Chapter 5

Weighted xgamma distribution

In the previous chapters, we have introduced and studied the xgamma distribution and its truncated versions mainly the upper truncated one. It is observed in Chapter 4 that the upper truncated version of xgamma distribution provides some flexibility over xgamma for modeling time-to-event data set. However, as mentioned in the last paragraph of Chapter 4, the applicability of upper truncated xgamma distribution is very specific. In this chapter, we study a weighted version of xgamma model and try to find an application in lifetime data. The study of weighted distributions is useful for two main purposes, it could provide a new understanding of the baseline distribution on which weight is considered and it might provide a method of extending the baseline distribution for added flexibility in fitting data. We discuss below the concept and applicability of weighted distributions.

The concept of weighted distributions can be traced back to Fisher (1934) in the study of the effect of methods of ascertainment upon estimation of frequencies. While extending the basic ideas of Fisher, Rao (1965, 1985) saw the need for a unifying concept by identifying various sampling situations that can be modeled by what he termed as weighted distributions. Zelen (1974) introduced weighted
Weighted xgamma distribution


In this chapter, the weighted version of xgamma distribution as a generalization of xgamma distribution (see Chapter 2 and Chapter 3) is studied, with special reference study is made to its length biased version. The method of moments and method of maximum likelihood estimation are proposed to estimate the unknown parameter of the length biased xgamma distribution. The length biased xgamma distribution is applied for modeling time-to-event data set and compared with other life distributions for complete sample situation.

5.1 Synthesis of weighted xgamma distribution

The form of the pdf of weighted distribution, by definition (see Patil, 1988), is given by

\[ f(x) = \frac{w(x)f_0(x)}{E[w(X)]}, \tag{5.1} \]

where \( w(x) \) is weight function which in non-negative and \( f_0(x) \) is a probability density function.
To synthesize the weighted version of xgamma distribution, we take \( w(x) = x^r \) for \( r = 1, 2, 3, \ldots \), and \( f_0(x) \) is taken as the pdf of xgamma distribution as in (2.2).

Note that, taking \( w(x) = x^r \) for \( r = 1, 2, 3, \ldots \), \( E[w(X)] \) is nothing but the \( r \)th order non-central (raw) moment of xgamma distribution given in (2.10), i.e.,

\[
E(X^r) = \frac{r!}{\theta^r(1+\theta)} \left[ \theta + \frac{(1+r)(2+r)}{2} \right], \quad \text{for } r = 1, 2, 3, \ldots.
\]

Now using (5.1) and putting the expressions for \( w(x) \), \( E[w(X)] \) and \( f_0(x) \), the pdf of \( r \)th order moment weighted version of xgamma distribution can be derived as

\[
f(x) = \frac{x^r \theta^2}{r! (1+\theta)} \left( 1 + \frac{\theta}{2} x^2 \right) e^{-\theta x}
\]

\[
= \frac{r!}{\theta^r(1+\theta)} \left[ \theta + \frac{(1+r)(2+r)}{2} \right] \left( x^r + \frac{\theta x^r}{2} \right) e^{-\theta x}.
\]

We have the following definition for the weighted xgamma distribution.

**Definition 5.1.** A non-negative continuous random variable, \( X \), is said to follow weighted xgamma (WXG) distribution with parameters \( r \) and \( \theta \) if its pdf is of the form

\[
f(x) = \frac{2\theta^{r+2}}{r! [2\theta + (1+r)(2+r)]} \left( x^r + \frac{\theta x^r}{2} \right) e^{-\theta x}, \quad x > 0, \theta > 0, r = 1, 2, 3, \ldots.
\]

(5.2)

It is denoted by \( X \sim WXG(r, \theta) \).

The Figure 5.1 shows the plot of density functions for weighted xgamma distribution for different values of \( r \) and \( \theta \).

**Non-central moments**

Now, we find the non-central moments for \( WXG(r, \theta) \).
Figure 5.1: Probability density curves of weighted xgamma distribution for different values of $\theta$ and $r$

The $k^{\text{th}}$, for $k = 1, 2, 3, \ldots$, order non-central moment of $WXG(r, \theta)$ can be obtained as

$$
\mu_k' = E[X^k] = \int_0^\infty x^k f(x) dx,
$$

$$
= \frac{2^{\theta+r+2}}{r![2\theta + (1 + r)(2 + r)]} \int_0^\infty x^k \left( x^r + \frac{\theta}{2} x^{r+2} \right) e^{-\theta x} dx,
$$

$$
= \frac{2^{\theta+r+2}}{r![2\theta + (1 + r)(2 + r)]} \left[ \int_0^\infty x^{r+k} e^{-\theta x} dx + \frac{\theta}{2} \int_0^\infty x^{r+k+2} e^{-\theta x} dx \right].
$$
Hence, we have,

$$\mu'_k = \frac{2\theta^{r+k+2}}{r![2\theta + (1 + r)(2 + r)]} \left[ \Gamma(r + k + 1) \frac{\theta^{r+k+1}}{\theta^{r+k+1}} + \frac{\theta}{2} \Gamma(r + k + 3) \right],$$

Here $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z}dz$ is the gamma function.

$$= \frac{2\theta^{r+k+2}}{r![2\theta + (1 + r)(2 + r)]} \left[ \frac{(r + k)!}{\theta^{r+k+1}} + \frac{(r + k + 2)!}{2\theta^{r+k+2}} \right],$$

$$= \frac{2\theta^{r+k+2}(r + k)!}{r![2\theta + (1 + r)(2 + r)]}\theta^{r+k+1} \left[ 1 + \frac{(r + k + 2)(r + k + 1)}{2\theta} \right],$$

$$= \frac{(r + k)!}{r!\theta^k} \left[ \frac{2\theta + (r + k + 2)(r + k + 1)}{\theta} \right],$$

$$= \frac{(r + 1)!\left[ 2\theta + (r + 2)(r + 3) \right]}{r!\theta \left[ 2\theta + (r + 1)(r + 2) \right]},$$

(5.3)

