CHAPTER 1
INTRODUCTION AND BASIC CONCEPTS

1.1 Introduction
1.1.1 General

Graph Theory is one of the most extensively researched branches of modern mathematics and computer applications. Graph theory may be said to have its beginning in 1736 when Euler considered the general case of the Konigsberg bridge problem. Euler begins his analysis of the bridge crossing problem by first replacing the map of the city by a simple diagram showing only the main features. In modern graph theory, this diagram has been simplified even further to include only points (representing land masses) and line segments (representing bridges). These points and line segments are referred to as vertices and edges respectively. The collection of vertices and edges together with the relationships between them is called a graph. Since then, graph theory has developed into an extensive and popular branch of Mathematics. Now it is being applied to many problems in mathematics. Computer science and other scientific areas including the not-so-scientific once.

Graphs can be used to model many types of relations and processes in physical, biological, social and information systems. Many practical problems can be represented by graphs. There are no standard notations for graph theoretical objects. This is natural, because the names one uses for the objects reflect the applications.

Thus, for instance, if a communications network is considered as a graph, then the computers taking part in this network, are called nodes rather than vertices or points. Graph theory is also used to study molecules in chemistry and physics. In condensed matter physics,
the three dimensional structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. In chemistry a graph makes a natural model for a molecule, where vertices represent atoms and edges represent bonds. In mathematics, graphs are useful in geometry and certain parts of topology such as knot theory. Algebraic graph theory has close links with group theory. A graph structure can be extended by assigning a weight to each edge of the graph. Graph with weights are used to represent structures in which pairwise connections have some numerical values. For example, if a graph represents a road network, the weights could represent the length of each road and so on.

1.1.2 Graph Domination

In 1990, Hedetniemi and Lasker note the domination problem. The domination problem was studied from 1950’s onwards, but the rate of research on domination significantly increase in the mid 1970’s.

One of the most interesting problems in graph theory is that of Domination theory. The earliest ideas of dominating sets are found in the classical problems of covering chess board with minimum number of chess pieces. Now a day’s domination theory ranks top among the most prominent areas of research in graph theory and combinatorics.

The theory of domination was formalized by claude Berge in his book “Theory of graphs and its application” in 1962. Qystein Ore published a book on Graph Theory. Berge mentions the problem of keeping a number of strategic locations under surveillance by a set of radar station. Ore was a first person to use the term domination number. The theory of domination boasts a host of applications to social network theory. Land surveying game theory interconnection network, parallel computing and image processing and so on.
1.1.3 Domination Polynomial

Graph Polynomials are a well developed area useful for analyzing properties of graphs. There are some polynomials associated to graphs. Chromatic polynomial, clique polynomial, characteristics polynomial and Tutte polynomial are some examples of these polynomials. Domination polynomial of a graph is a new graph polynomial which introduced in the literature as a Ph.D. thesis of Saeid Alikhani in 2009. This book introduce the domination polynomial of a graph, investigate its properties and establish some relationships between this polynomial and geometrical properties of graph. Using this basic concept of domination polynomial we create the strong domination polynomial, weak domination polynomial and private domination polynomial.

1.2 Preliminaries

Definition 1.2.1.

A graph is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of G, called edges. The vertex set and the edge set of G are respectively denoted by V (G) and E (G). A graph G with vertex set V (G) and edge set E (G) is denoted by G = (V, E).

If e = {u, v} is an edge, we write e = uv and we say e joins the vertices u and v; u and v are adjacent vertices, u and v are incident with e. If two vertices are not joined by an edge, then we say that they are non-adjacent. If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other.

Definition 1.2.2

The number of elements in the vertex set of a graph is called the order of G and is denoted by n. The number of elements in the edge set of graph is called the size of G and is
denoted by m. A graph with n vertices and m edges is called as (n, m) – graph. The (1, 0) – graph is called as trivial graph.

**Definition 1.2.3**

A graph $G_1$ is isomorphic to a graph $G_2$ if there is a bijection $f$ from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. If $G_1$ is isomorphic to $G_2$, we write $G_1 \cong G_2$.

