CHAPTER 4

STRONG, WEAK AND PRIVATE DOMINATION POLYNOMIAL OF CYCLE GRAPHS

4.1. Introduction

The concept of domination polynomial of cycle graph was introduced by Sahib Shayyal Kahat, Abdul Jalil M Khalaf and Roslan Hasni (2009) and also they construct recursive formula for \( d(C_m, i) \) using this concept we introduced the strong domination polynomial of cycle graph, weak domination of polynomial of cycle graph and private domination polynomial of cycle graph. Using recursive relation, we find the strong, weak and private domination polynomial of cycle graphs.

In first section, we discuss about strong dominating set of cycle graph and strong domination of polynomial of cycle graph.

In second section, we discuss about weak dominating set of cycle graph and weak domination of polynomial of cycle graph.

In third, we discuss about private dominating set of cycle graph and private domination of polynomial of cycle graph.

4.2. Strong Domination sets of Cycle graphs

Let \( C_m \), \( m \geq 3 \) be the cycle graph with \( m \) vertices, \( V [C_m] = [m] \) and \( E (C_m) = \{(v, u) : \forall u, v \in V (C_m)\} \). Let \( C^j_m \) be the family of strong dominating sets of \( C_m \) with the number of elements \( j \). We shall explore the strong dominating sets of cycle graphs. To prove our main results we need the following Lemmas.
Lemma 4.2.1

The following properties hold for cycles

(i) \( \gamma_{sd}(C_m) = \left\lfloor \frac{m}{3} \right\rfloor \)

(ii) \( C^i_j = \phi \) if and only if \( j > 1 \) or \( j < \left\lfloor \frac{i}{3} \right\rfloor \)

(iii) If a graph contains a simple path of length \( 3m - 1 \), the every strong dominating set of graph must contain at least \( m \) vertices of the path.

Lemma 4.2.2

(i) \( C^{i-1}_{m-1} = C^{j-1}_{m-3} = \phi \Rightarrow C^{j-1}_{m-2} = \phi \)

(ii) \( C^{i-1}_{m-1} = C^{i-1}_{m-3} = \phi \Rightarrow C^{i-1}_{m-2} \neq \phi \)

(iii) \( C^{j-1}_{m-1} = C^{j-1}_{m-2} = C^{j-1}_{m-3} \neq \phi \Rightarrow C^j_m = \phi \).

Lemma 4.2.3

Suppose \( C^j_m \neq \phi \), then we have

(i) \( C^{i-1}_{m-1} = C^{j-1}_{m-2} = \phi \) and \( C^{j-1}_{m-3} \neq \phi \) if and only if \( m = 3n \) and \( j = n \) for some \( n \in \mathbb{N} \).

(ii) \( C^{j-1}_{m-2} = C^{j-1}_{m-3} = \phi \) and \( C^{j-1}_{m-1} \neq \phi \) if and only if \( j = m \).

(iii) \( C^{j-1}_{m-1} = \phi \) and \( C^{j-1}_{m-2} = C^{j-1}_{m-3} \neq \phi \) if and only if \( m = 3n+2 \) and \( j = \left\lfloor \frac{3n+2}{3} \right\rfloor \) for some \( n \in \mathbb{N} \).

(iv) \( C^{j-1}_{m-1} = C^{j-1}_{m-2} \neq \phi \) and \( C^{j-1}_{m-3} \neq \phi \) if and only if \( j = m - 1 \).

(v) \( C^{j-1}_{m-1} = C^{j-1}_{m-2} = C^{j-1}_{m-3} \neq \phi \) if and only if \( \left\lfloor \frac{m-1}{3} \right\rfloor + 1 \leq j \leq m - 2 \).
Proof

(i) (⇒) Since $C_{m-1}^{j-1} = C_{m-2}^{j-1} = \phi$, by Lemma 4.2.1, we have $j - 1 > m - 1$ or $j - 1 < \left\lceil \frac{m - 2}{3} \right\rceil$. If $j - 1 > m - 1$, then $j > m$, and by Lemma 4.2.1, $C_m^j = \phi$, a contradiction. So we have $j - 1 < \left\lceil \frac{m - 2}{3} \right\rceil$, then $j < \left\lceil \frac{m - 2}{3} \right\rceil + 1$, we have $C_m^j \neq \phi$, together we have $\frac{m}{3} \leq j < \frac{m - 2}{3} + 1$, gives us $m = 3n$ and $j = n$ for some $n \in \mathbb{N}$.

(⇐) If $m = 3n$ and $j = n$ for some $n \in \mathbb{N}$, then by Lemma 4.2.1, we have $C_{m-1}^{j-1} = C_{m-2}^{j-1} = \phi$ and $C_{m-3}^{j-1} \neq \phi$.

