter effect measure matrix \( U^{(k)} \)

end

**Step-2:** Find the lower effect measure \( r^{(k)}_y \) and upper effect measure \( u^{(k)}_y \) and accomplish the consistent matrix of effect measure \( R^{(k)} = [r^{(k)}_y] \).
for k=1 to s do,

\( r_{ij}^{(k)} = \min_i \min_j \{ u_{ij}^{(k)} \} / u_{ij}^{(k)} , \)

\( r_{ij}^{(k)} = u_{ij}^{(k)} / \max_i \max_j \{ u_{ij}^{(k)} \} , \)

\[
R^{(k)} = \begin{bmatrix}
  r_{11}^{(k)} & r_{12}^{(k)} & \cdots & r_{1m}^{(k)} \\
  r_{21}^{(k)} & r_{22}^{(k)} & \cdots & r_{2m}^{(k)} \\
  \cdots & \cdots & \cdots & \cdots \\
  r_{n1}^{(k)} & r_{n2}^{(k)} & \cdots & r_{nm}^{(k)}
\end{bmatrix}
\]

end.

Step-3: Convert comprehensive matrix \( R^{(k)} \) of \( k \)th objective in minimization form for all objectives by subtracting each data of comprehensive matrix \( R \) of effect measure from 1.

Step-4: Find optimal solution to each objective.

Step-5: Find pay-off matrix by using each objective solution.

Step-6: Define linear as well as hyperbolic membership function using payoff matrix.

Step-7: Develope single objective optimization problem (Model 3.2.1 and Model 3.2.2.1) using fuzzy linear membership function and hyperbolic function.

Step-8: Solve model developed in step-7 and find compromise solution.

3.8 Numerical Illustrations

This section discusses several multi objective transportation problems and their solutions with Grey situation decision making theory and fuzzy programming technique based approach as well as its comparison with other approaches. In this section consider same numerical illustrations which have been discussed in Section 3.4.

Numerical illustration: 1(same as Numerical illustration-1 in section 3.4)

Solution:

(1) Effect measure matrices under goals are given below:

\[
U^{(1)} = \begin{bmatrix}
  1 & 2 & 7 & 7 \\
  1 & 9 & 3 & 4 \\
  8 & 9 & 4 & 6
\end{bmatrix}, \quad U^{(2)} = \begin{bmatrix}
  4 & 4 & 3 & 4 \\
  5 & 8 & 9 & 10 \\
  6 & 2 & 5 & 1
\end{bmatrix}
\]
For transporting a product goals are less than its better, hence, here we use lower effect measure. So the lower effect measure for first data is

$$i^{(1)}_{11} = \frac{\min \{u_{ij}\}}{u_{11}} = 1$$

Similarly, obtain lower effect measure for each data and convert comprehensive matrix $R^{(k)}$ of $k^{th}$ objective in minimization form for all objectives by subtracting each data of Comprehensive matrix $R$ of effect measure from 1 we get,

$R^{(1)} = \begin{bmatrix}
0 & 0.5 & 0.8571 & 0.8571 \\
0 & 0.8889 & 0.6667 & 0.75 \\
0.875 & 0.7778 & 0.25 & 0.3333
\end{bmatrix}$, $R^{(2)} = \begin{bmatrix}
0.25 & 0.5 & 0 & 0.75 \\
0.2 & 0.75 & 0.6667 & 0.9 \\
0.8333 & 0.5 & 0.8 & 0
\end{bmatrix}$

(2) Find each objective, solution for multi-objective transportation problem by using comprehensive matrix of effect measure with simplex method.

For first objective: The optimal allocations are

$x_{11} = 5, x_{12} = 3, x_{21} = 6, x_{23} = 13, x_{33} = 1, x_{34} = 16.$

Apply these allocations to first, second objective therefore we have

$Z_1(X^1) = 15.7499, Z_2(X^1) = 13.4171.$

For second objective: The optimal allocations are

$x_{13} = 8, x_{21} = 11, x_{22} = 2, x_{23} = 6, x_{32} = 1, x_{34} = 16.$

Apply these allocations to first, second objective therefore we have

$Z_1(X^2) = 18.7454, Z_2(X^2) = 8.2002.$

Hence, pay-off matrix

$$\begin{bmatrix}
Z_1(X^1) & Z_1(X^1) \\
Z_2(X^1) & Z_2(X^1)
\end{bmatrix} = \begin{bmatrix}
15.7499 & 13.4171 \\
18.7454 & 8.2002
\end{bmatrix}.$$ 