**Definition 1.2.4**

A graph $H$ is called a subgraph of a graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph of $G$ is a subgraph $H$ with $V(H) = V(G)$. For any set $S$ of vertices of $G$, the induced subgraph $< S >$ is the maximal subgraph of $G$ with vertex set. Therefore, two vertices of $S$ are adjacent in $< S >$ if and only if they are adjacent in $G$.

**Definition 1.2.5**

Let $G = (V, E)$ be a graph and $v$ be a vertex of $G$. $G - v$ is the induced subgraph $< V(G) - \{v\} >$ of $G$ and it is obtained from $G$ by removing $v$ and the edges incident with $v$. If $e \in E(G)$, $G - e$ is the spanning subgraph with edge set $E(G) - \{e\}$ and it is obtained from $G$ by removing the edge $e$ from $G$.

**Definition 1.2.6**

The degree of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$ and is denoted by $\deg_G(v)$ or $\deg(v)$ or $d(v)$, when there is no ambiguity. The maximum and the minimum degrees of the vertices of $G$ are respectively denoted by $\Delta(G)$ and $\delta(G)$. A vertex of degree 0 in $G$ is called an isolate vertex, and a vertex of degree 1 is called a pendent vertex or an end vertex of $G$. Any vertex adjacent to a pendent vertex is called a support. A support
is said to be trivial if it has only one pendent vertex adjacent to it. A support with two or more adjacent pendent vertices is called a non trivial support. The set of all pendent vertices adjacent to a support is denoted by \( N_0(u) \). A vertex of a graph \( G \) is said to be a full degree vertex or a dominating vertex if it is adjacent to all other vertices in \( G \).

**Definition 1.2.7**

A graph \( G \) is said to be regular graph of degree \( r \) if every vertex of \( G \) has degree \( r \). Such graphs are called \( r \) – regular graphs. A 3 – regular graph is called cubic graph.

**Definition 1.2.8**

A graph \( G \) is complete if every pair of its vertices are adjacent. A complete graph on \( n \) vertices is denoted by \( K_n \). A clique of a graph is a maximal complete subgraph.

**Definition 1.2.9**

The \( \overline{G} \) of a graph \( G \) is the graph with vertex set \( V(G) \) such that two vertices are adjacent in \( \overline{G} \) if and only if they are not adjacent in \( G \).

**Definition 1.2.10**

A bipartite graph is a graph \( G \) whose vertex set \( V(G) \) can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that every edge in \( G \) has one end vertex in \( V_1 \) and the other end vertex in \( V_2 \). \((V_1, V_2)\) is called a bipartition of \( G \). Further, if every vertex of \( V_1 \) is adjacent to every vertex of \( V_2 \), then \( G \) is called a complete bipartite graph. The complete bipartite graph with bipartition \((V_1, V_2)\) such that \(|V_1| = r \) and \(|V_2| = s \) is denoted by \( K_{r,s} \). \( K_{1,r} \) is called a star. When \( r \geq 2 \) the vertices of degree 1 of a star are called claws of the star and the vertex of degree greater than 1 is called the centre of the star. When \( r = 1 \), \( K_{1,1} \) becomes \( K_2 \) and in this case
any one of the two vertices of $K_2$ can be called a center. A double star is a graph obtained by
taking two stars and joining the vertices of maximum degree with an edge.

An $r$–partite graph (or) multipartite graph is a graph $G$ whose vertex set $V (G)$ can be
partitioned into $r$ subsets $V_1, V_2, \ldots, V_r$ such that every edge of $G$ has one end vertex in $V_i$
and the other end vertex in $V_j$, for $i \neq j$. $(V_1, V_2, \ldots, V_r)$ is called a $r$ – partition of $G$. If every
vertex of $V_i$ is adjacent to every vertex of $V_j$, $i \neq j$ for every $i$ and $j$, then $G$ is called a
complete $r$–partite graph (or) complete multipartite graph. The complete $r$-partite graph with
$r$-partition $(V_1, V_2, \ldots, V_r)$ such that $|V_i| = n_i$, $1 \leq i \leq r$, is denoted by $K_{n_1,n_2,\ldots,n_r}$.

