Chapter 4

Orthogonal Semiderivations on Semiprime Semirings

In this chapter the notion of orthogonal semiderivations and orthogonal generalized semiderivations on semiprime semirings have been defined and investigated some necessary and sufficient condition for two semiderivations and two generalized semiderivations to be orthogonal.

4.1 Introduction

The contents of this chapter is published in the journal IOSR Journal of Mathematics and in the journal Mathematical Sciences International Research Journal. We begin this section with the definition of two semiderivations to be orthogonal and give examples to illustrate them. This chapter is divided into three sections. In the first section we discuss the orthogonality property of semiderivations on semirings, in the second and third section we discuss some necessary and sufficient conditions for two
generalized semiderivations to be orthogonal.

4.2 Orthogonal Semiderivations

**Definition 4.2.1.** Let \( f_1 \) and \( f_2 \) be two semiderivations of a semiprime semiring \( S \) associated with functions \( g_1 : S \to S \) and \( g_2 : S \to S \) respectively. Then \( f_1 \) and \( f_2 \) are said to be **orthogonal** if 

\[
(f_1(x)Sf_2(y)) = 0 = (f_2(y)Sf_1(x)) \quad \text{for all} \quad x, y \in S.
\]

**Example 4.2.2.** Let \( S \) be a semiprime semiring. Let \( S = S_1 \oplus S_1 \). Define \( f_1 : S \to S \) by \( f_1(a, b) = (a, 0) \) and \( g_1 : S \to S \) by \( g_1(a, b) = (0, ab) \) for all \( a, b \in S_1 \). Also define \( f_2 : S \to S \) by \( f_2(a, b) = (0, b) \) and \( g_2 : S \to S \) by \( g_2(a, b) = (ab, 0) \) for all \( a, b \in S_1 \). Define addition and multiplication on \( S \) by 

\[
(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \quad \text{and} \quad (a_1, b_1).(a_2, b_2) = (a_1.a_2, b_1.b_2).
\]

Then it can be easily seen that \( f_1 \) and \( f_2 \) are semiderivations of \( S \) (with associated mappings \( g_1 \) and \( g_2 \) respectively) which are not derivations. Also it is clear that \( f_1 \) and \( f_2 \) are orthogonal semiderivations on \( S \).

**Example 4.2.3.** Let \( S \) be a semiprime semiring.

Let \( S = S_1 \oplus S_1 \). Let \( \alpha_1 : S_1 \to S_1 \) be an additive map, \( \alpha_2 : S_1 \to S_1 \) be a left and right \( S_1 \) module which is not a derivation. Define \( f_1 : S \to S \) by \( f_1(x_1, x_2) = (0, \alpha_2(x_2)) \) and \( g_1 : S \to S \) by \( g_1(x_1, x_2) = (\alpha_1(x_1), 0) \) for all \( x_1, x_2 \in S_1 \). Also define \( f_2 : S \to S \) by \( f_2(x_1, x_2) = (\alpha_2(x_1), 0) \) and \( g_2 : S \to S \) by \( g_2(x_1, x_2) = (0, \alpha_1(x_2)) \) for all \( x_1, x_2 \in S_1 \). Define addition and multiplication on \( S \) by 

\[
(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \quad \text{and} \quad (x_1, x_2).(y_1, y_2) = (x_1.y_1, x_2.y_2).
\]

Then it can be easily seen that \( f_1 \)
and $f_2$ are semiderivations of $S$ (with associated mappings $g_1$ and $g_2$ respectively) which are not derivations. Also it is clear that $f_1$ and $f_2$ are orthogonal semiderivations on $S$.

**Example 4.2.4.** Let $S$ be a semiprime semiring. Let $M_2(S) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in S \right\}$.

Define $f_1 : M_2(S) \to M_2(S)$ by $f_1 \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

Define $g_1 : M_2(S) \to M_2(S)$ by $g_1 \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$.

Define $f_2 : M_2(S) \to M_2(S)$ by $f_2 \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$.

Define $g_2 : M_2(S) \to M_2(S)$ by $g_2 \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$.

It is clear that $f_1$ and $f_2$ are not orthogonal.

**Lemma 4.2.5.** Let $S$ be a semiprime semiring, $a \in S$. If $S$ admits a semiderivation $f$ such that $af(x) = 0$ or $f(x)a = 0$ for all $x \in S$ then $a = 0$ or $f = 0$.

**Proof.** By hypothesis $af(x) = 0$ for all $x \in S$.

Replacing $x$ by $xy$ for all $x, y \in S$,

$af(xy) = 0$

$af(x)g(y) + axf(y) = 0$ for all $x, y \in S$, 49
\[ axf(y) = 0 \text{ for all } x, y \in S, \]
\[ asf(y) = 0 \text{ for all } y \in S, \]

Since \( S \) is prime \( a = 0 \) or \( f(y) = 0 \) for all \( y \in S \).

Hence \( a = 0 \) or \( f = 0 \).

Similarly we can prove for \( f(x)a = 0 \) \( \square \)

**Lemma 4.2.6.** Let \( S \) be a 2-torsion free semiprime semiring, \( a, b \in S \). Then the following are equivalent.

\( (i) \). \( aSb = 0 \)

\( (ii) \). \( bSa = 0 \)

\( (iii) \). \( aSb + bSa = 0 \).

If one of these conditions are fulfilled then \( ab = ba = 0 \)

**Proof.** Suppose \( asb = 0 \) for all \( s \in S \)

Premultiplying by \( bs \) and postmultiplying by \( sa \) we have

\( (bsa)s(bsa) = 0 \text{ for all } s \in S \)

Since \( S \) is semiprime we have \( bsa = 0 \)

Thus \( (i) \) implies \( (ii) \).