Now,

$U_1 = \max (15.7499, 18.7454) = 18.7454,$

$L_1 = \min (15.7499, 18.7454) = 15.7499,$

$U_1 - L_1 = 2.9955, U_1 + L_1 = 34.4953,$

$U_2 = \max (8.2002, 13.4171) = 13.4171,$

$L_2 = \min (8.2002, 13.4171) = 8.2002,$

$U_2 - L_2 = 5.2169, U_2 + L_2 = 21.6173$
Applying fuzzy linear membership function, we get the following model

Maximize $\lambda,$

$$0x_{11} + 0.5x_{12} + 0.8571x_{13} + 0.8571x_{14} + 0x_{21} + 0.8889x_{22} + 0.6667x_{23} + 0.75x_{24} + 0.875x_{31} + 0.7778x_{32} + 0.25x_{33} + 0.3333x_{34} + \lambda(2.9955) \leq 18.7454,$$

$$0.25x_{11} + 0.5x_{12} + 0x_{13} + 0.75x_{14} + 0.2x_{21} + 0.75x_{22} + 0.6667x_{23} + 0.9x_{24} + 0.8333x_{31} + 0.5x_{32} + 0.8x_{33} + 0x_{34} + \lambda(5.2169) \leq 13.4171,$$

Subject to the constraints:

$$x_{11} + x_{12} + x_{13} + x_{14} = 8,$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 19,$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 17,$$

$$x_{11} + x_{21} + x_{31} = 11,$$

$$x_{12} + x_{22} + x_{32} = 3,$$

$$x_{13} + x_{23} + x_{33} = 14,$$

$$x_{14} + x_{24} + x_{34} = 16,$$

$$x_{ij} \geq 0, i=1,2,3 \text{ and } j=1,2,3,4, \lambda \geq 0.$$

Solution of this model is:

The optimal allocations are

$$x_{11} = 0.02345065, x_{12} = 3, x_{13} = 4.976549, x_{21} = 10.97655,$$

$$x_{23} = 8.023451, x_{33} = 1, x_{34} = 16.$$

Using these allocations we have $Z_1 = 175.9062, Z_2 = 175.1173$

With degree of satisfaction $\lambda = \frac{0.6836805}{5}.$

Applying fuzzy hyperbolic membership function, we get the following model

Maximize $\lambda$

$$\lambda \leq \frac{1}{2} e^\left[\left(\frac{34.4953}{2}\right)^2 - 0x_{11} + 0.5x_{12} + 0.8571x_{13} + 0.8571x_{14} + 0x_{21} + 0.8889x_{22} + 0.6667x_{23} + 0.75x_{24} + 0.875x_{31} + 0.7778x_{32} + 0.25x_{33} + 0.3333x_{34}\right]$$

$$- e^\left[\left(\frac{34.4953}{2}\right)^2 - 0x_{11} + 0.5x_{12} + 0.8571x_{13} + 0.8571x_{14} + 0x_{21} + 0.8889x_{22} + 0.6667x_{23} + 0.75x_{24} + 0.875x_{31} + 0.7778x_{32} + 0.25x_{33} + 0.3333x_{34}\right]$$

$$+ e^\left[\left(\frac{34.4953}{2}\right)^2 - 0x_{11} + 0.5x_{12} + 0.8571x_{13} + 0.8571x_{14} + 0x_{21} + 0.8889x_{22} + 0.6667x_{23} + 0.75x_{24} + 0.875x_{31} + 0.7778x_{32} + 0.25x_{33} + 0.3333x_{34}\right]$$

$$+ \frac{1}{2}.$$
Subject to the constraints:

\[ x_{1i} + x_{12} + x_{13} + x_{14} = 8, \]
\[ x_{2i} + x_{22} + x_{23} + x_{24} = 19, \]
\[ x_{3i} + x_{32} + x_{33} + x_{34} = 17, \]
\[ x_{1i} + x_{12} + x_{31} = 11, \]
\[ x_{12} + x_{22} + x_{32} = 3, \]
\[ x_{13} + x_{23} + x_{33} = 14, \]
\[ x_{14} + x_{24} + x_{34} = 16, \]
\[ x_{ij} \geq 0, i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4, \lambda \geq 0. \]