**Definition 1.2.11**

Let $u$ and $v$ (not necessarily distinct) be vertices of a graph $G$. A $u – v$ walk of $G$ is
finite alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, e_k, u_k = v$ of vertices and edges beginning with
vertex $u$ and ending with vertex $v$ such that $e_i = u_{i-1}, u_i$, $i = 1, 2, \ldots, k$. The number $k$ is called
the length of the walk. The walk is said to be closed if $u = v$ and is open otherwise. A walk
$u_0, e_1, u_1, e_2, u_2, \ldots, e_k, u_k$ is determined by the sequence $u_0, u_1, u_2, \ldots, u_k$ of its vertices and
this walk is denoted by $u_0, u_1, u_2, \ldots, u_k$. A walk in which all the vertices are distinct is called
a path. A closed walk $u_0, u_1, u_2, \ldots, u_k$ in which the vertices $u_0, u_1, u_2, \ldots u_{k-1}$ are distinct is
called a cycle. A path on $k$ vertices is denoted by $P_k$ and a cycle $k$ vertices is denoted by $C_k$.

**Definition 1.2.12**

A graph $G$ is called a Hamiltonian graph if it has a spanning subgraph isomorphic to a
cycle.
**Definition 1.2.13**

A graph G is said to be connected if any two distinct vertices of G are joined by a path. A maximal connected subgraph of G is called a component of G. Thus a disconnected graph has at least two components.

**Definition 1.2.14**

The distance $d(u, v)$ between two vertices $u$ and $v$ in a graph G is the length of a shortest $u - v$ path in G. A shortest $u - v$ path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic. The diameter of G is denoted by $\text{diam}(G)$.

**Definition 1.2.15**

A cut-vertex (cut-edge) of a graph G is a vertex (edge) whose removal increases the number of components in G.

**Definition 1.2.16**

A graph is called acyclic, if it has no cycles. A connected acyclic graph is called a tree. A spider is a tree which has at most one vertex of degree $\geq 3$. A tree is a wounded spider if the tree is $K_{1,r}$, $r \geq 0$, in which at most $r - 1$ of the edges are subdivided. A star is also a wounded spider.

**Definition 1.2.17**

A subdivision of an edge $uv$ of a graph G is obtained by introducing a new vertex $w$ and replacing the edge $uv$ with edges $uw$ and $wv$. The graph obtained from G by subdividing each edge of G exactly once is called the subdivision graph (or subdivision) of G and is denoted by $S(G)$. 
Definition 1.2.18

A subset S of the vertex set in a graph G is said to be a independent if no two vertices in S are adjacent. The maximum number of vertices in an independent set of a graph G is called the independence number of G and is denoted by \( \beta_0(G) \) or \( \beta(G) \).

Definition 1.2.19

Let \( G = (V, E) \) be simple graph. A subset S of the vertex set \( V(G) \) is said to be a dominating set if every \( v \in V(G) \) is either an element of S or is adjacent to an element of S. The minimum cardinality of a dominating set is called the domination number of G and is denoted by \( \gamma(G) \).

Definition 1.2.20

Let \( G = (V, E) \) be simple graph. Let S be a subset of the vertex set \( V(G) \). Let \( v \in S \). Let \( Pvt[v, S] = N[v] - N[S - v] \). Elements of \( Pvt[v, S] \) are called private neighbours v with respect to S.

Definition 1.2.21

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be any two graphs. Then their union \( G_1 \cup G_2 \) is the graph whose vertex set is \( V_1 \cup V_2 \) and edge set is \( E_1 \cup E_2 \).

Definition 1.2.22

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be any two graphs. Then their join \( G_1 + G_2 \) is the graph whose vertex set is \( V_1 \cup V_2 \) and edge set is \( E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\} \).
Definition 1.2.23

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. The Cartesian Product $G_1 \times G_2$ is defined to be the graph whose vertex set is $V_1 \times V_2$ and edge set is $\{(u_1, v_1), (u_2, v_2) : \text{either } u_1 = v_2 \text{ and } v_1 \in E_2 \text{ or } v_1 = v_2 \text{ and } u_1 \in E_1\}$.