In particular, by putting $k = 1$ in (5.3), the mean of $WXG(r, \theta)$ distribution is obtained as

$$E(X) = \frac{(r + 1)!\left[ 2\theta + (r + 2)(r + 3) \right]}{r!\theta \left[ 2\theta + (r + 1)(r + 2) \right]},$$

(5.4)

Similarly, putting $k = 2$ in (5.3), the second order raw moment for $WXG(r, \theta)$ is obtained as

$$\mu'_2 = E(X^2) = \frac{(r + 1)(r + 2)\left[ 2\theta + (r + 3)(r + 4) \right]}{\theta^2 \left[ 2\theta + (r + 1)(r + 2) \right]},$$

(5.5)

Next, we find the expressions for cdf, survival function and hazard rate function of $WXG(r, \theta)$. 
Cumulative distribution function

The cdf of $W(XG(r, \theta))$ can be obtained as

$$F(x) = P(X \leq x),$$

$$= \int_0^x \frac{2\theta^{r+2}}{r!}[2\theta + (1 + r)(2 + r)] \left(t^r + \frac{\theta}{2} t^{r+2}\right) e^{-\theta t} dt,$$

Putting $\theta t = u$, we have,

$$= \frac{2\theta^{r+2}}{r!}[2\theta + (1 + r)(2 + r)] \left[\int_0^{\theta x} \left(\frac{u}{\theta}\right)^r e^{-u} du + \frac{\theta}{2} \int_0^{\theta x} \left(\frac{u}{\theta}\right)^{r+2} e^{-u} du\right],$$

$$= \frac{2\theta^{r+2}}{r!}[2\theta + (1 + r)(2 + r)] \left[\int_0^{\theta x} u^{r-1} e^{-u} du + \frac{1}{2\theta} \int_0^{\theta x} u^{r+2} e^{-u} du\right],$$

$$= \frac{2\theta}{r!}[2\theta + (1 + r)(2 + r)] \left[\gamma(r + 1, \theta x) + \frac{1}{2\theta} \gamma(r + 3, \theta x)\right],$$

(5.6)

where $\gamma(a, x) = \int_x^{\infty} u^{a-1} e^{-u} du$ is the lower incomplete gamma function.

Survival function

The survival function of $W(XG(r, \theta))$ can be derived as

$$S(x) = Pr(X > x),$$

$$= \int_x^{\infty} \frac{2\theta^{r+2}}{r!}[2\theta + (1 + r)(2 + r)] \left(u^r + \frac{\theta}{2} u^{r+2}\right) e^{-\theta u} du,$$

Putting $\theta t = u$, we have,

$$= \frac{2\theta^{r+2}}{r!}[2\theta + (1 + r)(2 + r)] \left[\int_{\theta x}^{\infty} \left(\frac{u}{\theta}\right)^r e^{-u} du + \frac{\theta}{2} \int_{\theta x}^{\infty} \left(\frac{u}{\theta}\right)^{r+2} e^{-u} du\right],$$

$$= \frac{2\theta^{r+2}}{r!}[2\theta + (1 + r)(2 + r)] \left[\int_{\theta x}^{\infty} u^{r-1} e^{-u} du + \frac{1}{2\theta} \int_{\theta x}^{\infty} u^{r+2} e^{-u} du\right],$$

$$= \frac{2\theta}{r!}[2\theta + (1 + r)(2 + r)] \left[\Gamma(r + 1, \theta x) + \frac{1}{2\theta} \Gamma(r + 3, \theta x)\right],$$

(5.7)

where $\Gamma(a, x) = \int_x^{\infty} u^{a-1} e^{-u} du$ is the upper incomplete gamma function.
Weighted xgamma distribution

Hazard rate or failure rate function
The failure rate or hazard rate function of \( WXG(r, \theta) \) is obtained as

\[
h(x) = \frac{f(x)}{S(x)} = \frac{\theta^{r+1} \left(x^r + \frac{\theta}{2} x^{r+2}\right) e^{-\theta x}}{\Gamma(r+1, \theta x) + \frac{1}{2\theta} \Gamma(r+3, \theta x)} ; x > 0, r = 1, 2, 3, \ldots \tag{5.8}
\]

The main emphasis is given in studying the length biased version of xgamma distribution hereafter. The rest of the chapter is organized as follows.

The length biased version for xgamma distribution is described along with its moments and related measures in section 5.2. Distributions of order statistics for length biased xgamma distribution are derived in section 5.3. Important entropy measures are described in section 5.4 and different survival properties are studied in section 5.5 for length biased version of xgamma distribution. Section 5.6 deals with the methods of estimation for the unknown parameter in length biased xgamma model for complete sample case. An algorithm for generating random samples from length biased xgamma along with a Monte-Carlo simulation study is presented in section 5.7. Real data illustration is described in section 5.8 for studying the application of length biased xgamma model. Finally, the section 5.9 summaries the chapter mentioning important finding and some open research problems for future investigation.

5.2 The length biased xgamma distribution

This section deals with the length biased version of xgamma distribution. The length biased version of xgamma distribution is obtained as a special case of weight xgamma distribution discussed in the previous section.

If we put \( r = 1 \) in (5.2), then we obtain so called length biased version of the xgamma distribution. We have the following definition for the length biased xgamma distribution.
Definition 5.2. A non-negative continuous random variable, \( X \), is said to follow the length biased xgamma (LBXG) distribution with parameter \( \theta \) if its pdf is of the form

\[
f(x) = \frac{\theta^3}{(\theta + 3)} \left( x + \frac{\theta}{2} x^3 \right) e^{-\theta x}, x > 0, \theta > 0.
\]

(5.9)

It is denoted by \( X \sim \text{LBXG}(\theta) \).

We note that the length biased xgamma distribution is a special mixture of \( \text{gamma}(2, \theta) \) and \( \text{gamma}(4, \theta) \) with mixing proportions \( \theta/(3 + \theta) \) and \( 3/(3 + \theta) \), respectively.