Solution of this model is:

The optimal allocations are

\[ x_{11} = 0.02345065, x_{12} = 3, x_{13} = 4.976549, x_{21} = 10.97655, \]
\[ x_{23} = 8.023451, x_{33} = 1, x_{34} = 16. \]

Using these allocations we have \( Z_1 = 175.9062, Z_2 = 175.1173. \)

With degree of satisfaction \( \lambda = 0.900623 \)

**Numerical illustration: 2 (same as Numerical illustration-2 in section 3.4)**

**Solution:**

(1) Effect measure matrices under goals are given below:

\[ U^{(2)} = \begin{bmatrix} 2 & 3 & 5 & 11 & 4 & 2 \\ 4 & 7 & 9 & 5 & 10 & 4 \\ 12 & 25 & 9 & 6 & 26 & 12 \\ 8 & 7 & 9 & 24 & 10 & 8 \end{bmatrix}, \quad \bar{U}^{(2)} = \begin{bmatrix} 1 & 2 & 7 & 7 & 4 & 4 \\ 1 & 9 & 3 & 4 & 5 & 8 \\ 8 & 9 & 4 & 6 & 6 & 2 \\ 3 & 4 & 9 & 10 & 5 & 1 \end{bmatrix} \]

(2) For transporting a product goals are less than its the batter, hence, here we use lower effect measure. So the lower effect measure for first data

\[ r_{11}^{(1)} = \frac{\min \{ u_{1i} \}}{u_{11}} = \frac{2}{2}. \]
Similarly, obtain lower effect measure for each data and convert comprehensive matrix \( R^{(k)} \) of \( k^{th} \) objective in minimization form for all objectives by subtracting each data of comprehensive matrix \( R \) of effect measure from 1 we get,

\[
R^{(1)} = \begin{bmatrix}
0 & 0.3333 & 0.6 & 0.8182 & 0.5 & 0 \\
0.5 & 0.5714 & 0.5556 & 0.2 & 0.6 & 0.5 \\
0.8333 & 0.88 & 0.4444 & 0.1667 & 0.8462 & 0.8333 \\
0.75 & 0.5714 & 0.4444 & 0.7917 & 0.6 & 0.75 \\
\end{bmatrix}
\]

\[
R^{(2)} = \begin{bmatrix}
0 & 0.5 & 0.8571 & 0.8571 & 0.75 & 0.75 \\
0 & 0.8889 & 0.6667 & 0.75 & 0.8 & 0.875 \\
0.875 & 0.7778 & 0.5 & 0.6667 & 0.6667 & 0.5 \\
0.6667 & 0.75 & 0.8889 & 0.9 & 0.8 & 0 \\
\end{bmatrix}
\]

(3) Find each objective solutions for multi-objective transportation problem by using comprehensive matrix of effect measure with simplex method.

For first objective: The optimal allocations are

\[ x_{11} = 1, x_{16} = 5, x_{21} = 2, x_{22} = 5, x_{25} = 1, x_{33} = 1, x_{34} = 6, x_{43} = 4, x_{45} = 3 \]

Apply these allocations to first, second objective therefore we have

\[ Z_1(X^1) = 9.4794, Z_2(X^1) = 19.45 \]

For second objective: The optimal allocations are

\[ x_{12} = 5, x_{15} = 1, x_{21} = 3, x_{24} = 5, x_{33} = 5, x_{34} = 1, x_{35} = 1, x_{45} = 2, x_{45} = 5 \]

Apply these allocations to first, second objective therefore we have

\[ Z_1(X^2) = 12.8517, Z_2(X^2) = 12.4333 \]

Hence, pay-off matrix

\[
\begin{bmatrix}
Z_1(X^1) & Z_2(X^1) \\
Z_1(X^2) & Z_2(X^2) \\
\end{bmatrix} = \begin{bmatrix}
9.4794 & 19.45 \\
12.8517 & 12.4333 \\
\end{bmatrix}
\]