Suppose \( bsa = 0 \) for all \( s \in S \) \( \cdots \cdots (1) \)

Premultiplying by \( as \) and postmultiplying by \( sb \) we have

\( (asb)s(asb) = 0 \text{ for all } s \in S \)

Since \( S \) is semiprime we have \( asb = 0 \) for all \( s \in S \) \( \cdots \cdots (2) \)

Adding (1) and (2) we get \( asb + bsa = 0 \) for all \( s \in S \)

Thus \( (ii) \) implies \( (iii) \).
Suppose \( asb + bsa = 0 \) for all \( s \in S \) .......(3)

Premultiplying (3) by \( bs \) we have

\[ bs(asb) + bs(bsa) = 0 \] for all \( s \in S \)

Again premultiplying by \( as \) we have,

\[ (asb)s(asb) + (asb)s(bsa) = 0 \] for all \( s \in S \) ............(4)

Post multiplying (3) by \( sa \),

\[ (asb)sa + (bxa)sa = 0 \] for all \( s \in S \)

Again post multiplying by \( sb \) we have

\[ (asb)s(asb) + (bxa)s(asb) = 0 \] for all \( s \in S \) ............(5)

Adding (4) and (5)

\[ 2((asb)s(asb)) = 0 \] for all \( s \in S \)

Since \( S \) is 2-torsion free we have

\[ (asb)s(asb) = 0 \] for all \( s \in S \)

Since \( S \) is semiprime, \( asb = 0 \) for all \( s \in S \)

Hence (iii) implies (i)

Now assume that \( asb = 0 \) for all \( s \in S \)

Premultiplying by \( b \) and post multiplying by \( a \),

\[ (ba)s(ba) = 0 \] for all \( s \in S \)

Since \( S \) is semiprime semiring, we have \( ba = 0 \)

Now \( asb = 0 \) implies \( bsa = 0 \) for all \( s \in S \)

Proceeding as above we can prove that \( ab = 0 \) \hfill \Box

**Lemma 4.2.7.** Let \( S \) be a 2-torsion free semiprime semiring, suppose that the additive mappings \( f \) and \( g \) of \( S \) into \( S \) satisfying \( f(x)Sg(x) = 0 \) for all \( x \in S \). Then \( f(x)Sg(y) = 0 \) for all \( x, y \in S \).
Proof. Suppose that \( f(x)Sg(x) = 0 \) for all \( x \in S \)

Hence \( f(x)sg(x) = 0 \) for all \( x, s \in S \) \hspace{1cm} \text{(1)}

Replacing \( x \) by \( x + y \), for all \( x, y, s \in S \), we have

\[
0 = f(x + y)sg(x + y).
\]

\[
= (f(x) + f(y))s(g(x) + g(y)).
\]

\[
= f(x)sg(x) + f(x)sg(y) + f(y)sg(x) + f(y)sg(y)
\]

\[
f(x)sg(y) + f(y)sg(x) = 0
\]

Premultiplying by \( f(x)sg(y)z \) for all \( x, y, z, s \in S \) we get,

\[
(f(x)sg(y)z)(f(x)sg(y) + f(y)sg(x)) = 0
\]

\[
(f(x)sg(y)z)(f(x)sg(y)) + (f(x)sg(y)z)(f(y)sg(x)) = 0
\]

\[
(f(x)sg(y))z(f(x)sg(y)) + f(x)(sg(y)zf(y)s)g(x) = 0
\]

\[
(f(x)sg(y))z(f(x)sg(y)) + f(x)ug(x) = 0,
\]

where \( u = sg(y)zf(y)s \in S \).

Using (1),

\[
(f(x)sg(y))z(f(x)sg(y)) = 0 \text{ for all } x, y, z, s \in S
\]

\[
(f(x)sg(y))S(f(x)sg(y)) = 0 \text{ for all } x, y, s \in S
\]

Since \( S \) is semiprime, \( f(x)Sg(y) = 0 \) for all \( x, y \in S \) \hspace{1cm} \Box

**Theorem 4.2.8.** Let \( S \) be a 2-torsion free semiprime semiring, and let \( f_1 \) and \( f_2 \) be semiderivations of \( S \) into \( S \) associated with functions \( g_1 : S \to S \) and \( g_2 : S \to S \) respectively. Then \( f_1 \) and \( f_2 \) are orthogonal if and only if

\[
f_1(x)f_2(y) + f_2(x)f_1(y) = 0 \text{ for all } x, y \in S.
\]

**Proof.** Suppose that \( f_1 \) and \( f_2 \) are orthogonal.

Then \( f_1(x)Sf_2(y) = 0 = f_2(y)Sf_1(x) \) for all \( x, y \in S \).
Since \( f_1(x)Sf_2(y) = 0 \) for all \( x, y \in S \) by Lemma 4.2.6 we have
\[
f_1(x)f_2(y) = 0 = f_2(x)f_1(y) \quad \text{for all } x, y \in S
\]
Hence \( f_1(x)f_2(y) + f_2(x)f_1(y) = 0 \) for all \( x, y \in S \).

Conversely assume that \( f_1(x)f_2(y) + f_2(x)f_1(y) = 0 \) for all \( x, y \in S \) .......(1)

Replace \( y \) by \( yx \), for all \( x, y \in S \)
\[
0 = f_1(x)f_2(yx) + f_2(x)f_1(yx)
\]
\[
= f_1(x)(f_2(y)g_2(x) + yf_2(x)) + f_2(x)(f_1(y)g_1(x) + yf_1(x))
\]
\[
= f_1(x)f_2(y)g_2(x) + f_1(x)yx f_2(x) + f_2(x)f_1(y)g_1(x) + f_2(x)yf_1(x)
\]

Since \( g_1 \) and \( g_2 \) are surjective we have,
\[
0 = f_1(x)f_2(y)x + f_1(x)yx f_2(x) + f_2(x)f_1(y)x + f_2(x)yf_1(x)
\]
\[
= 2f_1(x)yx f_2(x) \quad \text{for all } x, y \in S
\]
\[
= 2f_1(x)Sf_2(x) \quad \text{for all } x \in S
\]

Since \( S \) is 2-torsion free \( f_1(x)Sf_2(x) = 0 \) for all \( x \in S \)
By Lemma 4.2.7 we have \( f_1(x)Sf_2(y) = 0 \) for all \( x, y \in S \)
Hence by Lemma 4.2.6 we have
\[
f_1(x)Sf_2(y) = 0 = f_2(y)Sf_1(x) \quad \text{for all } x, y \in S
\]
Thus \( f_1 \) and \( f_2 \) are orthogonal. \( \square \)

**Theorem 4.2.9.** Let \( S \) be a 2-torsion free semiprime semiring, and let \( f_1 \) and \( f_2 \) be semiderivations of \( S \) into \( S \) associated with functions \( g_1 : S \to S \) and \( g_2 : S \to S \) respectively. Then \( f_1 \) and \( f_2 \) are orthogonal if and only if \( f_1f_2 = 0 = f_2f_1 \).