The probability density plots of \( \text{LBXG}(\theta) \) for different values of \( \theta \) is shown in Figure 5.2.
Now, for finding cdf of $LBXG(\theta)$, we calculate the followings.

$$F(x) = \Pr(X \leq x) = 1 - \Pr(X > x).$$

Now, we find $\Pr(X > x)$.

$$\Pr(X > x) = \int_x^\infty \frac{\theta^3}{(\theta + 3)} \left( t + \frac{\theta}{2} t^3 \right) e^{-\theta t} dt,$$

$$= \frac{\theta^3}{(\theta + 3)} \left[ \int_x^\infty t e^{-\theta t} dt + \frac{\theta}{2} \int_x^\infty t^3 e^{-\theta t} dt \right]. \tag{5.10}$$

Now, integrating by parts, we can have,

$$\int_x^\infty t^3 e^{-\theta t} dt = \frac{x^3 e^{-\theta x}}{\theta} + \frac{3}{\theta} \int_x^\infty t^2 e^{-\theta t} dt,$$

$$= \frac{x^3 e^{-\theta x}}{\theta} + \frac{3}{\theta} \left\{ \frac{x^2 e^{-\theta x}}{\theta} + \frac{2}{\theta} \left( \frac{x e^{-\theta x}}{\theta} + e^{-\theta x} \right) \right\},$$

$$= \frac{x^3 e^{-\theta x}}{\theta} + \frac{3x^2 e^{-\theta x}}{\theta^2} + \frac{6x e^{-\theta x}}{\theta^3} + \frac{6 e^{-\theta x}}{\theta^4}. \tag{5.11}$$

So, using (2.4) and (5.11), from (5.10), we have,

$$\Pr(X > x)$$

$$= \frac{\theta^3}{(\theta + 3)} \left[ \frac{x e^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2} + \frac{3}{2} \left( \frac{x^3 e^{-\theta x}}{\theta} + \frac{3x^2 e^{-\theta x}}{\theta^2} + \frac{6x e^{-\theta x}}{\theta^3} + \frac{6 e^{-\theta x}}{\theta^4} \right) \right],$$

$$= \frac{\theta^3}{(\theta + 3)} \left[ \frac{x e^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2} + \frac{x^3 e^{-\theta x}}{2 \theta} + \frac{3x^2 e^{-\theta x}}{2 \theta^2} + \frac{3x e^{-\theta x}}{\theta^3} + \frac{3 e^{-\theta x}}{\theta^4} \right],$$

$$= \frac{\theta^3 e^{-\theta x}}{(\theta + 3)} \left[ \frac{2\theta^2 x + 2 \theta x + \theta^3 x^3 + 3 \theta^2 x^2 + 6 \theta x + 6}{2 \theta^3} \right],$$

$$= \frac{e^{-\theta x}}{(\theta + 3)} \left[ (3 + \theta) + (3 + \theta) \theta x + \frac{3}{2} \theta^2 x^2 + \frac{1}{2} \theta^3 x^3 \right].$$

Hence, the cdf of $X \sim LBXG(\theta)$ is given by

$$F(x) = 1 - \frac{[(3 + \theta) + (3 + \theta) \theta x + \frac{3}{2} \theta^2 x^2 + \frac{1}{2} \theta^3 x^3]}{(\theta + 3)} e^{-\theta x}, x > 0. \tag{5.12}$$
The characteristic function of $LBXG(\theta)$ is obtained as

$$\phi_X(t) = E \left[ e^{itX} \right] = \frac{\theta^3}{(\theta + 3)} \int_0^\infty e^{itx} \left( x + \frac{\theta}{2} x^3 \right) e^{-\theta x} dx,$$

$$= \frac{\theta^3}{(\theta + 3)} \left[ \int_0^\infty xe^{-(\theta+it)x} dx + \frac{\theta}{2} \int_0^\infty x^3 e^{-(\theta+it)x} dx \right],$$

$$= \frac{\theta^3}{(\theta + 3)} \left[ \frac{\Gamma(2)}{(\theta - it)^2} + \frac{\theta \Gamma(4)}{2(\theta - it)^4} \right],$$

Here $\Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz$ is the gamma function.

$$= \frac{\theta^3}{(\theta + 3)} \left[ \frac{1}{(\theta - it)^2} + \frac{3\theta}{(\theta - it)^4} \right],$$

$$= \frac{\theta^3}{(\theta + 3)} \left[ (\theta - it)^{-2} + 3\theta(\theta - it)^{-4} \right]; i = \sqrt{-1}, t \in \mathbb{R}. \quad (5.13)$$

### 5.2.1 Moments and associated measures

Now, we find the moments and measures related to moments of $LBXG(\theta)$.

The $k^{th}$ order raw moment, $\mu_k'$ for $k = 1, 2, 3, \ldots$, of length-biased xgamma distribution can be obtained either directly using the pdf in (5.9) or by substituting $k = 1, 2, 3, \ldots$ in (5.3) after putting $r = 1$.

Hence, we have

$$\mu_k' = E(X^k) = \frac{(k+1)! [2\theta + (2 + k)(3 + k)]}{2\theta^k(\theta + 3)} \quad \text{for} \quad k = 1, 2, 3, \ldots \quad (5.14)$$

In particular,

$$E(X) = \frac{2(\theta + 6)}{\theta(\theta + 3)}; \quad E(X^2) = \frac{6(\theta + 10)}{\theta^2(\theta + 3)}. \quad (5.15)$$
So, we have the expression for second order central moment or the population variance for $X$ as

$$\text{Var}(X) = \mu_2 = \mu'_2 - \mu'_1,$$

$$= \frac{6(\theta + 10)}{\theta^2(\theta + 3)} - \left[ \frac{2(\theta + 6)}{\theta(\theta + 3)} \right]^2,$$

$$= \frac{6(\theta + 10)(\theta + 3) - 4(\theta + 6)}{\theta^2(\theta + 3)^2},$$

On simplification, we have,

$$= \frac{2(\theta^2 + 15\theta + 18)}{\theta^2(\theta + 3)^2} \quad (5.16)$$

so that the coefficient of variation (CV) becomes

$$\gamma = \sqrt{\text{Var}(X)} = \frac{\sqrt{2(\theta^2 + 15\theta + 18)}}{2(\theta + 6)}. \quad (5.17)$$

