CHAPTER 2

STRONG, WEAK AND PRIVATE DOMINATION NUMBER OF VARIETIES OF GRAPHS

In this chapter, we discussed about the private dominating set, weak dominating set, strong dominating set, private domination number, weak domination number and strong domination number for varieties of graphs like complete, Star, cycle, Bistar, $P_m^{(2)}$, path, friendship graph, fan graph, wheel, Helm, Gear graph and flower graph. Also, we discuss about the generalized private domination number, weak domination number and strong domination number. Using this various domination number we are calculating the domination polynomial in the subsequent chapters.

2.1. Private Domination Number for varieties of graph

Definition 2.1.1

Let $G = (V, E)$ be a simple graph of order $m$. A set $S \subseteq V$ is said to be private dominating set of a graph $G$ if it is a dominating set and for $u$ in $S$ there exists an external private neighbor $v \in V - S$ (B. Jaya Prasad et al., 2013).

Cycle graph 2.1.2

A cycle graph or circular graph is a graph that consists of a single cycle, or every vertex has exactly two edges incident with it.
The Private domination number for cycle graphs,

\[ \Gamma_{pvt}(C_4) = 2, \quad \Gamma_{pvt}(C_5) = 2, \quad \Gamma_{pvt}(C_6) = 2 \]

\[ \Gamma_{pvt}(C_7) = 3, \quad \Gamma_{pvt}(C_8) = 3, \quad \Gamma_{pvt}(C_9) = 3, \quad \Gamma_{pvt}(C_{10}) = 4, \quad \text{etc.} \]

The generalized Private domination number for cycle graph is

\[ \Gamma_{pvt}(C_m) = \left\lceil \frac{m}{3} \right\rceil, \quad m \geq 3. \]
**Complete graph 2.1.3**

A simple graph with \( m \) mutual vertices is called a complete graph and if it is denoted by \( K_m \). In the graph, a vertex should have edges with all other vertices, then it called a complete graph.

In other words, if a vertices connected to all other vertices in a graph then it is called a complete graph.

![Complete Graph](image)

**Figure 2.2 Complete graph**

The Private domination number for complete graphs \( K_m \) is \( \Gamma_{pvt}(K_4) = 1, \Gamma_{pvt}(K_7) = 1 \) and so on.

The generalized private domination number of complete graph \( \Gamma_{pvt}(K_m) = 1, \) for all \( m \).

**Bistar graph 2.1.4.**

The Bistar graph \( B_{m,n} \) is the graph obtained from \( K_2 \) by joining \( m \) pendant edges to one of \( K_2 \) and \( n \) pendant edges to the other end of \( K_2 \), \( (m, n \geq 2) \).

![Bistar Graph](image)
The Private domination number for Bistar graphs

\[ \Gamma_{\text{pvt}}(B_{2, 2}) = 2, \Gamma_{\text{pvt}}(B_{2, 3}) = 2, \Gamma_{\text{pvt}}(B_{3, 3}) = 2, \]

\[ \Gamma_{\text{pvt}}(B_{3, 4}) = 2, \Gamma_{\text{pvt}}(B_{4, 4}) = 2, \Gamma_{\text{pvt}}(B_{4, 5}) = 2, \]

\[ \Gamma_{\text{pvt}}(B_{5, 5}) = 2, \Gamma_{\text{pvt}}(B_{5, 6}) = 2, \text{ and so on.} \]

The generalized Private domination number of Bistar graph

\[ \Gamma_{\text{pvt}}(B_{m, n}) = 2, \forall m, n \]

Path graph 2.1.5

A path is a trail in which all vertices are distinct. A path between two vertices \( u \) and \( v \) is called a \( u - v \) path.
The Private domination number for Bistar graphs

\[ \Gamma_{pvt}(P_2) = 1, \quad \Gamma_{pvt}(P_3) = 1, \quad \Gamma_{pvt}(P_4) = 2, \quad \Gamma_{pvt}(P_5) = 2, \]

\[ \Gamma_{pvt}(P_6) = 3, \quad \Gamma_{pvt}(P_7) = 3, \quad \Gamma_{pvt}(P_8) = 4 \quad \text{and so on.} \]