**Proof.** Suppose that \( f_1 \) and \( f_2 \) are orthogonal.

Then \( f_1(x)Sf_2(y) = 0 = f_2(y)Sf_1(x) \) for all \( x, y \in S \)
Then by Lemma 4.2.6,
\[ f_1(x)f_2(y) = 0 \text{ for all } x, y \in S \] ......(1)
\[ 0 = f_1(x)f_2(y) \text{ for all } x, y \in S \]
\[ = f_1(f_1(x)f_2(y)) \text{ for all } x, y \in S \]
\[ = f_1(f_1(x))g_1(f_2(y)) + f_1(x)f_1(f_2(y)) \text{ for all } x, y \in S \]
\[ = f_1(f_1(x))(f_2(y)) + f_1(x)f_1(f_2(y)) \text{ since } g_1 \text{ is surjective.} \]
\[ = f_1(x)f_1(f_2(y)) \text{ for all } x, y \in S, \text{ by (1)} \]
Premultiply by \( f_1f_2(y) \),
\[ 0 = f_1f_2(y)f_1(x)f_1(f_2(y)) \text{ for all } x, y \in S \]
\[ 0 = f_1f_2(y)Sf_1f_2(y) \text{ for all } y \in S \]
By the semiprimeness of \( S \),
\[ f_1f_2(y) = 0 \text{ for all } y \in S \]
Hence \( f_1f_2 = 0 \)
Similarly we can prove \( f_2f_1 = 0 \)
Conversely assume that \( f_1f_2 = 0 = f_2f_1 \)
\[ 0 = f_1f_2(xy) \text{ for all } x, y \in S \]
\[ = f_1(f_2(x)g_2(y) + xf_2(y)) \text{ for all } x, y \in S \]
\[ = f_1f_2(x)g_1g_2(y) + f_2(x)f_1(g_2(y)) + f_1(x)g_1(f_2(y)) + xf_1f_2(y) \]
\[ = f_2(x)f_1(g_2(y)) + f_1(x)g_1(f_2(y)) \text{ for all } x, y \in S \]
\[ = f_2(x)f_1(y) + f_1(x)f_2(y) \text{ for all } x, y \in S \]
Now by Theorem 4.2.8 \( f_1 \) and \( f_2 \) are orthogonal. \( \square \)

**Theorem 4.2.10.** Let \( S \) be a 2-torsion free semiprime semiring, and let \( f_1 \) and \( f_2 \) be semiderivations of \( S \) into \( S \) associated with functions \( g_1 : S \rightarrow S \) and \( g_2 : S \rightarrow S \) respectively. Then \( f_1 \) and \( f_2 \) are orthogonal if and only if
\[ f_1 f_2 + f_2 f_1 = 0. \]

**Proof.** Suppose that \( f_1 \) and \( f_2 \) are orthogonal.

Then \( f_1(x) S f_2(y) = 0 = f_2(y) S f_1(x) \) for all \( x, y \in S \)

Since \( f_1 \) and \( f_2 \) are orthogonal, by Theorem 4.2.9

\[ f_1 f_2 = 0 = f_2 f_1 \]

Hence \( f_1 f_2 + f_2 f_1 = 0 \)

Conversely assume \( f_1 f_2 + f_2 f_1 = 0 \)

\[ 0 = (f_1 f_2 + f_2 f_1)(x y) \text{ for all } x, y \in S \]

\[ = f_1 f_2(x y) + f_2 f_1(x y) \text{ for all } x, y \in S \]

\[ = f_1(f_2(x) g_2(y) + x f_2(y)) + f_2(f_1(x) g_1(y) + x f_1(y)) \text{ for all } x, y \in S \]

\[ = f_1(f_2(x)) g_1(g_2(y)) + f_2(x f_1(x) g_1(y)) + f_1(x) g_1(f_2(y)) + \]

\[ + x f_1(f_2(y)) + f_2(f_1(x)) g_2(g_1(y)) + f_1(x) f_2(g_1(y)) + \]

\[ + f_2(x) g_2(f_1(y)) + x f_2(f_1(y)) \text{ for all } x, y \in S \]

\[ = f_1(f_2(x)) y + f_2(x) f_1(y) + f_1(x) f_2(y) + x f_1(f_2(y)) + \]

\[ + f_2(f_1(x)) y + f_1(x) f_2(y) + f_2(x)(f_1(y)) + x f_2(f_1(y)) \text{ for all } x, y \in S \]

\[ = 2(f_1(x) f_2(y) + f_2(x) f_1(y)) \text{ for all } x, y \in S \]

Since \( S \) is 2-torsion free,

\[ f_1(x) f_2(y) + f_2(x) f_1(y) = 0 \text{ for all } x, y \in S \]

Now by Theorem 4.2.8 \( f_1 \) and \( f_2 \) are orthogonal. \( \Box \)

**Theorem 4.2.11.** Let \( S \) be a 2-torsion free semiprime semiring, and let \( f_1 \) and \( f_2 \) be semiderivations of \( S \) into \( S \) associated with functions \( g_1 : S \to S \) and \( g_2 : S \to S \) respectively. Then \( f_1 \) and \( f_2 \) are orthogonal if and only if \( f_1 f_2 \) or \( f_2 f_1 \) is a semiderivation associated with the function \( g_1 g_2 \) or \( g_2 g_1 \) respectively.
Proof. Suppose that $f_1$ and $f_2$ are orthogonal.

Then $f_1(x)Sf_2(y) = 0 = f_2(y)Sf_1(x)$ for all $x, y \in S$

Also by Theorem 4.2.8 we have

$$f_1(x)f_2(y) + f_2(x)f_1(y) = 0 \text{ for all } x, y \in S$$

...............(1)

Also by Theorem 4.2.9 we have $f_1f_2 = 0$

\[
0 = f_1f_2(xy) \text{ for all } x, y \in S \\
= f_1(f_2(x)g_2(y) + xf_2(y)) \text{ for all } x, y \in S \\
= f_1(f_2(x))g_1(g_2(y)) + f_2(x)f_1(g_2(y)) + f_1(x)g_1(f_2(y)) \\
+ xf_1(f_2(y)) \text{ for all } x, y \in S \\
= f_1f_2(x)g_1g_2(y) + f_2(x)f_1(y) + f_1(x)f_2(y) \\
+ xf_1f_2(y) \text{ for all } x, y \in S \text{ and } g \text{ is surjective.} \\
= f_1f_2(x)g_1g_2(y) + xf_1f_2(y) \text{ for all } x, y \in S
\]

Also $g_1g_2(f_1f_2(x)) = f_1f_2(x) = f_1f_2(g_1g_2(x))$ for all $x, y \in S$

Hence $f_1f_2$ is a semiderivation associated with the function $g_1g_2 : S \to S$.

Similarly starting with $f_2f_1 = 0$ we can prove $f_2f_1$ is a semiderivation associated with the function $g_2g_1 : S \to S$.

Conversely assume that $f_1f_2$ is a semiderivation associated with the function $g_1g_2 : S \to S$.