The moment generating function of $X$ is derived as

$$M_X(t) = E \left[ e^{tX} \right],$$

$$= \frac{\theta^3}{(\theta + 3)} \int_0^\infty e^{tx} \left( x + \frac{\theta}{2} x^3 \right) e^{-\theta x} dx,$$

$$= \frac{\theta^3}{(\theta + 3)} \left[ \int_0^\infty xe^{-(\theta-t)x} dx + \frac{\theta}{2} \int_0^\infty x^3 e^{-(\theta-t)x} dx \right],$$

$$= \frac{\theta^3}{(\theta + 3)} \left[ \Gamma(2) \left( \frac{\theta}{(\theta-t)^2} \right) + \frac{\theta \Gamma(4)}{2(\theta-t)^4} \right],$$

Here $\Gamma(a) = \int_0^\infty z^{a-1}e^{-z} dz$ is the gamma function.

$$= \frac{\theta^3}{(\theta + 3)} \left[ \frac{1}{(\theta-t)^2} + \frac{3\theta}{(\theta-t)^4} \right],$$

$$= \frac{\theta^3}{(\theta + 3)} \left[ (\theta-t)^{-2} + 3\theta(\theta-t)^{-4} \right]; t \in \mathbb{R}. \quad (5.18)$$
The cumulant generating function of $X$ is obtained as

$$K_X(t) = \ln M_X(t),$$

$$= \ln \frac{\theta^3}{(\theta + 3)}[(\theta - t)^{-2} + 3\theta(\theta - t)^{-4}],$$

$$= \ln \frac{\theta^3}{(\theta + 3)(\theta - t)^2} + \ln \left[1 + 3\theta(\theta - t)^{-2}\right]; t \in \mathbb{R}. \quad (5.19)$$

5.3 Distributions of order statistics

In this section, we find the distributions of extreme order statistics for $LBXG(\theta)$.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ drawn from $X \sim LBXG(\theta)$. Denote $X_{j:n}$ as the $j^{th}$ order statistic. Then $X_{1:n}$ and $X_{2:n}$ denote the smallest and largest order statistics for a sample of size $n$ drawn from length-biased xgamma distribution with parameter $\theta$, respectively.

For any $x > 0$, the pdf of $X_{1:n}$ is derived as

$$f_{X_{1:n}}(x) = n[1 - F(x)]^{n-1}f(x),$$

$$= \frac{n\theta^3}{(\theta + 3)^n} \left[(3 + \theta) + (3 + \theta)\theta x + \frac{3}{2} \theta^2 x^2 + \frac{1}{2} \theta^3 x^3\right]^{n-1} e^{-n\theta x}. \quad (5.20)$$

Similarly, for any $x > 0$, the pdf of $X_{n:n}$ is obtained as

$$f_{X_{n:n}}(x) = n[F(x)]^{n-1}f(x),$$

$$= \frac{n\theta^3}{(\theta + 3)} \left[1 - \frac{(3 + \theta) + (3 + \theta)\theta x + \frac{3}{2} \theta^2 x^2 + \frac{1}{2} \theta^3 x^3 e^{-\theta x}}{(\theta + 3)}\right]^{n-1} e^{-\theta x},$$

after simple arrangements,

$$= \frac{n\theta^3}{(\theta + 3)^n} \left[(\theta + 3)\{1 - e^{-\theta x}(1 + \theta x)\} - \frac{1}{2} \theta^2 x^2 e^{-\theta x}(3 + \theta x)\right]^{n-1} e^{-\theta x}. \quad (5.21)$$
5.4 Entropy measures

We first derive the Rényi entropy measure when \( X \sim LBXG(\theta) \). We derive,

\[
\int_0^\infty f^\gamma(x)dx
\]

\[
= \int_0^\infty \left[ \frac{\theta^3}{(\theta + 3)^\gamma} \left( x + \frac{\theta}{2} x^3 \right) e^{-\theta x} \right] dx, \quad \text{for} \; \gamma > 0 (\neq 1),
\]

\[
= \frac{\theta^{3\gamma}}{(\theta + 3)^\gamma} \int_0^\infty \left( x + \frac{\theta}{2} x^3 \right)^\gamma e^{-\theta x}dx,
\]

Putting \( 1 + \frac{\theta}{2} x^2 = \sum_{j=0}^{\gamma} \frac{\gamma_j}{j} \left( \frac{\theta x^2}{2} \right)^j \),

\[
= \frac{\theta^{3\gamma}}{(\theta + 3)^\gamma} \sum_{j=0}^{\gamma} \frac{\gamma_j}{j} \int_0^\infty \left( \left( \frac{\theta x^2}{2} \right)^j \right) x^{2j+\gamma} e^{-\theta x} dx,
\]

\[
= \frac{\theta^{3\gamma}}{(\theta + 3)^\gamma} \sum_{j=0}^{\gamma} \frac{\gamma_j}{j} \frac{\Gamma(2j + \gamma + 1)}{2^{2j+\gamma+1} \theta^j \Gamma(\gamma + 1) \gamma^{2j+\gamma+1}},
\]

Here \( \Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz \) is the gamma function.

\[
= \frac{\theta^{3\gamma}}{(\theta + 3)^\gamma} \sum_{j=0}^{\gamma} \frac{\gamma_j}{j} \frac{\Gamma(2j + \gamma + 1)}{2^{2j+\gamma+1} \theta^j \Gamma(\gamma + 1) \gamma^{2j+\gamma+1}}
\]

to obtain Rényi entropy as

\[
H_R(\gamma) = \frac{1}{1-\gamma} \ln \left[ \int_0^\infty f^\gamma(x)dx \right],
\]

\[
= \frac{1}{1-\gamma} \left[ 3\gamma \ln \theta - \gamma \ln(\theta + 3) \right] + \frac{1}{1-\gamma} \ln \left[ \sum_{j=0}^{\gamma} \frac{\gamma}{j} \frac{\Gamma(2j + \gamma + 1)}{2^{2j+\gamma+1} \theta^j \Gamma(\gamma + 1) \gamma^{2j+\gamma+1}} \right].
\]