The generalized Private domination number of path graph is

\[ \Gamma_{pvt} = \left\lfloor \frac{m}{2} \right\rfloor \]

**P\textsubscript{m}^2** graph 2.1.6.

**P\textsubscript{m}^2** is the graph obtained from the path **P\textsubscript{m}**, having vertices \(v_1, v_2, \ldots, v_m\) together with the edges \(v_i v_{i+2}\) for \(1 \leq i \leq m - 1\).
The Private dominate on number of $P_m^2$ graphs

$$\Gamma_{pvt}(P_3^2) = 1,$$
$$\Gamma_{pvt}(P_4^2) = 1,$$
$$\Gamma_{pvt}(P_5^2) = 1,$$
$$\Gamma_{pvt}(P_6^2) = 2,$$
$$\Gamma_{pvt}(P_7^2) = 2,$$
$$\Gamma_{pvt}(P_8^2) = 2,$$
$$\Gamma_{pvt}(P_9^2) = 2,$$
$$\Gamma_{pvt}(P_{10}^2) = 2,$$
$$\Gamma_{pvt}(P_{11}^2) = 3,$$
$$\Gamma_{pvt}(P_{12}^2) = 3,$$
$$\Gamma_{pvt}(P_{13}^2) = 4,$$ and so on.

The generalized Private domination number of $P_m^2$ is

$$\Gamma_{pvt}(P_m^2) = \left\lceil \frac{m}{5} \right\rceil$$

**Friendship graph 2.1.7**

The Friendship graphs $F_m$ ($m \geq 1$) consists of $m$ triangles with a common vertex.

![Friendship graph](image)

**Figure 2.6 Friendship graph**

The Private domination number of Friendship graph $F_m$

$$\Gamma_{pvt}(F_1) = 1,$$
$$\Gamma_{pvt}(F_2) = 2,$$
$$\Gamma_{pvt}(F_3) = 3,$$
$$\Gamma_{pvt}(F_4) = 4,$$ and so on.

The generalized Private domination number of Friendship graph is $\Gamma_{pvt}(F_m) = m, m \geq 1$

**Fan graph 2.1.8**

The Fan graph $F_m$ ($m \geq 2$) is obtained by joining all vertices of $P_m$ to a further vertex called center and contains $m + 1$ vertices $2m - 1$ edges.
The Private domination number of Fan graph $F_m (P_{m+1} + k_1)$

$\Gamma_{pvt} (F_1) = 1, \quad \Gamma_{pvt} (F_2) = 1, \quad \Gamma_{pvt} (F_3) = 2, \quad \Gamma_{pvt} (F_4) = 2$ and so on.

The generalized Private domination number of Fan graph

$\Gamma_{pvt} (F_m) = \Gamma_{pvt} (P_{m+1} + K_1) = \left\lceil \frac{m}{2} \right\rceil, \quad m \geq 1$.

**Wheel graph 2.1.9**

The Wheel $W_m$ $(m \geq 4)$ is obtained by joining all vertices of $C_m$ to a further vertex called the centre and contains $m + 1$ vertices and $2m$ edges.
The Private domination number of wheel graph $W_m$ is

$$
\Gamma_{\text{pvt}} (W_4) = 1, \quad \Gamma_{\text{pvt}} (W_5) = 1, \quad \Gamma_{\text{pvt}} (W_6) = 2, \quad \Gamma_{\text{pvt}} (W_7) = 2, \quad \Gamma_{\text{pvt}} (W_8) = 3, \quad \Gamma_{\text{pvt}} (W_9) = 3,
$$

$$
\Gamma_{\text{pvt}} (W_{10}) = 4, \quad \Gamma_{\text{pvt}} (W_{11}) = 4 \text{ and so on.}
$$

The generalized Private domination number of wheel graph is

$$
\Gamma_{\text{pvt}} (W_m) = \left\lfloor \frac{m}{2} - 1 \right\rfloor, \quad m \geq 4.
$$

**Helm graph 2.110**

The helm graph $H_m$ is the graph obtained from a wheel by attaching a pendent edge at each vertex of the $m$-cycle.
The Private domination number of Helm graph $H_m$ is

$\Gamma_{pvt}(H_3) = 3$,  $\Gamma_{pvt}(H_4) = 4$,  $\Gamma_{pvt}(H_6) = 6$  and so on.