Now,

\[ U_1 = \max (9.4794, 12.8517) = 12.8517, \]

\[ L_1 = \min (9.4794, 12.8517) = 9.4794, \]

\[ U_1 - L_1 = 3.3723, U_1 + L_1 = 22.3311, \]

\[ U_2 = \max (12.4333, 19.45) = 19.45, \]

\[ L_2 = \min (12.4333, 19.45) = 12.4333, \]

\[ U_2 - L_2 = 7.0167, U_2 + L_2 = 31.8833 \]

Applying fuzzy linear membership function, we get the following model
Maximize $\lambda$, 
\[ 0x_{11} + 0.3333x_{12} + 0.6x_{21} + 0.8182x_{23} + 0.5x_{45} + 0.5x_{16} + 0.5x_{25} + 0.5714x_{22} + 0.5556x_{23} + 0.2x_{34} + 0.6x_{25} + 0.5x_{36} + 0.8333x_{31} + 0.88x_{32} + 0.4444x_{33} + 0.1667x_{34} + 0.8462x_{35} + 0.8333x_{36} + 0.75x_{41} + 0.5714x_{42} + 0.4444x_{43} + 0.7917x_{44} + 0.6x_{45} + 0.75x_{46} + \lambda(3.3723) \leq 12.8517, \]

Subject to the constraints:
\begin{align*}
\text{Constraint } 1: & \quad x_{11} + x_{21} + x_{31} + x_{41} + x_{16} = 6, \\
\text{Constraint } 2: & \quad x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} = 8, \\
\text{Constraint } 3: & \quad x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} = 7, \\
\text{Constraint } 4: & \quad x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} = 7, \\
\text{Constraint } 5: & \quad x_{11} + x_{21} + x_{31} + x_{41} = 3, \\
\text{Constraint } 6: & \quad x_{12} + x_{22} + x_{32} + x_{42} = 5, \\
\text{Constraint } 7: & \quad x_{13} + x_{23} + x_{33} + x_{43} = 5, \\
\text{Constraint } 8: & \quad x_{14} + x_{24} + x_{34} + x_{44} = 6, \\
\text{Constraint } 9: & \quad x_{15} + x_{25} + x_{35} + x_{45} = 4, \\
\text{Constraint } 10: & \quad x_{16} + x_{26} + x_{36} + x_{46} = 5, \\
\text{Constraint } 11: & \quad x_{ij} \geq 0, i = 1, 2, 3, 4 \text{ and } j = 1, 2, 3, 4, 5, 6, \lambda \geq 0.
\end{align*}

**Solution of this model:**

The optimal allocations are
\[ x_{11} = 3, x_{12} = 1.105526, x_{16} = 1.894474, x_{24} = 4, x_{25} = 4, x_{33} = 5, x_{34} = 2, x_{42} = 3.894474, x_{46} = 3.105526. \]

Using these allocations we have $Z_1 = 182.211052, Z_2 = 99.47237$.

With degree of satisfaction $\lambda = 0.6444804$. 

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Applying fuzzy hyperbolic membership function, we get the following model

Maximize $\lambda$

$$\begin{align*}
\lambda & \leq 1 - e^1 \\
\lambda & \leq 1 - e^2
\end{align*}$$

Subject to the constraints

$$\begin{align*}
x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} & = 6, \\
x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} & = 8, \\
x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} & = 7, \\
x_{41} + x_{42} + x_{43} + x_{44} + x_{45} + x_{46} & = 7, \\
x_{11} + x_{21} + x_{31} + x_{41} & = 3, \\
x_{12} + x_{22} + x_{32} + x_{42} & = 5, \\
x_{13} + x_{23} + x_{33} + x_{43} & = 5, \\
x_{14} + x_{24} + x_{34} + x_{44} & = 6, \\
x_{15} + x_{25} + x_{35} + x_{45} & = 4, \\
x_{16} + x_{26} + x_{36} + x_{46} & = 5,
\end{align*}$$

$x_{ij} \geq 0, i=1,2,3,4$ and $j=1,2,3,4,5,6, \lambda \geq 0$.

Solution of this model:

The optimal allocations are

$$\begin{align*}
x_{11} & = 3, x_{12} = 1.105526, x_{16} = 1.894474, x_{24} = 4, x_{25} = 4, \\
x_{33} & = 5, x_{34} = 2, x_{42} = 3.894474, x_{46} = 3.105526.
\end{align*}$$

Using these allocations we have $Z_1 = 182.211052, Z_2 = 99.47237$.

With degree of satisfaction $\hat{\lambda} = 0.849893397$. 