Definition 1.2.24

Let $G_1$ be a $(n_1, m_1)$–graph and let $G_2$ be $(n_2, m_2)$–graph. Then the corona $G_1 \circ G_2$ is defined as the graph $G$ obtained by taking one copy of $G_1$ and $n_1$ copies of $G_2$, and joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$.

Definition 1.2.25

The graph $K_{1,3}$ is called a claw and the graph $K_3 \circ K_1$ is called a net.

Definition 1.2.26

For any real number $x$, $\lceil x \rceil$ denotes the smallest integer greater than or equal to $\lfloor x \rfloor$, $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$.

Definition 1.2.27

The open neighbourhood $N(v)$ of a vertex $v$ in a graph $G$ is the set of all vertices adjacent to $v$ in $G$. The closed neighbourhood $N[v]$ of $v$ is the set $N(v) \cup \{v\}$.

Definition 1.2.28

The open neighbourhood $N(D)$ of a set $D$ of vertices is the set of all vertices adjacent to vertices in $D$. The closed neighbourhood $N[D]$ of $D$ is the set $N(D) \cup D$. If $x \in D$, a private neighbor of $x$ with respect to $D$ is a vertex $v$ such that $v \in N[x] - N[D - \{x\}]$. 
Definition 1.2.29

Let $G = (V, E)$ be a graph and $u, v \in V$. If $uv \in E$, it is said that $u$ and $v$ dominate each other. A subset $D$ of $V$ is a dominating set of $G$ if every vertex $v$ in $V - D$ is dominated by some $u \in D$.

A dominating set $D$ of $G$ is called an independent dominating set if the vertices in $D$ are independent. A dominating set $D$ of $G$ is called a minimal dominating set if $D - \{u\}$ is not a dominating set for any $u$ in $D$. Let $\gamma(G)$ and $\Gamma(G)$ denote the cardinality of smallest and largest minimal dominating sets of $G$ respectively. The number $\gamma(G)$ is called the domination number and $\Gamma(G)$ is called the upper domination number. The independent dominating number $i(G)$ of $G$ is the cardinality of its smallest independent dominating set.

A dominating set $D$ of $G$ is called a minimum dominating set or $\gamma$-set if $D$ is a dominating set with cardinality $\gamma(G)$.

Definition 1.2.30

A set $s \subseteq V$ is a strong dominating set of $G$ if for every $u \in V - s$, there exists a $v \in s$ such that $uv \in E$ and $\deg(u) \leq \deg(v)$. The minimum cardinality of strong dominating set is called minimum strong dominating number and is denoted by $\gamma_{sd}(G)$.

Theorem 1.2.31

A dominating set $D$ of a graph $G$ is minimal if and only if for every $u \in D$ one of the following conditions holds.

(i) $N(u) \cap D = \emptyset$

(ii) There is a vertex $u \in V - D$ such that $N(v) \cap D = \{u\}$. 
Theorem 1.2.32

Every connected graph \( G \) of order \( n \geq 2 \) has a dominating set \( D \) whose complement \( V - D \) is also a dominating set.

Theorem 1.2.33

If \( g \) is a graph with no isolated vertices, then the complement \( V - D \) of every minimal dominating set \( D \) is a dominating set.

Corollary 1.2.34

If \( G \) is a graph without isolated vertices on \( n \) vertices, then \( \gamma(G) < \left\lceil \frac{n}{2} \right\rceil \).

Theorem 1.2.35

For any graph \( G \) with even order \( n \) and no isolated vertices, \( \gamma(G) = \frac{n}{2} \) if and only if the components of \( G \) are the cycle \( C_4 \) or the corona \( H \circ K_1 \) for any connected graph \( H \).

Definition 1.2.36

If there is only one minimum dominating set for a graph \( G \), then we say that \( G \) has a unique minimum dominating set.

Theorem 1.2.37

The following are some classes of graphs having unique minimum dominating set.

(i) All graph \( G \) having only one vertex of full degree.

(ii) All graph \( G \) such that every component \( G_i \) of \( G \) is either trivial or is a star \( K_{1,r_i} \) with \( r_i \geq 2 \).