The generalized Private domination number of Helm graph $\Gamma_{pvt}(H_m) = m$, $m \geq 3$

**Gear graph 2.1.11**

A gear graph is obtained from the wheel by adding a vertex between every pair of adjacent vertices of the cycle.
The Private domination number of Gear graph $G_m$ is

$$
\Gamma_{pvt}(G_3) = 2, \quad \Gamma_{pvt}(G_4) = 3, \quad \Gamma_{pvt}(G_5) = 4, \quad \Gamma_{pvt}(G_6) = 4, \quad \Gamma_{pvt}(G_7) = 5,
$$

$$
\Gamma_{pvt}(G_8) = 6, \quad \Gamma_{pvt}(G_9) = 6 \text{ and so on.}
$$

**Flower graph 2.1.12**

The graph obtained from a helm by joining each pendant vertex to the central vertex of the helm.
The Private domination number of Flower graph

\( \Gamma_{\text{pvt}} (F_3) = 3 \), \( \Gamma_{\text{pvt}} (F_4) = 4 \), \( \Gamma_{\text{pvt}} (F_5) = 5 \), \( \Gamma_{\text{pvt}} (F_8) = 8 \) and so on.

The generalized Private domination number of Flower graph is \( \Gamma_{\text{pvt}} (F_m) = m, m \geq 3 \).

**Star Graph 2.1.13**

A star graph \( S_m \) is the complete bipartitue graph \( K_{1,m} \): a tree with 1 internal node and \( k \) leaves.

In other words, \( S_m \) to be the tree of order \( m \) with maximum diameter two; in which a case of star of \( m > 2 \) has \( m-1 \) leaves.

The Private domination number for complete graphs \( S_m \) is \( \Gamma_{\text{pvt}} (S_4) = 1 \), \( \Gamma_{\text{pvt}} (S_7) = 1 \) and so on.

The generalized private domination number of complete graph \( \Gamma_{\text{pvt}} (S_m) = 1 \), for all \( m \).
Figure 2.12 Star Graph

2.2. Weak Domination number for varieties of graph

Definition 2.2.1

A set $s \subseteq v$ is a weak dominating set of $G$ if for every $u \in v - s$, there exists a $v \in s$ such that $uv \in E$ and $\deg(u) \geq \deg(v)$. The weak domination number $\gamma_w(G)$ is the minimum cardinality of a weak dominating set of $G$ (S.K. Vaidhya and S.H. Karkar, 2017).

Cycle graph 2.2.2
The weak domination number of cycle graph

\[ \gamma_{wd}(C_3) = 1, \quad \gamma_{wd}(C_4) = 2, \quad \gamma_{wd}(C_5) = \gamma_{wd}(C_6) = 2, \]

\[ \gamma_{wd}(C_7) = \gamma_{wd}(C_8) = \gamma_{wd}(C_9) = 3 \]

\[ \gamma_{wd}(C_{10}) = 4, \quad \gamma_{wd}(C_{14}) = 5, \quad \gamma_{wd}(C_{18}) = 6 \]

The generalized weak domination number of cycle graph

\[ \gamma_{wd}(C_m) = \left\lceil \frac{m}{3} \right\rceil, \quad m \geq 3. \]

Complete graph 2.2.3

The weak domination number of Complete graph

\[ \gamma_{wd}(K_3) = \gamma_{wd}(K_5) = \gamma_{wd}(K_7) = 1, \quad \gamma_{wd}(K_9) = 1 \text{ etc.} \]
The generalized Weak domination number of Complete graph is $\gamma_{od}(K_m) = 1$, for all $m$

**Bistar graph 2.2.4**

![Bistar graph images]

**Figure 2.15 Bistar graph of weak domination**

The weak domination number of Bistar graph $B_{m,n}$ in

- $\gamma_{od}(B_{2,1}) = 3$, $\gamma_{od}(B_{2,2}) = 4$, $\gamma_{od}(B_{2,3}) = 5$, $\gamma_{od}(B_{3,3}) = 6$,
- $\gamma_{od}(B_{3,4}) = 7$, $\gamma_{od}(B_{4,4}) = 8$ and so on.