Then $f_1f_2(xy) = f_1f_2(x)g_1g_2(y) + xf_1f_2(y) \text{ for all } x, y \in S$  \hspace{1cm} ...........(2)

Also

$f_1f_2(xy) = f_1(f_2(x)g_2(y) + xf_2(y)) \text{ for all } x, y \in S$
\[
\begin{align*}
&= f_1(f_2(x))g_1(g_2(y)) + f_2(x)f_1(g_2(y)) + \\
&\quad + f_1(x)g_1f_2(y) + xf_1(f_2(y)) \text{ for all } x, y \in S \\
&= f_1f_2(x)g_1g_2(y) + f_2(x)f_1(y) + f_1(x)f_2(y) + xf_1f_2(y)
\end{align*}
\]
for all \(x, y \in S\), \(g_1\) and \(g_2\) are surjective .............(3)

Comparing (2) and (3)

\[f_1(x)f_2(y) + f_2(x)f_1(y) = 0 \text{ for all } x, y \in S\]

Hence by Theorem 4.2.8 \(f_1\) and \(f_2\) are orthogonal.

Similarly we can prove if \(f_2f_1\) is a semiderivation associated with the function \(g_2g_1 : S \to S\), then \(f_1\) and \(f_2\) are orthogonal. \(\square\)

**Corollary 4.2.12.** Let \(S\) be a 2-torsion free semiprime semiring, and let \(f_1\) and \(f_2\) be semiderivations of \(S\) into \(S\) associated with functions \(g_1 : S \to S\) and \(g_2 : S \to S\) respectively. If \(f_1^2 = f_2^2\), then \(f_1 + f_2\) and \(f_1 - f_2\) are orthogonal.

**Proof.** By hypothesis, \(f_1^2 = f_2^2\)

Hence \(f_1^2 - f_2^2 = 0\)

Thus we have \((f_1 + f_2)(f_1 - f_2) = 0 = (f_1 - f_2)(f_1 + f_2)\)

Hence by Theorem 4.2.9

we have \(f_1 + f_2\) and \(f_1 - f_2\) are orthogonal. \(\square\)

**Theorem 4.2.13.** Let \(S\) be a 2-torsion free semiprime semiring, and let \(f_1\) and \(f_2\) be semiderivations of \(S\) into \(S\) associated with functions \(g_1 : S \to S\) and \(g_2 : S \to S\) respectively. Then \(f_1\) and \(f_2\) are orthogonal if and only if their exist \(a, b \in S\) such that \(f_1f_2(x) = ax + xb\) for all \(x \in S\).
Proof. Suppose that $f_1$ and $f_2$ are orthogonal.

Then by Theorem 4.2.9 $f_1 f_2 = 0$

Hence $f_1 f_2(x) = 0$

Choosing $a = b = 0$ we get $f_1 f_2(x) = ax + xb$ for all $x \in S$.

Conversely assume that $f_1 f_2(x) = ax + xb$ for all $x \in S$.

Replace $x$ by $xy$, for all $x, y \in S$

$f_1 f_2(xy) = axy + xyb$

$f_1(f_2(x)g_2(y) + xf_2(y)) = axy + xyb$

$f_1(f_2(x))g_1(g_2(y)) + f_2(x)f_1(g_2(y)) + f_1(x)g_1(f_2(y)) +
+ x f_1(f_2(y)) = axy + xyb$

$f_1 f_2(xy) + f_2(x)f_1(y) + f_1(x)f_2(y) + x f_1 f_2(y) = axy + xyb$,

$g_1$ and $g_2$ are surjective.

$(ax + xb)y + f_2(x)f_1(y) + f_1(x)f_2(y) + x(ay + yb) = axy + xyb$

$axy + xay + f_2(x)f_1(y) + f_1(x)f_2(y) = 0$  \ldots \ldots (1)$

Replace $y$ by $yx$, for all $x, y \in S$

$xyx + xayx + f_2(x)f_1(yx) + f_1(x)f_2(yx) = 0$

$xyx + xayx + f_2(x)f_1(y)g_1(x) + f_2(x)yf_1(x) + f_1(x)f_2(y)g_2(x) +
+ f_1(x)yf_2(x) = 0$

$(xyy + xay + f_2(x)f_1(y) + f_1(x)f_2(y))x + 2f_1(x)yf_2(x) = 0$

$2f_1(x)yf_2(x) = 0$ by(1)$

$2f_1(x)Sf_2(x) = 0$

Since $S$ is 2-torsion free $f_1(x)Sf_2(x) = 0$ for all $x \in S$

Then by Lemma 4.2.7 $f_1(x)Sf_2(y) = 0$ for all $x, y \in S$

Hence $f_1(x)Sf_2(y) = 0 = f_2(y)Sf_1(x)$ for all $x, y \in S$
Hence $f_1$ and $f_2$ are orthogonal. \hfill \Box

**Theorem 4.2.14.** Let $S$ be a 2-torsion free semiprime semiring, and let $f_1$ and $f_2$ be semiderivations of $S$ into $S$ associated with functions $g_1 : S \to S$ and $g_2 : S \to S$ respectively such that $f_1f_2$ is a semiderivation associated with the function $g_1g_2 : S \to S$. Then either $f_1$ or $f_2$ is zero.

**Proof.** By hypothesis $f_1f_2$ is a semiderivation

Then $f_1f_2(xy) = f_1f_2(x)g_1g_2(y) + xf_1f_2(y)$ for all $x, y \in S$  \hspace{1cm} (1)

Also,

\[
f_1f_2(xy) = f_1(f_2(x)g_2(y) + xf_2(y)) \text{ for all } x, y \in S
\]

\[= f_1(f_2(x))g_1(g_2(y)) + f_2(x)f_1(g_2(y)) +
\]

\[+ f_1(x)g_1(f_2(y)) + xf_1f_2(y) \text{ for all } x, y \in S
\]

\[= f_1f_2(x)g_1g_2(y) + f_2(x)f_1(y) + f_1(x)f_2(y) +
\]

\[+ xf_1f_2(y) \text{ for all } x, y \in S \text{ and } g_1 \text{ and } g_2 \text{ are surjective.}  \hspace{1cm} (2)
\]

Comparing (1) and (2)

\[f_1(x)f_2(y) + f_2(x)f_1(y) = 0 \text{ for all } x, y \in S  \hspace{1cm} (3)
\]

Now by Theorem 4.2.8 $f_1$ and $f_2$ are orthogonal.

Hence $f_1(x)Sf_2(y) = 0 = f_2(y)Sf_1(x)$ for all $x, y \in S$  \hspace{1cm} (4)

Since $S$ is prime from (4) we have,

$f_1(x) = 0$ or $f_2(y) = 0$ for all $x, y \in S$

Hence $f_1 = 0$ or $f_2 = 0$ \hfill \Box

**Corollary 4.2.15.** Let $S$ be a 2-torsion free semiprime semiring, and let $f_1$ and $f_2$ be semiderivations of $S$ into $S$ associated with functions $g_1 : S \to S$
and \( g_2 : S \to S \) respectively such that \( f_1^2 \) is a semiderivation associated with the function \( g_1^2 : S \to S \). Then \( f_1 \) is zero.