(5.22)

Now, when \( X \sim LBXG(\theta) \), to obtain Tallis measure of entropy, defined by

\[
S_q(X) = \frac{1}{q - 1} \ln \left[ 1 - \int_0^\infty f^q(x)dx \right] \quad \text{for} \; q > 0 (\neq 1),
\]
we calculate,

\[
\int_0^\infty f^q(x)dx
\]

\[
= \int_0^\infty \left[ \frac{\theta^3}{(\theta + 3)^q} \left( x + \frac{\theta}{2} x^3 \right) e^{-q\theta x} \right] dx,
\]

\[
= \frac{\theta^3}{(\theta + 3)^q} \int_0^\infty \left( x + \frac{\theta}{2} x^3 \right)^q e^{-q\theta x} dx,
\]

\[
= \frac{\theta^3}{(\theta + 3)^q} \int_0^\infty \frac{q}{x^q} \left( 1 + \frac{\theta}{2} x^2 \right)^q e^{-q\theta x} dx,
\]

Putting \( \left( 1 + \frac{\theta}{2} x^2 \right)^q = \sum_{j=0}^{\infty} \binom{q}{j} \left( \frac{\theta x^2}{2} \right)^j \),

\[
= \frac{\theta^3}{(\theta + 3)^q} \sum_{j=0}^{\infty} \binom{q}{j} \frac{\Gamma(2j + q + 1)}{2^{2j} j! \theta^{j+q+1} q^{2j+q+1}},
\]

Here \( \Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz \) is the gamma function.

\[
= \frac{\theta^3}{(\theta + 3)^q} \sum_{j=0}^{\infty} \binom{q}{j} \frac{\Gamma(2j + q + 1)}{2^{2j} j! \theta^{j+q+1} q^{2j+q+1}}.
\]

Hence, the final form of Tsallis entropy is given by

\[
S_q(x) = \frac{1}{1 - q} \left[ 1 - \frac{\theta^3}{(\theta + 3)^q} \sum_{j=0}^{\infty} \binom{q}{j} \frac{\Gamma(2j + q + 1)}{2^{2j} j! \theta^{j+q+1} q^{2j+q+1}} \right]. \quad (5.23)
\]

### 5.5 Survival properties

In this section we study survival properties of \( LBXG(\theta) \).

The survival function of \( X \sim LBXG(\theta) \) is obtained as

\[
S(x) = \Pr(X > x) = \frac{[3 + \theta + (3 + \theta)dx + \frac{3\theta^2 x^2 + 2\theta^3 x}{(\theta + 3)}]}{e^{-\theta x}}, \quad x > 0. \quad (5.24)
\]
5.5.1 Hazard rate or failure rate function

The hazard rate (or failure rate) function is obtained as

\[ h(x) = \frac{f(x)}{S(x)} = \frac{\theta^3 \left( x + \frac{\theta}{2} x^3 \right)}{\left[ (3 + \theta) + (3 + \theta) \theta x + \frac{3}{2} \theta^2 x^2 + \frac{1}{2} \theta^3 x^3 \right]}, \quad x > 0. \quad (5.25) \]

The hazard rate plots for different values of \( \theta \) is shown in the Figure 5.3. It is observed that the hazard rate function in (5.25) is increasing in \( \theta \) and \( x \).

![Figure 5.3: Hazard rate function of length biased xgamma distribution for different values of \( \theta \).](image-url)
5.5.2 MRL function

When $X \sim LBXG(\theta)$, the MRL function can be derived as below.

\[
m(x) = \frac{1}{S(x)} \int_x^\infty S(t)dt,
\]

\[
= \frac{1}{(\theta + 3)S(x)} \int_x^\infty \left[ (3 + \theta) + (3 + \theta)\theta t + \frac{3}{2} \theta^2 t^2 + \frac{1}{2} \theta^3 t^3 \right] e^{-\theta t}dt.
\]

(5.26)

Now,

\[
\int_x^\infty \left[ (3 + \theta) + (3 + \theta)\theta t + \frac{3}{2} \theta^2 t^2 + \frac{1}{2} \theta^3 t^3 \right] e^{-\theta t}dt
\]

\[
= (3 + \theta) \int_x^\infty e^{-\theta t}dt + (3 + \theta) \theta \int_x^\infty te^{-\theta t}dt + \frac{3}{2} \theta^2 \int_x^\infty t^2 e^{-\theta t}dt + \frac{\theta^3}{2} \int_x^\infty t^3 e^{-\theta t}dt,
\]

Using the expressions of integration in (2.3), (2.4), (2.5) and (??), we have,

\[
= (3 + \theta) \frac{e^{-\theta x}}{\theta} + (3 + \theta) \theta \left( \frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2} \right) + \frac{3\theta^2}{2} \left\{ \frac{x^2e^{-\theta x}}{\theta} + \frac{2}{\theta} \left( \frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2} \right) \right\} + \frac{\theta^3}{2} \left( \frac{x^3e^{-\theta x}}{\theta} + \frac{3x^2e^{-\theta x}}{\theta^2} + \frac{6xe^{-\theta x}}{\theta^3} + \frac{6e^{-\theta x}}{\theta^4} \right),
\]

\[
= e^{-\theta x} \left[ \frac{\theta + 3}{\theta} + \frac{(\theta + 3)(1 + \theta x)}{\theta} + \frac{3(\theta^2x^2 + 2\theta x + 2)}{2\theta} + \frac{(\theta^3x^3 + 3\theta^2x^2 + 6\theta x + 6)}{2\theta} \right],
\]