The generalized weak domination number of Bistar graph is

$$\gamma_{od}(B_{m,n}) = m + n, \text{ if } n = 1, 2, 3, \ldots, \text{ and } m = 1, 2, 3, \ldots$$

**Path graph 2.2.5**

![Path graph images]

**Figure 2.16 Path graph of weak domination**
The weak domination number of path graph $P_m$

$\gamma_{\text{od}}(P_2) = 1$, $\gamma_{\text{od}}(P_3) = 2$, $\gamma_{\text{od}}(P_4) = 2$, $\gamma_{\text{od}}(P_5) = 3$, $\gamma_{\text{od}}(P_6) = 3$,

$\gamma_{\text{od}}(P_7) = 3$, $\gamma_{\text{od}}(P_8) = 4$, $\gamma_{\text{od}}(P_9) = 4$, $\gamma_{\text{od}}(P_{10}) = 4$ and so on.

$P_m^{(2)}$ graph 2.2.6

![Path Graph $P_m^{(2)}$](image)

**Figure 2.17** $P_m^{(2)}$ graph of weak domination

The weak domination number of $P_m^{(2)}$ graph

$\gamma_{\text{od}}(P_2^{(2)}) = 1$, $\gamma_{\text{od}}(P_3^{(2)}) = 1$, $\gamma_{\text{od}}(P_4^{(2)}) = 2$, $\gamma_{\text{od}}(P_5^{(2)}) = 2$,

$\gamma_{\text{od}}(P_6^{(2)}) = 2$, $\gamma_{\text{od}}(P_7^{(2)}) = 3$, $\gamma_{\text{od}}(P_8^{(2)}) = 4$, $\gamma_{\text{od}}(P_9^{(2)}) = 5$,

$\gamma_{\text{od}}(P_{10}^{(2)}) = 5$, $\gamma_{\text{od}}(P_{11}^{(2)}) = 5$, $\gamma_{\text{od}}(P_{12}^{(2)}) = 5$, $\gamma_{\text{od}}(P_{13}^{(2)}) = 6$,

$\gamma_{\text{od}}(P_{14}^{(2)}) = 6$, $\gamma_{\text{od}}(P_{17}^{(2)}) = 6$, $\gamma_{\text{od}}(P_{18}^{(2)}) = 7$, and so on.

Friendship graph 2.2.7

![Friendship Graph](image)
The weak domination number of friendship graph $F_m$

\[
\gamma_{od}(F_1) = 1, \quad \gamma_{od}(F_2) = 2, \quad \gamma_{od}(F_3) = 3, \quad \gamma_{od}(F_4) = 4, \quad \gamma_{od}(F_5) = 5,
\]

\[
\gamma_{od}(F_6) = 6, \quad \gamma_{od}(F_7) = 7, \quad \gamma_{od}(F_8) = 8 \quad \text{and so on.}
\]

The generalized weak domination number of Friendship graph $F_m$ is

\[
\gamma_{od}(F_m) = m, \text{ if } m = 1, 2, 3, \ldots
\]

**Fan graph 2.2.8**

The weak domination number of Fan graph

\[
\gamma_{od}(P_4 + K_1) = 2, \quad \gamma_{od}(P_5 + K_1) = 3, \quad \gamma_{od}(P_6 + K_1) = 3, \quad \gamma_{od}(P_7 + K_1) = 4,
\]

\[
\gamma_{od}(P_8 + K_1) = 4 \quad \text{and so on.}
\]
The generalized weak domination number of Fan graph \( F_m(P_m + K_1) \) is

\[
\gamma_{wd}(P_m + K_1) = \begin{cases} 
\frac{m + 2}{2} & \text{if } m \text{ is even} \\
\frac{m + 1}{2} & \text{if } m \text{ is odd}
\end{cases}
\]