**Proof.** In Theorem 4.2.14 replacing \( f_2 \) by \( f_1 \) we get the result \( f_1 = 0 \). ⌣

### 4.3 Orthogonal Generalized Semiderivations

In this section we define orthogonal generalized semiderivations on semiprime semirings and investigated some results for two generalized semiderivations to be orthogonal. Motivated by some results on orthogonal generalized derivations on semiprime rings by Nurcan Argac, Atsushi Nakajima and Emine Albas, [40] in this section orthogonality of two generalized semiderivations on semiprime semirings has been defined. Moreover, we extend some known results on orthogonal generalized derivations on semiprime rings to orthogonal generalized semiderivations on semiprime semirings. These results are generalization of results of M. Bresar and J. Vukman, [8] which are related to a Theorem of E.Posner for the product of derivations on a prime ring. Also the necessary and sufficient condition for the product of two generalized semiderivations to be a generalized semiderivation in terms of orthogonal semiderivation are proved.

**Definition 4.3.1.** Let \( f_1 \) and \( f_2 \) be two semiderivations of a semiprime semiring \( S \) associated with functions \( g_1 : S \to S \) and \( g_2 : S \to S \) respectively. Let \((F_1, f_1)\) and \((F_2, f_2)\) be two generalized semiderivations on \( S \), Then \((F_1, f_1)\) and \((F_2, f_2)\) are said to be *orthogonal* if
\[ F_1(x)SF_2(y) = 0 = F_2(y)SF_1(x) \] for all \( x, y \in S \).

**Example 4.3.2.** Let \( S_1 \) be a semiprime semiring. Let \( S = S_1 \oplus S_1 \). Define \( f_1 : S \to S, f_2 : S \to S, g_1 : S \to S, g_2 : S \to S, F_1 : S \to S \) and \( F_2 : S \to S \) as follows: 

- \( f_1(x_1, x_2) = (x_1, 0) \), \( f_2(x_1, x_2) = (0, x_2) \),
- \( g_1(x_1, x_2) = (0, x_1x_2) \), \( g_2(x_1, x_2) = (x_1x_2, 0) \),
- \( F_1(x_1, x_2) = f_1(x_1, x_2) + (x_1, x_2) \), \( F_2(x_1, x_2) = f_2(x_1, x_2) + (x_1, x_2) \).

Define addition and multiplication on \( S \) by 

\[
(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \quad \text{and} \quad (x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2).
\]

Then it can be easily seen that \( f_1 \) and \( f_2 \) are semiderivations of \( S \) (with associated mappings \( g_1 \) and \( g_2 \) respectively) which are orthogonal. But \( F_1 \) and \( F_2 \) are generalised semiderivations which are not orthogonal.

**Example 4.3.3.** Let \( S \) be a semiprime semiring.

Let \( M_3(S) = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a, b \in S \right\} \).

Define \( f_1 : M_3(S) \to M_3(S) \) by 
\[
f_1 \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

and \( g_1 : M_3(S) \to M_3(S) \) by 
\[
g_1 \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
\[ f_2 : M_3(S) \rightarrow M_3(S) \text{ by } f_2 \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ g_2 : M_3(S) \rightarrow M_3(S) \text{ by } g_2 \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Define \( F_1 : M_3(S) \rightarrow M_3(S) \) by \( F_1 \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a + b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \)

\[ F_2 : M_3(S) \rightarrow M_3(S) \text{ by } F_2 \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a + b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

It is clear that \( f_1 \) and \( f_2 \) are orthogonal semiderivations. Also \( F_1 \) and \( F_2 \) are orthogonal generalized semiderivations.

**Example 4.3.4.** Let \( S \) be a semiprime semiring.

Let \( M_3(S) = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} : a, b \in S \right\}. \)

Define \( f_1 : M_3(S) \rightarrow M_3(S) \) by \( f_1 \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \)
$g_1 : M_3(S) \rightarrow M_3(S)$ by $g_1 \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$f_2 : M_3(S) \rightarrow M_3(S)$ by $f_2 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & a & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$g_2 : M_3(S) \rightarrow M_3(S)$ by $g_2 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & a & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Define $F_1 : M_3(S) \rightarrow M_3(S)$ by $F_1 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & a & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

$F_2 : M_3(S) \rightarrow M_3(S)$ by $F_2 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & a & b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

It is clear that $f_1$ and $f_2$ are orthogonal semiderivations. Also $F_1$ and $F_2$ are orthogonal generalized semiderivations.

**Lemma 4.3.5.** Let $S$ be a 2-torsion free semiprime semiring and let $(F_1, f_1)$ and $(F_2, f_2)$ be orthogonal generalized semiderivations of $S$ into $S$ associated with functions $g_1 : S \rightarrow S$ and $g_2 : S \rightarrow S$ respectively. Then the following relations hold.

(i) $F_1(x)F_2(y) = F_2(x)F_1(y) = 0$ and hence $F_1(x)F_2(y) + F_2(x)F_1(y) = 0$

for all $x, y \in S$. 

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(ii). $f_1$ and $F_2$ are orthogonal and $f_1(x)F_2(y) = F_2(x)f_1(x) = 0$ for all $x, y \in S$

(iii). $f_2$ and $F_1$ are orthogonal and $f_2(x)F_1(y) = F_1(x)f_2(y) = 0$ for all $x, y \in S$

(iv). $f_1$ and $f_2$ are orthogonal semiderivations

(v). $f_1F_2 = F_2f_1 = 0$ and $f_2F_1 = F_1f_2 = 0$

(vi). $F_1F_2 = F_2F_1 = 0$

**Proof.** (i). By hypothesis $(F_1, f_1)$ and $(F_2, f_2)$ are orthogonal.

Then $F_1(x)SF_2(y) = 0 = F_2(y)SF_1(x)$ for all $x, y \in S$.

Then by Lemma 4.2.6, we have

$F_1(x)F_2(y) = 0 = F_2(y)F_1(x)$ for all $x, y \in S$.

Hence $F_1(x)F_2(y) + F_2(y)F_1(x) = 0$ for all $x, y \in S$.

(ii). By hypothesis $(F_1, f_1)$ and $(F_2, f_2)$ are orthogonal.

Then $F_1(x)SF_2(y) = 0 = F_2(y)SF_1(x)$ for all $x, y \in S$.

Then by Lemma 4.2.6, we have

$F_1(x)F_2(y) = 0 = F_2(y)F_1(x)$ for all $x, y \in S$.