On simplification,

\[
= e^{-\theta x} \left[ \frac{4\theta + 24 + 2\theta^2 x + 18\theta x + 6\theta^2 x^2 + \theta^3 x^3}{2\theta} \right],
\]

\[
= e^{-\theta x} \left[ \frac{2\theta + 12 + \theta^2 x + 9\theta x + 3\theta^2 x^2 + \frac{1}{2} \theta^3 x^3}{\theta} \right].
\]

Using (5.26), we have then,

\[
m(x) = \frac{1}{(\theta + 3)S(x)} \frac{e^{-\theta x}}{\theta} \left[ \frac{2\theta + 12 + \theta^2 x + 9\theta x + 3\theta^2 x^2 + \frac{1}{2} \theta^3 x^3}{\theta} \right],
\]

\[
= \frac{2\theta + 12 + \theta^2 x + 9\theta x + 3\theta^2 x^2 + \frac{1}{2} \theta^3 x^3}{\theta \left[ (3 + \theta) + (3 + \theta)\theta x + \frac{3}{2} \theta^2 x^2 + \frac{1}{2} \theta^3 x^3 \right]}.
\]
On adjustment of the numerator, we have,

$$m(x) = \frac{((3 + \theta) + (3 + \theta)\theta x + \frac{3}{2}\theta^2x^2 + \frac{1}{2}\theta^3x^3) + (\theta + 9 + 6\theta x + \frac{3}{2}\theta^2x^2)}{\theta [(3 + \theta) + (3 + \theta)\theta x + \frac{3}{2}\theta^2x^2 + \frac{1}{2}\theta^3x^3]},$$

$$= \frac{1}{\theta} + \frac{\theta + 9 + 6\theta x + \frac{3}{2}\theta^2x^2}{\theta [(3 + \theta) + (3 + \theta)\theta x + \frac{3}{2}\theta^2x^2 + \frac{1}{2}\theta^3x^3]}.$$

Hence, the MRL function is given by

$$m(x) = \frac{1}{\theta} + \frac{(\theta + 3) + 6(1 + \theta x) + \frac{3}{2}\theta^2x^2}{\theta [(\theta + 3) + (\theta + 3)\theta x + \frac{3}{2}\theta^2x^2 + \frac{1}{2}\theta^3x^3]}.$$

The plots of MRL function for different values of \( \theta \) is shown in the Figure 5.4.

![Figure 5.4: Mean residual life function of length biased xgamma distribution for different values of \( \theta \) ](image-url)
The following points are noted.

(i) It is clear that the hazard rate is increasing function in $x$ ($>0$). The fact can easily be identified as the length biased distribution given in (5.9) is log-concave.

(ii) The MRL function in (5.27) is bounded below by $1/\theta$ and bounded above by $\frac{2(\theta+6)}{\theta(\theta+3)} = E(X)$ and is decreasing in $x$.

(iii) Therefore, the distribution posses increasing failure rate (IFR) and decreasing mean residual life (DMRL) property.

### 5.5.3 Reversed hazard rate function

The reversed hazard rate function of $X \sim LBXG(\theta)$ is given by (see Figure 5.5 for the plots of reversed hazard rate function for selected values of $\theta$)

$$r(x) = \frac{f(x)}{F(x)},$$

$$= \frac{\theta^3 \left( x + \frac{\theta}{2}x^3 \right) e^{-\theta x}}{(\theta + 3) - \left((\theta + 3) + (\theta + 3)\theta x + \frac{3}{2}\theta^2 x^2 + \frac{1}{2}\theta^3 x^3 \right) e^{-\theta x}},$$

$$= \frac{\theta^3 \left( x + \frac{\theta}{2}x^3 \right) e^{-\theta x}}{(\theta + 3) \{1 - (1 + \theta x)e^{-\theta x} \} - \frac{1}{2}\theta^2 x^2 (3 + \theta x)e^{-\theta x}}, x > 0. \quad (5.28)$$

### 5.5.4 Stochastic ordering

In this sub-section, we study stochastic order relationship of length biased xgamma random variables.

The following theorem shows that length biased xgamma random variables possess strong stochastic ordering depending the value of the parameter.
Theorem 5.3. If $X \sim \text{LBXG}(\theta_1)$ and $Y \sim \text{LBXG}(\theta_2)$, then for $\theta_1 > \theta_2$, $X$ is smaller than $Y$ in hazard rate order (i.e., $X \leq_{HR} Y$) and thereby in mean residual life order ($X \leq_{MRL} Y$) and stochastic order ($X \leq_{ST} Y$), respectively.

Proof. For $t > 0$, we have the ratio of the hazard functions of $X$ and $Y$ as

$$\frac{h_X(t)}{h_Y(t)} = \left(\frac{\theta_1}{\theta_2}\right)^3 \frac{(2t + \theta_1 t^3)}{(2t + \theta_2 t^3)} \left[\frac{(3 + \theta_2) + (3 + \theta_2)\theta_2 t + \frac{3}{2}\theta_2^2 t^2 + \frac{1}{2}\theta_2^3 t^3}{(3 + \theta_1) + (3 + \theta_1)\theta_1 t + \frac{3}{2}\theta_1^2 t^2 + \frac{1}{2}\theta_1^3 t^3}\right],$$

which is more than unity if $\theta_1 > \theta_2$ (see Figure 5.6 for the plots of $\frac{h_X(t)}{h_Y(t)}$ for selected values of $\theta_1$ and $\theta_2$). Hence, $h_X(t) > h_Y(t)$ for $\theta_1 > \theta_2$ and $t > 0$. So, $X \leq_{HR} Y$.

Again by Shaked and Shanthikumar (1994), $X \leq_{HR} Y \Rightarrow X \leq_{MRL} Y$ and $X \leq_{HR} Y \Rightarrow X \leq_{ST} Y$, and hence the proof.
5.6 Parameter estimation

In this section, method of moments and method maximum likelihood are been proposed for estimating $\theta$ when $X \sim LBXG(\theta)$. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ drawn from $LBXG(\theta)$.