Wheel graph 2.2.9

The weak domination number of wheel graph

\[
\gamma_{wd}(W_3) = 1, \quad \gamma_{wd}(W_4) = 2, \quad \gamma_{wd}(W_5) = 2, \quad \gamma_{wd}(W_6) = 2, \\
\gamma_{wd}(W_7) = 3, \quad \gamma_{wd}(W_8) = 10, \quad \gamma_{wd}(W_9) = 4, \quad \gamma_{wd}(W_{13}) = 5
\]

and so on.
Figure 2.20 Wheel graph of weak domination

The generalized weak domination number of wheel graph is \( \gamma_{wd}(W_m) = \left\lfloor \frac{m}{3} \right\rfloor, \quad m \geq 3 \)

Helm graph 2.2.10

The weak domination number of Helm graph \( H_m \)

\( \gamma_{wd}(H_3) = 4, \quad \gamma_{wd}(H_4) = 5, \quad \gamma_{wd}(H_5) = 6, \quad \gamma_{wd}(H_6) = 7, \quad \gamma_{wd}(H_7) = 8, \)

\( \gamma_{wd}(H_8) = 9, \) and so on.

The generalized weak domination number of Helm graph is

\( \gamma_{wd}(H_m) = m + 1 \) if \( m \geq 3 \).
Figure 2.21 Helm graph of weak domination

Gear graph 2.2.11
Figure 2.22 Gear graph of weak domination

The weak domination number of Gear graph

\[ \gamma_{\text{wd}}(G_3) = 4, \quad \gamma_{\text{wd}}(G_4) = 5, \quad \gamma_{\text{wd}}(G_5) = 6, \quad \gamma_{\text{wd}}(G_6) = 7, \quad \gamma_{\text{wd}}(G_7) = 8, \]

\[ \gamma_{\text{wd}}(G_7) = 8, \quad \gamma_{\text{wd}}(G_8) = 9 \quad \text{and so on.} \]

The generalized weak domination number of Gear graph is

\[ \gamma_{\text{wd}}(G_m) = \begin{cases} 
  m+1 & \text{if } m \geq 3 \\
  2 & \text{if } m = 2 
\end{cases} \]

Flower graph 2.2.12

The weak domination number of Flower graph

\[ \gamma_{\text{wd}}(F_3) = 3, \quad \gamma_{\text{wd}}(F_4) = 4, \quad \gamma_{\text{wd}}(F_5) = 5, \quad \gamma_{\text{wd}}(F_6) = 6, \quad \gamma_{\text{wd}}(F_7) = 7, \]

\[ \gamma_{\text{wd}}(F_8) = 8 \quad \text{and so on.} \]
The generalized weak domination number of Flower graph is

\[ \gamma_{\text{od}}(F_{\ell m}) = m, \quad \text{if } m \geq 3. \]

**Star Graph 2.2.13**

The weak domination number of star graph \( S_m \)

\[ \gamma_{\text{od}}(S_4) = 3, \quad \gamma_{\text{od}}(S_8) = 7, \quad \gamma_{\text{od}}(S_{10}) = 9, \quad \gamma_{\text{od}}(S_{12}) = 11, \quad \gamma_{\text{od}}(S_{13}) = 12 \quad \text{and so on.} \]
The generalized weak domination number of star graph is
\[ \gamma_w(S_m) = m - 1 \text{ for all } m. \]

2.3. Strong Domination number for varieties of graph

**Definition 2.3.1**

A set \( s \subseteq V \) is a strong dominating set of \( G \) if for every \( u \in V - s \), there exists a \( v \in s \) such that \( uv \in E \) and \( \deg(u) \leq \deg(v) \). The minimum cardinality of strong dominating set is called minimum strong dominating number and is denoted by \( \gamma_{sd}(G) \) (S.K. Vaidhya and S.H. Karkar, 2017).

**Cycle graph 2.3.2**

The strong domination number of cycle graph
\[ \gamma_{sd}(C_3) = 1, \quad \gamma_{sd}(C_4) = 2, \quad \gamma_{sd}(C_5) = 2, \quad \gamma_{sd}(C_6) = 2, \quad \gamma_{sd}(C_7) = 3, \]
\[ \gamma_{sd}(C_8) = 3, \quad \gamma_{sd}(C_9) = 3, \quad \gamma_{sd}(C_{10}) = 4, \quad \gamma_{sd}(C_{11}) = 4, \quad \gamma_{sd}(C_{12}) = 4, \]
\[ \gamma_{sd}(C_{13}) = 5, \quad \gamma_{sd}(C_{14}) = 5, \quad \text{and so on.} \]
The generalized strong domination number of cycle graph is \( \gamma_{sd}(C_m) = \left\lfloor \frac{m}{3} \right\rfloor, \ m \geq 3 \).