$0 = F_1(x)F_2(y)$ for all $x, y \in S$.

Replace $x$ by $zx$

$0 = F_1(zx)F_2(y)$ for all $x, y, z \in S$.

$= F_1(z)xF_2(y) + g_1(z)f_1(x)F_2(y)$ for all $x, y, z \in S$.

$= g_1(z)f_1(x)F_2(y)$ for all $x, y, z \in S$, since $F_1$ and $F_2$ are orthogonal.

Premultiplying by $f_1(x)F_2(y)$ we get,
\[ 0 = f_1(x)F_2(y)g_1(z)f_1(x)F_2(y) \text{ for all } x, y, z \in S \]

\[ 0 = f_1(x)F_2(y)Sf_1(x)F_2(y) \text{ for all } x, y \in S \]

Since \( S \) is semiprime we get

\[ f_1(x)F_2(y) = 0 \text{ for all } x, y \in S \] ..........................(1)

Replace \( x \) by \( xz \)

\[ f_1(xz)F_2(y) = 0 \text{ for all } x, y, z \in S \]

\[ 0 = f_1(xz)F_2(y) \text{ for all } x, y, z \in S \]

\[ = f_1(x)g_1(z)F_2(y) + xf_1(z)F_2(y) \text{ for all } x, y, z \in S \]

\[ = f_1(x)g_1(z)F_2(y) \text{ for all } x, y, z \in S \]

\[ = f_1(x)SF_2(y) \text{ for all } x, y \in S \]

Thus \( f_1(x)SF_2(y) = 0 = F_2(y)Sf_1(x) \text{ for all } x, y \in S \)

Thus \( f_1 \) and \( F_2 \) are orthogonal.

Also by (1) we have,

\[ f_1(x)F_2(y) = 0 = F_2(x)f_1(y) \text{ for all } x, y \in S \]

\((iii)\). Proof is as similar as \((ii)\)

\((iv)\). By hypothesis \((F_1, f_1)\) and \((F_2, f_2)\) are orthogonal.

Then \( F_1(x)SF_2(y) = 0 = F_2(y)SF_1(x) \text{ for all } x, y \in S \).

Then by Lemma 4.2.6, we have

\[ F_1(x)F_2(y) = 0 = F_2(y)F_1(x) \text{ for all } x, y \in S. \]

\[ 0 = F_1(x)F_2(y) \text{ for all } x, y \in S. \]

Replace \( x \) by \( xz \) and \( y \) by \( yw \)

\[ 0 = F_1(xz)F_2(yw) \text{ for all } x, y, z, w \in S. \]

\[ = xf_1(z)yg_2(w) \text{ for all } x, y, z, w \in S \text{ and } g_1, g_2 \text{ are surjective.} \]

Premultiplying by \( f_1(z)yg_2(w), \)
0 = f_1(z)yf_2(w)x f_1(z)yf_2(w) for all \(x, y, z, w \in S\)

0 = f_1(z)yf_2(w)Sf_1(z)yf_2(w) for all \(y, z, w \in S\)

Since \(S\) is semiprime

\(f_1(z)yf_2(w) = 0\) for all \(y, z, w \in S\)

Hence \(f_1(z)Sf_2(w) = 0\) for all \(z, w \in S\)

In particular,

\(f_1(x)Sf_2(y) = 0 = f_2(y)Sf_1(x)\) for all \(x, y \in S\)

Hence \(f_1\) and \(f_2\) are orthogonal.

\((v)\). From \((ii)\) we have \(f_1\) and \(F_2\) are orthogonal.

\(f_1(x)SF_2(y) = 0 = F_2(y)Sf_1(x)\) for all \(x, y \in S\)

In particular, for all \(x, y, z \in S\)

\[0 = f_1(x)zF_2(y)\]

\[= F_2(f_1(x)zF_2(y))\]

\[= F_2(f_1(x))zF_2(y) + g_2(f_1(x))f_2(zF_2(y))\]

\[= F_2(f_1(x))zF_2(y), \text{ since } f_1 \text{ and } f_2 \text{ are orthogonal and } g_2 \text{ is surjective.}\]

\[= F_2(f_1(x))SF_2(y)\]

Replace \(y\) by \(f_1(x)\)

\(F_2(f_1(x))SF_2(f_1(x)) = 0\) for all \(x \in S\)

Since \(S\) is semiprime, \(F_2(f_1(x)) = 0\) for all \(x \in S\)

Hence \(F_2f_1 = 0\)

Similarly from \(f_1(F_2(x)Sf_1(y)) = 0\), we get \(f_1F_2 = 0\)

From \(F_1(f_2(x)SF_1(y)) = 0\) and \(f_2(F_1(x)Sf_2(y)) = 0\)

we get \(F_1f_2 = 0 = f_2F_1\)

\((vi)\). From \(F_2(F_1(x)SF_2(y)) = 0\) and \(F_1(F_2(x)SF_1(y)) = 0\)
we get \( F_1F_2 = 0 = F_2F_1 = 0 \)

**Theorem 4.3.6.** Let \( S \) be a 2-torsion free semiprime semiring and let \((F_1, f_1)\) and \((F_2, f_2)\) be generalized semiderivations of \( S \) into \( S \) associated with functions \( g_1 : S \to S \) and \( g_2 : S \to S \) respectively. Then the following conditions are equivalent.

(i). \((F_1, f_1)\) and \((F_2, f_2)\) are orthogonal.

(ii). (a). \( F_1(x)F_2(y) + F_2(x)F_1(y) = 0 \) and

(b). \( f_1(x)F_2(y) + f_2(x)F_1(y) = 0 \) for all \( x, y \in S \)

(iii). \( F_1(x)F_2(y) = 0 = f_1(x)F_2(y) \) for all \( x, y \in S \)

(iv). \( F_1(x)F_2(y) = 0 = f_1(x)f_2(y) \) for all \( x, y \in S \) and

\[ f_1F_2 = 0 = f_1f_2 \]

(v). \((F_1F_2, f_1f_2)\) is a generalized semiderivation and

\[ F_1(x)F_2(y) = 0 \]

**Proof.** (i) implies (ii), (iii), (iv) are proved by Lemma 4.3.5 and Theorem 4.2.9

To Prove (i) implies (v)

By Lemma 4.3.5, it is clear that \((F_1, f_1)\) and \((F_2, f_2)\) are orthogonal.