5.6.1 Method of moments

If $\bar{X}$ denotes the sample mean, then by applying the method of moments, we have

$$\bar{X} = \frac{2(\theta + 6)}{\theta(\theta + 3)}.$$
which gives a quadratic equation in \( \theta \) as

\[
\bar{X}\theta^2 + (3\bar{X} - 2)\theta - 12 = 0. \tag{5.29}
\]

Denoting \( \hat{\theta}_M \) as the method of moment estimator for \( \theta \). \( \hat{\theta}_M \) is the solution of (5.29) and is obtained as

\[
\hat{\theta}_M = \frac{-(3\bar{X} - 2) + \sqrt{(3\bar{X} - 2)^2 + 48\bar{X}}}{2\bar{X}} \text{ for } \bar{X} > 0. \tag{5.30}
\]

### 5.6.2 Method of maximum likelihood

Let \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) be a particular realization on \( X_1, X_2, \ldots, X_n \). The likelihood function of \( \theta \) given \( \mathbf{x} \) is then written as

\[
L(\theta|\mathbf{x}) = \prod_{i=1}^{n} \theta \left( x_i + \frac{\theta}{2} x_i^3 \right) e^{-\theta x_i} = \frac{\theta^n}{(\theta + 3)^n} e^{-\theta \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \left( x_i + \frac{\theta}{2} x_i^3 \right). \]

The log-likelihood function is given by

\[
\ln L(\theta|\mathbf{x}) = 3n \ln \theta - n \ln(\theta + 3) - \theta \left( \sum_{i=1}^{n} x_i \right) + \sum_{i=1}^{n} \ln \left( x_i + \frac{\theta}{2} x_i^3 \right). \tag{5.31}
\]

Differentiating (5.31) with respect to \( \theta \) and equating with zero, the log-likelihood equation is

\[
\frac{\partial}{\partial \theta} \ln L(\theta|\mathbf{x}) = 0 \Rightarrow \frac{3n}{\theta} - \frac{n}{(\theta + 3)} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{x_i^2/2}{1 + \theta/2 x_i^2} = 0. \tag{5.32}
\]

Differentiating (5.31) twice with respect to \( \theta \), we have

\[
\frac{\partial^2}{\partial \theta^2} \ln L(\theta|\mathbf{x}) = \frac{n}{(\theta + 3)^2} - \frac{3n}{\theta^2} - \sum_{i=1}^{n} \left( \frac{x_i^2/2}{1 + \theta/2 x_i^2} \right)^2. \tag{5.33}
\]
The equation (5.32) can not be solved analytically, hence for finding the maximum likelihood estimator, say \( \hat{\theta} \), of \( \theta \) numerical method like Newton-Raphson is applied.

### 5.7 Simulation study

The procedure for simulating random sample of specific size from \( LBXG(\theta) \) is discussed in this section along with a simulation study.

The fact that length biased xgamma distribution is a special mixture of \( \text{gamma}(2, \theta) \) and \( \text{gamma}(4, \theta) \) with mixing proportions \( \theta/(3 + \theta) \) and \( 3/(3 + \theta) \), respectively, is utilized for constructing the simulation algorithm from the distribution.

If \( X \sim LBXG(\theta) \), then for generating a random sample of size \( n \) we can have the following algorithm.

1. Generate \( U_i \sim \text{uniform}(0, 1); i = 1, 2, \ldots, n \).
2. Generate \( V_i \sim \text{gamma}(2, \theta); i = 1, 2, \ldots, n \).
3. Generate \( W_i \sim \text{gamma}(4, \theta); i = 1, 2, \ldots, n \).
4. If \( U_i \leq \frac{\theta}{\theta + 3} \), then set \( X_i = V_i \), otherwise set \( X_i = W_i \).

A Monte-Carlo simulation study is carried out by considering \( N = 10,000 \) times for selected values of \( n \) and \( \theta \). Samples of sizes 20, 40, 60 and 100 are considered and values of \( \theta \) are taken as 0.1, 0.5, 1.0, 1.5, 3, 4.5 and 6. The required numerical evaluations are carried out using \( R \) software. The following two measures are been computed.

(i) Average estimate of \( \theta \):
\[
\hat{\theta} = \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}_i, \text{ where } \hat{\theta}_i\text{'s are simulated estimates.}
\]

(ii) Mean Square Error (MSE) of the simulated estimates \( \hat{\theta}_i, i = 1, 2, \ldots, N \):
\[
\frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2.
\]
The results of the simulation study is presented in Table 5.1. The following observations are made from the simulation study.

1. For a given value of $\theta$, the average mean square error (MSE) decreases as sample size $n$ increases.

2. For a larger given value of $\theta$, MSE gets higher and follow the similar trends as indicated in (i) above.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$n = 20$</th>
<th>$n = 40$</th>
<th>$n = 60$</th>
<th>$n = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>MSE</td>
<td>Estimate</td>
<td>MSE</td>
</tr>
<tr>
<td>0.1</td>
<td>0.09989</td>
<td>0.00013</td>
<td>0.09976</td>
<td>0.00006</td>
</tr>
<tr>
<td>0.5</td>
<td>0.48862</td>
<td>0.00326</td>
<td>0.48647</td>
<td>0.00171</td>
</tr>
<tr>
<td>1.0</td>
<td>0.95012</td>
<td>0.01445</td>
<td>0.94499</td>
<td>0.00884</td>
</tr>
<tr>
<td>1.5</td>
<td>1.39521</td>
<td>0.03706</td>
<td>1.38425</td>
<td>0.02610</td>
</tr>
<tr>
<td>3.0</td>
<td>2.64595</td>
<td>0.22346</td>
<td>2.6256</td>
<td>0.18895</td>
</tr>
<tr>
<td>4.5</td>
<td>3.80880</td>
<td>0.68941</td>
<td>3.78011</td>
<td>0.61890</td>
</tr>
<tr>
<td>6.0</td>
<td>4.91580</td>
<td>1.53158</td>
<td>4.88066</td>
<td>1.42160</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$n = 60$</th>
<th>$n = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>MSE</td>
</tr>
<tr>
<td>0.1</td>
<td>0.09963</td>
<td>0.00004</td>
</tr>
<tr>
<td>0.5</td>
<td>0.48404</td>
<td>0.00126</td>
</tr>
<tr>
<td>1.0</td>
<td>0.94193</td>
<td>0.00714</td>
</tr>
<tr>
<td>1.5</td>
<td>1.38165</td>
<td>0.02240</td>
</tr>
<tr>
<td>3.0</td>
<td>2.61816</td>
<td>0.17740</td>
</tr>
<tr>
<td>4.5</td>
<td>3.76977</td>
<td>0.59767</td>
</tr>
<tr>
<td>6.0</td>
<td>4.85885</td>
<td>1.41716</td>
</tr>
</tbody>
</table>