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Numerical illustration: 3 (same as Numerical illustration-3 in section 3.4)

Solution:

(1) Effect measure matrices under goals are given below:

\[ U^{(1)} = \begin{bmatrix} 5 & 9 & 3 \\ 4 & 6 & 2 \\ 2 & 1 & 1 \end{bmatrix}, \quad U^{(2)} = \begin{bmatrix} 4 & 7 & 3 \\ 2 & 1 & 5 \\ 7 & 6 & 6 \end{bmatrix} \]

(2) For transporting a product goals are less than its better, hence, here we use lower effect measure. So the lower effect measure for first data is

\[ r_{11}^{(1)} = \frac{\min \min \{u_{11}\}}{u_{11}} = 2 \]

Similarly, obtain lower effect measure for each data and convert comprehensive matrix \( R_k^{(k)} \) of \( k \)th objective in minimization form for all objectives by subtracting each data of comprehensive matrix \( R \) of effect measure from 1 we get,

\[ R^{(1)} = \begin{bmatrix} 0.6 & 0.8889 & 0.6667 \\ 0.5 & 0.8333 & 0.5 \\ 0.5 & 0 & 0 \end{bmatrix}, \quad R^{(2)} = \begin{bmatrix} 0.5 & 0.8571 & 0 \\ 0.5 & 0 & 0.8 \\ 0.7143 & 0.8333 & 0.5 \end{bmatrix}. \]

(3) Find each objective solutions for multi-objective transportation problem by using comprehensive matrix of effect measure with simplex method.

For first objective:

\[ x_{11} = 10, x_{15} = 5, x_{23} = 20, x_{32} = 4, x_{33} = 6, \]

Apply these allocations to first, second objective therefore we have

\[ Z_1(X^1) = 19.335, Z_2(X^1) = 27.3333 \]

For second objective:

\[ x_{13} = 15, x_{21} = 10, x_{22} = 4, x_{23} = 6, x_{33} = 10, \]

Apply these allocations to first, second objective therefore we have

\[ Z_1(X^2) = 21.3333, Z_2(X^2) = 14.8 \]

Hence, pay-off matrix=

\[ \begin{bmatrix} Z_1(X^1) & Z_2(X^1) \\ Z_1(X^2) & Z_2(X^2) \end{bmatrix} = \begin{bmatrix} 19.335 & 27.3333 \\ 21.3333 & 14.8 \end{bmatrix}, \]
Now,

\[ U_1 = \max (19.3335, 21.3333) = 21.3333, \]
\[ L_1 = \min (19.3335, 21.3333) = 19.3335, \]
\[ U_1 - L_1 = 1.9998, U_1 + L_1 = 40.6668, \]
\[ U_2 = \max (27.3333, 14.8) = 27.3333, \]
\[ L_2 = \min (27.3333, 14.8) = 14.8, \]
\[ U_2 - L_2 = 12.5333, U_2 + L_2 = 42.1333 \]

Applying fuzzy linear membership function, we get the following model

Maximize \( \lambda \)

\[
0.6x_{11} + 0.8889x_{12} + 0.6667x_{13} + 0.5x_{21} + 0.8333x_{22} + 0.5x_{23} \\
+ 0.5x_{31} + 0x_{32} + 0x_{33} + \lambda (1.9998) \leq 21.3333, \\
0.5x_{11} + 0.8571x_{12} + 0x_{13} + 0.5x_{21} + 0x_{22} + 0.8x_{23} \\
+ 0.7143x_{31} + 0.8333x_{32} + 0.5x_{33} + \lambda (12.5333) \leq 27.3333, 
\]

Subject to the constraints:

\[
x_{11} + x_{12} + x_{13} = 15, \\
x_{21} + x_{22} + x_{23} = 20, \\
x_{31} + x_{32} + x_{33} = 10, \\
x_{11} + x_{21} + x_{31} = 10, \\
x_{12} + x_{22} + x_{32} = 4, \\
x_{13} + x_{23} + x_{33} = 31, \\
x_{ij} \geq 0, \forall i, j, \lambda \geq 0. 
\]

**Solution of this model:**

The optimal allocations are

\[
x_{11} = 15, x_{21} = 10, x_{22} = 0.10953, x_{23} = 9.89047, \\
x_{32} = 3.89047, x_{33} = 6.10953. 
\]