Hence by Theorem 4.2.11, \( f_1f_2 \) is a semiderivation associated with the function \( g_1g_2 : S \to S \)

Now,
\[ F_1F_2(xy) = F_1(F_2(x)y + g_2(x)f_2(y)) \text{ for all } x, y \in S \]
\[ = F_1F_2(x)y + g_1(F_2(x))f_1(y) + F_1(g_2(x))f_2(y) + \]
\[ + g_1g_2(x)f_1f_2(y) \text{ for all } x, y \in S \]
\[ = F_1F_2(x)y + g_1g_2(x)f_1f_2(y) \text{ for all } x, y \in S, \]

and since \( f_1 \) and \( F_2 \) are orthogonal, \( f_2 \) and \( F_1 \) are orthogonal by Lemma 4.3.5.

Thus \((F_1F_2, f_1f_2)\) is a generalized semiderivation on \( S \).

To prove (\( ii \)) implies (\( i \))

By hypothesis, \( F_1(x)F_2(y) + F_2(x)F_1(y) = 0 \) for all \( x, y \in S \)

Replace \( x \) by \( xz \),
\[ 0 = F_1(xz)F_2(y) + F_2(xz)F_1(y) \text{ for all } x, y, z \in S \]
\[ = F_1(x)zf_2(y) + g_1(x)f_1(z)F_2(y) + \]
\[ + F_2(x)zf_1(y) + g_2(x)f_2(z)F_1(y) \text{ for all } x, y, z \in S \]
\[ = F_1(x)zf_2(y) + F_2(x)zf_1(y) \text{ for all } x, y, z \in S \]
\[ = F_1(x)SF_2(y) + F_2(x)SF_1(y) \text{ for all } x, y \in S \]

Replace \( y \) by \( x \)
\( F_1(x)SF_2(x) + F_2(x)SF_1(x) = 0 \) for all \( x \in S \)

By Lemma 4.2.6, \( F_1(x)SF_2(x) = 0 = F_2(x)SF_1(x) \) for all \( x \in S \)

By Lemma 4.2.7, \( F_1(x)SF_2(y) = 0 = F_2(y)SF_1(x) \) for all \( x, y \in S \)

Hence \((F_1, f_1)\) and \((F_2, f_2)\) are orthogonal.

To prove (\( iii \)) implies (\( i \))

By hypothesis, \( F_1(x)F_2(y) = 0 = f_1(x)F_2(y) \) for all \( x, y \in S \)
\( F_1(x)F_2(y) = 0 \)
Replace $x$ by $xz$

$$0 = F_1(xz)F_2(y)$$

$$= F_1(x)zF_2(y) + g_1(x)f_1(z)F_2(y) \text{ for all } x, y, z \in S$$

$$= F_1(x)zF_2(y) \text{ for all } x, y, z \in S$$

$$= F_1(x)SF_2(y) \text{ for all } x, y \in S$$

Hence, $F_1(x)SF_2(y) = 0 = F_2(y)SF_1(x) \text{ for all } x, y \in S$

Thus $(F_1, f_1)$ and $(F_2, f_2)$ are orthogonal.

To prove $(iv)$ implies $(i)$

By hypothesis, $F_1(x)F_2(y) = 0 = F_2(x)F_1(y) = 0 \text{ for all } x, y \in S$

Also $f_1F_2 = 0 = f_1f_2$

$$0 = f_1F_2(xy) \text{ for all } x, y \in S$$

$$= f_1(F_2(x)y + g_2(x)f_2(y)) \text{ for all } x, y \in S$$

$$= f_1(F_2(x))g_1(y) + F_2(x)f_1(y) + f_1(g_2(x))g_1(f_2(y)) +$$

$$+ g_2(x)f_1f_2(y) \text{ for all } x, y \in S$$

$$= F_2(x)Sf_1(y) \text{ for all } x, y \in S$$

$$= F_2(x)f_1(y) \text{ for all } x, y \in S$$

By $(iii)$ $(F_1, f_1)$ and $(F_2, f_2)$ are orthogonal.

To prove $(v)$ implies $(i)$

By hypothesis, $(F_1F_2, f_1f_2)$ is a generalized semiderivation on $S$

Then, $F_1F_2(xy) = F_1(F_2(x)y + g_1g_2(x)f_1f_2(y)) \text{ for all } x, y \in S \quad \text{............(1)}$

Also, $F_1F_2(xy) = F_1(F_2(x)y + g_2(x)f_2(y)) \text{ for all } x, y \in S$

$$= F_1F_2(x)y + g_1(F_2(x))f_1(y) + F_1(g_2(x))f_2(y) +$$

$$+ g_1g_2(x)f_1f_2(y) \text{ for all } x, y \in S \quad \text{............(2)}$$

Comparing $(1)$ and $(2)$ we get
\[ g_1(F_2(x))f_1(y) + F_1(g_2(x))f_2(y) = 0 \text{ for all } x, y \in S \]

Since \( g_1 \) and \( g_2 \) are surjective,

\[ F_2(x)f_1(y) + F_1(x)f_2(y) = 0 \text{ for all } x, y \in S, \quad \text{ ..........(3)} \]

Also we have \( F_1(x)F_2(y) = 0 \)

Replace \( y \) by \( yz \)

\[ 0 = F_1(x)F_2(yz) \text{ for all } x, y, z \in S \]

\[ = F_1(x)F_2(y)z + F_1(x)g_2(y)f_2(z) \text{ for all } x, y, z \in S \]

\[ = F_1(x)g_2(y) f_2(z) \text{ for all } x, y, z \in S \]

\[ = F_1(x)Sf_2(z) \text{ for all } x, z \in S \]

By Lemma 4.2.6, \( F_1(x)f_2(z) = 0 \) for all \( x, z \in S \)

Replace \( z \) by \( y \)

\[ F_1(x)f_2(y) = 0 \text{ for all } x, y \in S \]

Now (3) implies \( F_2(x)f_1(y) = 0 \) for all \( x, y \in S \)

Hence by (iii) we get \( (F_1, f_1) \) and \( (F_2, f_2) \) are orthogonal. \( \blacksquare \)

**Theorem 4.3.7.** Let \( S \) be a 2-torsion free semiprime semiring and let \( (F_1, f_1) \) be generalized semiderivations of \( S \) into \( S \) associated with functions \( g_1 : S \rightarrow S \). If \( F_1(x)F_1(y) = 0 \) for all \( x, y \in S \), then \( F_1 = f_1 = 0 \).