## 5.8 Application

In this section, a real life data set is analyzed to illustrate the applicability of length biased xgamma distribution.
Fatigue is an important factor in determining the service life of ball bearings. Bearing manufacturers are therefore constantly engaged in fatigue-testing operations in order to obtain information relating fatigue life to load and other factors. The data set of 23 fatigue life for deep-groove ball bearings, compiled by American Standards Association and reported in Lieblein and Zelen (1956) is used to illustrate the applicability of the length biased xgamma model.

The data set (given in Table 5.2) is positively skewed (skewness=0.94 and kurtosis=0.49) with mean value 72.22, median 67.80 and is unimodal (mode at 50).

For comparison purpose, besides length biased xgamma distribution with parameter $\theta$, five other different life distributions, namely, exponential with rate $\theta$, gamma distribution with shape $\alpha$ and rate $\theta$, Weibull distribution with shape $\alpha$ and scale $\beta$, xgamma distribution with parameter $\theta$ and length biased weighted exponential distribution with parameters $\alpha$ and $\lambda$, i.e., $LBWE(\alpha, \lambda)$ (Das and Kundu, 2016), are considered.

In order to compare lifetime models, criteria like, negative log-likelihood, AIC and BIC are taken. The better fitted distribution corresponds to smaller negative log-likelihood, AIC and BIC values. Maximum likelihood estimates (MLEs) are obtained for the parameters involved in the distributions considered for the purpose. Statistical software R is utilized for computation.

Table 5.3 shows the estimates of the model parameter(s) with standard error(s) of estimates in parenthesis and different model selection criteria. From Table 5.3, it is observed that $LBXG(\theta)$ better fits the data as compared to the other models. Moreover, added flexibility over xgamma distribution is observed in real data application.
Table 5.3: MLEs of model parameters and model selection criteria for fatigue lives of ball bearing data.

<table>
<thead>
<tr>
<th>Distributions</th>
<th>Estimate(Std. Error)</th>
<th>-Log-likelihood</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential(θ)</td>
<td>θ=0.0138 (0.0029)</td>
<td>121.435</td>
<td>244.870</td>
<td>246.005</td>
</tr>
<tr>
<td>Gamma(α, θ)</td>
<td>̂α=4.0260 (1.1396)</td>
<td>113.029</td>
<td>230.059</td>
<td>232.330</td>
</tr>
<tr>
<td></td>
<td>̂θ=0.0557 (0.0168)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Weibull(α, β)</td>
<td>̂α=2.1021 (0.3286)</td>
<td>113.691</td>
<td>231.383</td>
<td>233.654</td>
</tr>
<tr>
<td></td>
<td>̂β=81.8683 (8.5986)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Xgamma(θ)</td>
<td>̂θ=0.0407 (0.0049)</td>
<td>113.966</td>
<td>229.931</td>
<td>231.067</td>
</tr>
<tr>
<td>LBWE(α, λ)</td>
<td>̂α=0.0251 (0.8960)</td>
<td>113.522</td>
<td>231.045</td>
<td>233.326</td>
</tr>
<tr>
<td></td>
<td>̂λ=0.0410 (0.0182)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LBXG(θ)</td>
<td>̂θ=0.0549 (0.0057)</td>
<td>113.086</td>
<td>228.171</td>
<td>229.307</td>
</tr>
</tbody>
</table>

The Figure 5.7 shows the plot of histogram and fitted exponential, gamma, Weibull, xgamma, \(LBWE(α, λ)\) and \(LBXG(θ)\) curves for fatigue lives data.

![Figure 5.7: Plot of histogram and fitted lifetime models for fatigue lives data.](image)

Figure 5.7: Plot of histogram and fitted lifetime models for fatigue lives data.
5.9 Conclusion

Owing the importance of weighted distributions in statistical literature, the weighted xgamma distribution, considering a special non-negative weight function, is proposed and studied in this chapter as a generalization of xgamma distribution. As a special case of weighted xgamma distribution, length biased version of xgamma distribution is obtained and its different distributional and survival properties are studied in detail. Method of moments and method of maximum likelihood are proposed for estimating unknown parameter in the length biased xgamma distribution. Real data are analyzed to show the applicability of the proposed model and compared with other life distributions. The following important findings are obtained in this chapter.

1. It is observed that the length biased xgamma is a special case of weighted xgamma distribution and is a special finite mixture of $\text{gamma}(2, \theta)$ and $\text{gamma}(4, \theta)$.

2. Length biased xgamma distribution is unimodal and holds IFR and DMRL property.

3. Length biased xgamma random variable possesses strong hazard rate, mean residual life and stochastic ordering for certain restriction on parameter.

4. Simulation study shows that the estimator of the unknown parameter in length biased xgamma distribution behaves satisfactorily for larger sample. Real data illustration shows that the length biased xgamma distribution is a potential model in describing real life time-to-event data and can be utilized as a flexible lifetime model against the standard lifetime models available in the literature.

This chapter opens some further scope for future research on the distribution proposed.
Open research problems:

Listed below some future research problems one could be interested in.

- Investigation for a suitable method of discriminating between xgamma distribution and length biased xgamma distribution for a given sample data.

- Bayesian estimation aspects for length biased xgamma distribution for different loss functions and under different censoring schemes could be potential research interest.

- Bivariate and multivariate extensions of length biased xgamma distribution could be interesting generalizations.