Using these allocations we have \( Z_1 = 115.4381, Z_2 = 174.5619. \)

With degree of satisfaction \( \lambda = 0.64821116. \)
Applying fuzzy hyperbolic membership function, we get the following model

Maximize $\lambda$

\[
\lambda \leq \frac{1}{2} e^{\left(\frac{40.6668}{2}\right)} \left[0.6 x_{11} + 0.8889 x_{12} + 0.6667 x_{13} + 0.5 x_{21} + 0.5 x_{31} + 0.5 x_{12} + 0.8333 x_{13} + 0.5 x_{22} + 0.5 x_{32} + 0.5 x_{13} + 0.5 x_{23} + 0.5 x_{33}\right] - \frac{1}{2} e^{\left(\frac{40.6668}{2}\right)} \left[0.6 x_{11} + 0.8889 x_{12} + 0.6667 x_{13} + 0.5 x_{21} + 0.5 x_{31} + 0.5 x_{12} + 0.8333 x_{13} + 0.5 x_{22} + 0.5 x_{32} + 0.5 x_{13} + 0.5 x_{23} + 0.5 x_{33}\right] + 1
\]

Subject to the constraints

\[
\begin{align*}
x_{11} + x_{12} + x_{13} &= 15, \\
x_{21} + x_{22} + x_{23} &= 20, \\
x_{31} + x_{32} + x_{33} &= 10, \\
x_{11} + x_{21} + x_{31} &= 10, \\
x_{12} + x_{22} + x_{32} &= 4, \\
x_{13} + x_{23} + x_{33} &= 31, \\
x_{ij} &\geq 0, \forall i, j, \lambda &\geq 0.
\end{align*}
\]

Solution of this model:

The optimal allocations are

\[
\begin{align*}
x_{13} &= 15, x_{21} = 10, x_{22} = 0.10953, x_{23} = 9.89047, \\
x_{32} &= 3.89047, x_{33} = 6.10953.
\end{align*}
\]

Using these allocations we have $Z_1 = 115.4381, Z_2 = 174.5619$.

With degree of satisfaction $\lambda = 0.855516$.

3.9 Comparison of Grey Situation Decision Making Theory Approach by using Membership Function with Other Approaches:

This section discusses comparison of grey situation decision making theory based approach by using membership function with other developed approaches. The comparison table shows that the grey situation decision making theory based approach by using membership function provides very efficient alternative approach to find solution of multi-objective transportation problem.
<table>
<thead>
<tr>
<th>Sr.no.</th>
<th>Multi-objective Grey situation decision(MGSD) making theory approach(using membership function)</th>
<th>Other approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Numerical Illustration: 1</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Z_1 = 175.9062$, $Z_2 = 175.1173$. using linear membership function, $\lambda = 0.6836805$. using hyperbolic membership function, $\lambda = 0.900623$.</td>
<td>By Waiel et.al. [107]: $Z_1 = 186$, $Z_2 = 173$.</td>
</tr>
<tr>
<td></td>
<td><strong>Numerical Illustration: 2</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Z_1 = 182.211052$, $Z_2 = 99.47237$. using linear membership function, $\lambda = 0.644804$. using hyperbolic membership function, $\lambda = 0.84989397$.</td>
<td>By W.Ritha et.al. [108]: $Z_1 = 171$, $Z_2 = 101$.</td>
</tr>
<tr>
<td></td>
<td><strong>Numerical Illustration: 3</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$Z_1 = 115.4381$, $Z_2 = 174.5619$. using linear membership function, $\lambda = 0.64821116$. using hyperbolic membership function, $\lambda = 0.855516$.</td>
<td>By Abbas et.al.[109]: $Z_1 = 161$, $Z_2 = 199$.</td>
</tr>
</tbody>
</table>

**Table 3.6: Comparison of numerical illustrations with other approach:**

Table 3.6 contains comparison of multi-objective transportation problems by using membership function with different approaches. Table 3.6 shows that degree of satisfaction using hyperbolic membership function is better than linear membership function. It provides alternative compromised solution of MOTP.

**Conclusion:**

Chapter has discussed Grey situation decision making theory and fuzzy programming technique based model to find the solution of Multi-Objective Transportation Problem and provide an alternative approach to find the solution of MOTP as well as other multi objective problems.