**Proof.** By hypothesis \( F_1(x)F_1(y) = 0 \) for all \( x, y \in S \)

Replace \( y \) by \( yz \),

\[ 0 = F_1(x)F_1(yz) \text{ for all } x, y, z \in S \]

\[ 0 = F_1(x)F_1(y)z + F_1(x)g_1(y)f_1(z) \text{ for all } x, y, z \in S \]

\[ 0 = F_1(x)g_1(y)f_1(z) \text{ for all } x, y, z \in S \]

\[ 0 = F_1(x)Sf_1(z) \text{ for all } x, z \in S \]
By Lemma 4.2.6 \( F_1(x)f_1(z) = 0 = f_1(z)F_1(x) \) for all \( x, z \in S \)

Replace \( x \) by \( xz \)

\[
0 = f_1(z)F_1(xz) \quad \text{for all} \quad x, z \in S
\]

\[
= f_1(z)F_1(x)z + f_1(z)g_1(x)f_1(z) \quad \text{for all} \quad x, z \in S
\]

\[
= f_1(z)g_1(x)f_1(z) \quad \text{for all} \quad x, z \in S
\]

\[
= f_1(z)Sf_1(z) \quad \text{for all} \quad z \in S
\]

Since \( S \) is semiprime, \( f_1(z) = 0 \) for all \( z \in S \)

Hence \( f_1 = 0 \).

Replace \( x \) by \( yx \) in \( F_1(x)F_1(y) = 0 \)

\[
0 = F_1(yx)F_1(y) \quad \text{for all} \quad x, y \in S
\]

\[
= F_1(y)xF_1(y) + g_1(y)f_1(x)F_1(y) \quad \text{for all} \quad x, y \in S
\]

\[
= F_1(y)xF_1(y) \quad \text{for all} \quad x, y \in S
\]

\[
= F_1(y)SF_1(y) \quad \text{for all} \quad y \in S
\]

Since \( S \) is semiprime, \( F_1(y) = 0 \) for all \( y \in S \)

Hence \( F_1 = 0 \).

Thus \( F_1 = f_1 = 0 \) \( \square \)

**Theorem 4.3.8.** Let \( S \) be a 2-torsion free semiprime semiring and let \( (F_1, f_1) \) and \( (F_2, f_2) \) be two generalized semiderivations of \( S \) into \( S \) associated with functions \( g_1 : S \to S \) and \( g_2 : S \to S \) respectively. Then \( (F_1F_2, f_1f_2) \) is a generalized semiderivation if and only if \( F_1 \) and \( f_2, F_2 \) and \( f_1 \) are orthogonal semiderivations.

**Proof.** Suppose that \( (F_1F_2, f_1f_2) \) is a generalized semiderivation.

Then by Theorem 4.3.6, \( (F_1, f_1) \) and \( (F_2, f_2) \) are orthogonal.
Hence by Lemma 4.3.5, $F_1$ and $f_2$, $F_2$ and $f_1$ are orthogonal semiderivations.

Conversely assume that $F_1$ and $f_2$, $F_2$ and $f_1$ are orthogonal semiderivations.

Then,

$$F_1(x)Sf_2(y) = 0 = f_2(y)SF_1(x), \text{ for all } x, y \in S$$

$$F_2(x)Sf_1(y) = 0 = f_1(y)SF_2(x), \text{ for all } x, y \in S$$

By Lemma 4.2.6 $F_1(x)Sf_2(y) = 0$ for all $x, y \in S$ implies

$$F_1(x)f_2(y) = 0 \text{ for all } x, y \in S$$

Replace $x$ by $zx$

$$0 = F_1(zx)f_2(y) \text{ for all } x, y, z \in S$$

$$= F_1(z)x f_2(y) + g_1(z)f_1(x)f_2(y) \text{ for all } x, y, z \in S$$

$$= g_1(z)f_1(x)f_2(y) \text{ for all } x, y, z \in S$$

Premultiplying by $f_1(x)f_2(y)$ on both sides,

$$0 = f_1(x)f_2(y) + g_1(z)f_1(x)f_2(y) \text{ for all } x, y, z \in S$$

$$= f_1(x)f_2(y)Sf_1(x)f_2(y) \text{ for all } x, y \in S$$

Since $S$ is semiprime,

$$f_1(x)f_2(y) = 0 \text{ for all } x, y \in S$$

Replace $x$ by $xz$

$$0 = f_1(xz)f_2(y) \text{ for all } x, y, z \in S$$

$$= f_1(x)g_1(z)f_2(y) + x f_1(z)f_2(y) \text{ for all } x, y, z \in S$$

$$= f_1(x)g_1(z)f_2(y), \text{ for all } x, y, z \in S$$

$$= f_1(x)Sf_2(y), \text{ for all } x, y \in S$$

Thus $f_1(x)Sf_2(y) = 0 = f_2(y)Sf_1(x)$ for all $x, y \in S$

Hence $f_1$ and $f_2$ are orthogonal.

Thus by Theorem 4.2.11, $f_1f_2$ is a semiderivation associated with the
function \( g_1g_2 : S \to S \).

Now,

\[
F_1F_2(xy) = F_1(F_2(x)y + g_2(x)f_2(y)) \quad \text{for all } x, y \in S
\]

\[
= F_1F_2(x)y + g_1(F_2(x))f_1(y) + F_1(g_2(x))f_2(y) + g_1g_2(x)f_1f_2(y)
\]

\[
= F_1F_2(x)y + g_1g_2(x)f_1f_2(y) \quad \text{for all } x, y \in S, \text{ since } f_1 \text{ and } F_2, f_2
\]

and \( F_1 \) are orthogonal

Hence \((F_1F_2, f_1f_2)\) is a generalized semiderivation

\[\square\]

**Corollary 4.3.9.** Let \( S \) be a 2-torsion free semiprime semiring and let \((F_1, f_1)\)

and \((F_2, f_2)\) be two generalized semiderivations of \( S \) into \( S \) associated with

functions \( g_1 : S \to S \) and \( g_2 : S \to S \) respectively such that \( F_1 \) and \( f_2, F_2 \)

and \( f_1 \) are orthogonal semiderivations. Then \( F_1 = f_1 = 0 \) or \( F_2 = f_2 = 0 \).

**Proof.** Since \( F_1 \) and \( f_2 \) are orthogonal

\[
F_1(x)Sf_2(y) = 0 = f_2(y)SF_1(x) \quad \text{for all } x, y \in S
\]

Since \( S \) is prime, \( F_1(x) = 0 \) or \( f_2(y) = 0 \) for all \( x, y \in S \)

Hence \( F_1 = 0 \) or \( f_2 = 0 \).

Since \( F_2 \) and \( f_1 \) are orthogonal,

\[
F_2(x)Sf_1(y) = 0 = f_1(y)SF_2(x) \quad \text{for all } x, y \in S
\]

Since \( S \) is prime, \( F_2(x) = 0 \) or \( f_1(y) = 0 \) for all \( x, y \in S \)

Hence \( F_2 = 0 \) or \( f_1 = 0 \).

Thus, \( F_1 = f_1 = 0 \) or \( F_2 = f_2 = 0 \). \[\square\]