**Complete graph 2.3.3**

![Figure 2.26 Complete graph of strong domination](image)

The strong domination number of Complete graphs are

\( \gamma_{sd}(K_3) = \gamma_{sd}(K_5) = \gamma_{sd}(K_7) = 1, \ \gamma_{sd}(K_9) = 1 \) etc.

The generalized strong domination number of Complete graph is \( \gamma_{sd}(K_m) = 1 \), for all m

**Bistar graph 2.3.4**

![Figure 2.27 Bistar graph of strong domination](image)
The strong domination number of Bistar graph

\[ \gamma_{sd}(B_{2, 1}) = 2, \quad \gamma_{sd}(B_{2, 2}) = 2, \quad \gamma_{sd}(B_{3, 2}) = 2, \quad \gamma_{sd}(B_{3, 3}) = 2, \quad \gamma_{sd}(B_{4, 3}) = 2, \]
\[ \gamma_{sd}(B_{4, 4}) = 2, \quad \text{and so on.} \]

The generalized strong domination number of Bistar graph in \( \gamma_{sd}(B_{m, n}) = 2 \) \( m, n. \)

Path graph 2.3.5

![Path graph of strong domination](image)

The strong domination number of path graph \( P_m \)

\[ \gamma_{sd}(P_2) = 1, \quad \gamma_{sd}(P_3) = 1, \quad \gamma_{sd}(P_4) = 2, \quad \gamma_{sd}(P_5) = 2, \quad \gamma_{sd}(P_6) = 2, \]
\[ \gamma_{sd}(P_7) = 3, \quad \gamma_{sd}(P_8) = 3, \quad \gamma_{sd}(P_9) = 3, \quad \gamma_{sd}(P_{10}) = 4, \quad \gamma_{sd}(P_{12}) = 4, \]
\[ \gamma_{sd}(P_{13}) = 5, \quad \gamma_{sd}(P_{15}) = 5 \quad \text{and so on.} \]

The generalized strong domination number of path graph \( P_m \) is \( \gamma_{sd}(P_m) = \left\lceil \frac{m}{3} \right\rceil \quad m \geq 2. \)

\( P_m^{(2)} \) graph 2.3.6
The strong domination number of $P_m^{(2)}$ graph

\[
\gamma_{sd}(P_2^{(2)}) = \gamma_{sd}(P_3^{(2)}) = \gamma_{sd}(P_4^{(2)}) = \gamma_{sd}(P_5^{(2)}) = 1,
\]
\[
\gamma_{sd}(P_6^{(2)}) = \gamma_{sd}(P_7^{(2)}) = \gamma_{sd}(P_8^{(2)}) = \gamma_{sd}(P_9^{(2)}) = \gamma_{sd}(P_{10}^{(2)}) = 2,
\]
\[
\gamma_{sd}(P_{11}^{(2)}) = \gamma_{sd}(P_{12}^{(2)}) = \gamma_{sd}(P_{13}^{(2)}) = \gamma_{sd}(P_{14}^{(2)}) = \gamma_{sd}(P_{15}^{(2)}) = 3,
\]
\[
\gamma_{sd}(P_{16}^{(2)}) = \gamma_{sd}(P_{17}^{(2)}) = \gamma_{sd}(P_{18}^{(2)}) = \gamma_{sd}(P_{19}^{(2)}) = \gamma_{sd}(P_{20}^{(2)}) = 4 \text{ and so on.}
\]

The generalized strong domination number of $P_m^{(2)}$ graph is $\gamma_{sd}(P_m^{(2)}) = \left\lfloor \frac{m}{5} \right\rfloor$, $m \geq 2$.

Friendship graph 2.3.7

Figure 2.29 $P_m^{(2)}$ graph of strong domination

Figure 2.30 Friendship graph of strong domination
The strong domination number of Friendship graph $F_m$, 
\[ \gamma_{sd}(F_1) = 1, \quad \gamma_{sd}(F_2) = 1, \quad \gamma_{sd}(F_3) = 1, \quad \gamma_{sd}(F_4) = 1, \quad \gamma_{sd}(F_5) = 1 \text{ and so on.} \]

The generalized strong domination number of Friendship graph $F_m$ is $\gamma_{sd}(F_m) = 1$.