(iii) \( C_r \circ C_s \) for all values of \( r \) and \( s \) with \( r \equiv 0 \pmod{3} \) and \( s \equiv 0 \pmod{3} \)

(iv) all paths \( P_k \) of length \( k \) with \( k \equiv 2 \pmod{3} \)
(v) All unicyclic graphs G such that the degree of every vertex on the unicycle of G is at least 4 and the degree of every vertex not on the cycle is one.

**Theorem 1.2.38**

For any graph G, $d(G) \geq 2$ if and only if G has no isolated vertices.

**Definition 1.2.39**

A set $s \subseteq V$ is a weak dominating set of G if for every $u \in V - s$, there exists a $v \in s$ such that $uv \in E$ and $\deg(u) \geq \deg(v)$. The weak domination number $\gamma_w(G)$ is the minimum cardinality of a weak dominating set of G.

**Theorem 1.2.40**

If G is a graph with n vertices and domination number $\gamma(G)$, where $2 \leq \gamma(G) \leq n$, then $m(G) \leq \left\lfloor \frac{1}{2} (n - \gamma(G)) (n - \gamma(G) + 2) \right\rfloor$. Equality occurs if and only if G is the disjoint union of a graph of $\gamma(G) - 2$ isolated vertices and a graph obtained by removing from $(n - \gamma(G) - 2)$ isolated vertices and a graph obtained by removing from $(n - \gamma(G) + 2)$ clique, the edges belonging to a minimum edge cover of G.

**Theorem 1.2.41**

If G is a graph with n vertices and domination number $\gamma(G)$, then $m(G) \leq \left\lfloor \frac{1}{2} (n - \gamma(G)) (n - \gamma(G) + 2) - \frac{1}{2} (n - \gamma(G) - \Delta) \right\rfloor$.

**Theorem 1.2.42**

If G is a graph with n vertices, domination number $\gamma(G)$ and $\Delta(G) \leq n - \gamma(G) - 1$, then $m(G) \leq \frac{1}{2} (n - \gamma(G) - 1) (n - \gamma(G))$ and this result is best possible.
**Definition 1.2.43**

Kneser Graph Let \( k, n \) be two positive integers, such that \( 2 < k < n \). Let \( M \) be a set with \( n \) elements. The Kneser graph \( K(n, k) \) is defined as the graph with vertex set \( V \) to be the set of all subsets of \( n \) of cardinality \( k \). Two vertices of \( K(n, k) \) are adjacent if and only if the corresponding sets are disjoint. This concept was introduced by Kneser in 1978. When \( n = 2k + 1 \) the Kneser graph is also called odd graph by Mulder.

**Definition 1.2.44**

Generalised Petersen Graph \( P(n, k) \) Let \( V(G) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\} \). Let \( E(G) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} \pmod{n}\} \)

**Definition 1.2.45**

Shadow graph of a graph Let \( G = (V, E) \) be simple graph. Let \( V(G) = \{v_1, v_2, ..., v_n\} \). Add vertices \( v'_1, v'_2, ..., v'_n \) to \( G \) and make \( v'_i \) adjacent to the neighbours of \( v_i \) in \( G \), \( 1 \leq i \leq n \). The resulting graph is called the shadow graph of \( G \). The order of the shadow graph is \( 2n \) and the size of the shadow graph is \( 2m \), where \( m \) is the size of \( G \).

**Definition 1.2.46**

Let \( G = (V, E) \) be a simple graph of order \( m \). A set \( S \subseteq V \) is said to be private dominating set of a graph \( G \) if it is a dominating set and for \( u \) in \( S \) there exists an external private neighbor \( v \in V - S \).
Definition 1.2.47

A dominating set $S$ is minimal dominating set if no proper subset of $S$ is a dominate set.

Figure 1.1 $\{2, 5, 8\}$ is a minimal dominating set

Definition 1.2.48

A graph $G(V, E)$ a set of vertices $S \subseteq V$ is independent if no two vertices in $S$ are adjacent.