**Fan graph 2.3.8**

![Fan graph of strong domination](image)

Figure 2.31 Fan graph of strong domination

The strong domination number of Fan graph $F_m(P_m + K_1)$
\[ \gamma_{sd}(P_2 + K_1) = 1, \quad \gamma_{sd}(P_3 + K_1) = 1, \quad \gamma_{sd}(P_4 + K_1) = 1, \]
\[ \gamma_{sd}(P_5 + K_1) = 1, \quad \gamma_{sd}(P_6 + K_1) = 1, \text{ and so on.} \]

The generalized strong domination number of Fan graph $F_m(P_m + K_1)$ is $\gamma_{sd}(P_m + K_1) = 1$.

**Wheel graph 2.3.9**

![Wheel graph](image)
The strong domination number of wheel graph $W_m$

\[ \gamma_{sd}(W_3) = 1, \quad \gamma_{sd}(W_4) = 1, \quad \gamma_{sd}(W_5) = 1, \quad \gamma_{sd}(W_6) = 1 \text{ and so on.} \]

The generalized strong domination number of wheel graph $W_m$ is $\gamma_{sd}(W_m) = 1, m \geq 3$.

**Helm graph 2.3.10**

The strong domination number of Helm graph $H_m$

\[ \gamma_{sd}(H_3) = 3, \quad \gamma_{sd}(H_4) = 4, \quad \gamma_{sd}(H_5) = 6, \quad \gamma_{sd}(H_6) = 7, \quad \gamma_{sd}(H_7) = 8, \quad \gamma_{sd}(H_8) = 9, \text{ and so on.} \]
The generalized strong domination number of Helm graph $H_m$ is

$$\gamma_{sd}(H_m) = \begin{cases} m, & m = 3, 4 \\ m + 1, & m \geq 5 \end{cases}$$

**Gear graph 2.3.11**

The strong domination number of Gear graph $G_3$ and $G_4$ is $3$,

$$\gamma_{sd}(G_3) = \gamma_{sd}(G_4) = 3,$$

$\gamma_{sd}(G_5)$ and $\gamma_{sd}(G_6)$ is $4$,

$$\gamma_{sd}(G_5) = \gamma_{sd}(G_6) = 4,$$

$\gamma_{sd}(G_7)$ and $\gamma_{sd}(G_8)$ is $5$,

$$\gamma_{sd}(G_7) = \gamma_{sd}(G_8) = 5,$$
\[ \gamma_{sd}(G_9) = \gamma_{sd}(G_{10}) = 6 \quad \text{and so on.} \]

The generalized strong domination number of Gear graph \( G_m \) is

\[
\gamma_{sd}(G_m) = \begin{cases} 
\frac{m+2}{2}, & \text{if } \, m \text{ is even } & \text{and } m \geq 4 \\
\frac{m+3}{2}, & \text{if } \, m \text{ is odd } & \text{and } m \geq 3
\end{cases}
\]

**Flower graph 2 3 12**

The strong domination number of Flower graph \( F\ell_m \)

\[ \gamma_{sd}(F\ell_3) = 1, \quad \gamma_{sd}(F\ell_4) = 1, \quad \gamma_{sd}(F\ell_5) = 1, \quad \gamma_{sd}(F\ell_6) = 1 \quad \text{and so on.} \]

The generalized strong domination number of Flower graph \( F\ell_m \) is

\[ \gamma_{sd}(F\ell_m) = 1, \quad \blacktriangleleft \, m \geq 3. \]
The strong domination number of star graph $S_m$

$$\gamma_{sd}(S_3) = 1, \quad \gamma_{sd}(S_4) = 1, \quad \gamma_{sd}(S_5) = 1, \quad \gamma_{sd}(S_6) = 1 \text{ and so on.}$$

The generalized strong domination number of star graph $S_m$ is

$$\gamma_{sd}(S_m) = 1, \quad \forall m \geq 3.$$