Figure 1.2 $\{4, 9\}$ is a Independent set

Definition 1.2.49

A maximal independent set is an independent set that is not properly contained in any independent set.
Figure 1.3 \{1, 4, 9\} is a maximal Independent set

**Definition 1.2.50**

The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for $G$.

**Definition 1.2.51**

Let $d(G, i)$ be the number of dominating sets of size $i$ in a graph $G$, then the domination polynomial $D(G, x)$ of $G$ in the variable $x$ is defined as $D(G, x) = \sum_{i=\gamma(G)}^{d(G,i)} D(G,i)x^i$, where $\gamma(G)$ is the domination number of $G$.

**Lemma 1.2.52**

The following properties are hold by definition of combination

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \quad \text{(or)} \quad n^i = \frac{n!}{i!(n-i)!}$$

for all $n \in \mathbb{Z}^+$.

(i) $\binom{n}{n} = 1$
(ii) $\binom{n}{n-1} = n$
(iii) $\binom{n}{1} = n$

(iv) $\binom{n}{0} = 1$
(v) $\binom{n}{i} = 0$ if $i > n$.

We use $\binom{n}{i}$ or $nc_i$ for the combination $n$ to $i$ and we denote the set \{1, 2, ..., $n$\} simply by $[n]$
1.3 Thesis Origin

Most graph domination polynomial trace their origin to one introduced by S. Alikhani in 2009. S. Alikhani introduce a domination polynomial of a graph $G$. The domination polynomial of a graph $G$ of order $n$ is the polynomial of a graph $G$ of order $n$ is the polynomial

$$D(G, x) = \sum_{i=0}^{n} d(G, i) x^i,$$

where $d(G, i)$ is the number of dominating sets of $G$ of size $i$.

Saeid Alikhani (2010) developed the characterization of graphs using domination polynomials. In the domination polynomial, a root of $D(G, x)$ is called a domination root of $G$. The set of distinct domination roots by $Z(D, (G, x))$. Two graphs $G$ and $H$ are said to be D-equivalent if $D(G, x) = D(H, x)$. The O-equivalence class of $G$ is $[G] = \{H : H \sim G\}$. Also a graph $G$ is said to be D-unique if $[G] = \{G\}$. He find if a graph $G$ has two distinct domination roots then $Z(D(G, x)) = \{-2, 0\}$. If $G$ is a graph with no pendant vertex and has three distinct domination roots, then $Z(D(G, x)) \subseteq \{0, -2 \pm \sqrt{3}i, \frac{-3 \pm \sqrt{3}i}{2}\}$. Also, if $n \equiv 0, 2 \pmod{3}$, then $C_n$ is D-unique, and if $n \equiv 0 \pmod{3}$, then $[P_n]$ consists of exactly two graphs.


In 2010, S. Alikhani an P. Yee-hock, introduced dominating sets and domination polynomial of certain graphs II.
In 2013, S. Alikhani introduced on the domination polynomial of some graph operations. The domination polynomial of $G$ is the polynomial $D(G, \lambda) = \sum_{i=0}^{n} d(G, i) \lambda^i$, where $d(G, i)$ is the number of dominating sets of $G$ of size $i$. Every root of $D(G, \lambda)$ is called the domination root of $G$.

A. Vijayan, K. Lal Gipson (2013) developed the dominating sets and Domination polynomial of square of paths. They construct the polynomial $D(P_n^2, x) = \sum_{i=|x|^2}^{n} d(P_n^2, i) x^i$, where $D(P_n^2, x)$ denote the family of all dominating sets of $P_n^2$ with cardinality $i$.

Sahib Shayyal Kahat, Abdul Jalil M. Khalaf, and Roslan Hasni in 2014, introduced the domination sets and domination polynomial of stars. They construct recursive formula for $d(S_n, i)$. The polynomial is $D(S_n, x) = \sum_{i=1}^{n} d(S_n, i) x^i$.

Sahib Sh. Kahat, Abdul Jalil M. Khalaf in 2014 again developed the dominating sets and domination polynomial of wheels. Using recursive formula, they construct the polynomial $D(W_n, x) = \sum_{i=1}^{n} d(W_n, i) x^i$, where $D(W_n, x)$ denote the family of all dominating sets of $W_n$ with cardinality $i$.

B. Jaya Prasad, T. Tamizh Chelvam and S. Robinson Chelladurai (2013) introduced the private domination number of graph. A set $S \subseteq V$ is said to be private dominating set of a graph $G$ if it is a dominating set and for every $u$ in $S$ there exists an external private neighbor.
v \in V - S. The maximum cardinality of these sets is called the private domination number and is denoted by $\Gamma_{pv1}(G)$. Also they introduced private domination number for some graphs.

S. Alikhani, J.I. Brown and S. Jahari developed on the domination polynomial of friendship graphs in 2016. Let $F_n$ be the friendship graph with $2n + 1$ vertices and $3n$ edges, formed by the join of $k_1$ with $nk_2$. Also they developed domination polynomial of this family of graphs and in particular examine the domination roots of the family and find the limiting curve for the roots.

S.K. Vaidya and S.H. Karkar, developed on strong domination number of graphs in 2017. A subset $S$ of a vertex $V$ is called a dominating set of graph $G$ if every vertex of $V - S$ is dominated by some element of set $S$. If $e$ is an edge with end vertices $u$ and $v$ and degree of $u$ is greater than or equal to degree of $v$ then we say $u$ strongly dominates $v$. If every vertex of $V - S$ is strongly dominated by some vertex of $S$ then $S$ is called strong dominating set. The minimum cardinality of a strong dominating set is called the strong domination number of graph. Also they find strong domination number for some graphs.

1.4 Thesis Outline

This thesis aims at introducing three domination polynomial in graph theory. The thesis consists of six chapters motivated by the concepts of strong domination polynomial, weak domination polynomial and private domination polynomial of complete, cycle, wheel and star graphs.

Chapter 1 : The first chapter is introductory in nature, also it gives some basic definitions and terminologies and literature review which we needed for the subsequent chapters.
Chapter 2: In this chapter we collect the strong domination number, weak domination number and private domination number of various graphs. Using this various domination number we are calculating the domination polynomial in the subsequent chapters.

Chapter 3: The Third chapter consists of strong dominating sets and Strong domination polynomial of complete graphs, weak dominating sets and weak domination polynomial of complete graphs, Private dominating set and private domination polynomial of complete graphs.

Let Sd (K_m, j), Wd (K_m, j) and Pd (K_m, j) are the families of strong, weak and private dominating sets of complete graphs. Also, Let SD (K_m, x), WD (K_m, x) and PD (K_m, x) are the strong, weak and private domination polynomial of complete graphs.

Chapter 4: The Fourth chapter consists of strong domination polynomial of cycle graph. Weak domination polynomial and private domination polynomial of cycle graphs.

Let Sd (C_m, j), Wd (C_m, j) and Pd (C_m, j) are the families of strong, weak and private dominating sets of cycle graphs. Also, let SD (C_m, x), WD (C_m, x) and PD (C_m, x) are the strong, weak and private domination polynomial of cycle graphs. Also, we discussed about some properties of strong, weak and private domination polynomial of cycle graphs.

Chapter 5: The fifth chapter consists of strong domination polynomial, weak domination polynomial and private domination polynomial of wheel graphs.
Let $S_d(W_m, j)$, $W_d(W_m, j)$ and $P_d(W_m, j)$ are the families of strong, weak and private dominating sets of wheel graphs. Also, let $S_d(W_m, x)$, $W_d(W_m, x)$ and $P_d(W_m, x)$ are the strong, weak and private domination polynomial of wheel graphs. In this chapter, also we discuss about the some characteristics of polynomial.

**Chapter 6:** The sixth chapter, consist of strong domination polynomial, weak domination polynomial and private domination polynomial of star graphs.

Let $S_d(S_m, j)$, $W_d(S_m, j)$ and $P_d(S_m, j)$ are the families of strong, weak and private dominating sets and $S_d(S_m, x)$, $W_d(S_m, x)$ and $P_d(S_m, x)$ are the strong domination polynomial, weak domination polynomial and private domination polynomial of star graphs. Also we discuss about the some characteristics of polynomial.