(ii) (⇒) Since $C_{m-2}^{j-1} = C_{m-3}^{j-1} = \phi$, using the Lemma 4.2.1, we have $j - 1 > m - 2$ or $j - 1 < \left\lceil \frac{m - 3}{3} \right\rceil$. If $j - 1 < \left\lceil \frac{m - 3}{3} \right\rceil$, then $j - 1 < \left\lceil \frac{m - 1}{3} \right\rceil$, and hence $C_{m-1}^{j-1} = \phi$, a contradiction. So we have $j - 1 > m - 2$, that is $j > m - 1$. Also since $C_{m-1}^{j-1} \neq \phi$, we have $j - 1 \leq m - 1$, Therefore, we have $j = m$.

(⇐) If $j = m$, then by Lemma 4.2.1, we have $C_{m-2}^{j-1} = C_{m-3}^{j-1} = \phi$ and $C_{m-1}^{j-1} \neq \phi$.

(iii) (⇒) Since $C_{m-1}^{j-1} = \phi$, using the Lemma 4.2.1, we have $j - 1 > m - 1$ or $j - 1 < \left\lceil \frac{m - 1}{3} \right\rceil$. If $j - 1 > m - 1$, then $j - 1 > m - 2$ and using Lemma 4.2.1, we have $C_{m-2}^{j-1} = C_{m-3}^{j-1} = \phi$, a contradiction. So we must have $j < \left\lceil \frac{m - 1}{3} \right\rceil + 1$. But we have $j - 1 \geq \left\lceil \frac{m - 2}{3} \right\rceil$, since $C_{m-2}^{j-1} \neq \phi$. Hence we have $\frac{m - 2}{3} + 1 \leq j < \frac{m - 1}{3} + 1$. Therefore, $m = 3n + 2$ and $j = n + 1 = \left\lceil \frac{3n + 2}{3} \right\rceil$, for some $n \in \mathbb{N}$.
(⇐) If \( m = 3n+2 \) and \( j = \left\lceil \frac{3n+2}{3} \right\rceil \) for some \( n \in \mathbb{N} \), then using Lemma 4.2.1, we get
\[
C_{m-1}^{j-1} = C_{3n+1}^{n} = \phi, \quad C_{m-2}^{j-1} = C_{m-3}^{j-1} \neq \phi.
\]

(iv) (⇒) Since \( C_{m-3}^{j-1} = \phi \), using Lemma 4.2.1, we have \( j - 1 > m - 3 \) or \( j - 1 < \left\lceil \frac{m-3}{3} \right\rceil \). Since \( C_{m-2}^{j-1} \neq \phi \), by Lemma 4.2.1, we have \( \left\lceil \frac{m-2}{3} \right\rceil + 1 \leq j < m - 1 \). Therefore \( j - 1 < \left\lceil \frac{m-3}{3} \right\rceil \) is not possible. Hence we must have \( j - 1 > m - 3 \). Thus \( j = m - 1 \) or \( m \). But \( j \neq m \) because \( C_{m-2}^{j-1} \neq 0 \), So \( j = m - 1 \).

(⇐) If \( j = m - 1 \), then by Lemma 4.2.1, we have \( C_{m-1}^{j-1} \neq \phi, C_{m-2}^{j-1} \neq \phi \) and \( C_{m-3}^{j-1} = \phi \).

(v) (⇒) Since \( C_{m-1}^{j-1} = C_{m-2}^{j-1} \neq \phi \) and \( C_{m-3}^{j-1} = \phi \) then using Lemma 4.2.1, we have \( \left\lceil \frac{m-3}{3} \right\rceil \leq j - 1 \leq m - 2 \), and \( \left\lceil \frac{m-2}{3} \right\rceil \leq j - 1 \leq m - 3 \). So \( \left\lceil \frac{m-3}{3} \right\rceil \leq j - 1 \leq m - 3 \) and hence \( \left\lceil \frac{m-1}{3} \right\rceil + 1 \leq j \leq m - 2 \).

(⇐) If \( \left\lceil \frac{m-1}{3} \right\rceil + 1 \leq j \leq m - 2 \), then using the Lemma 4.2.1, we have
\[
C_{m-1}^{j-1} = C_{m-2}^{j-1} = C_{m-3}^{j-1} \neq \phi.
\]

**Theorem 4.2.4**

For every \( m \geq 4 \), and \( j \geq \left\lceil \frac{m}{3} \right\rceil \).

(i) If \( C_{m-2}^{j-1} = C_{m-3}^{j-1} = \phi \) and \( C_{m-1}^{j-1} \neq \phi \), then \( C_{m}^{j} = \{[m]\} \).

(ii) If \( C_{m-1}^{j-1} = C_{m-2}^{j-1} = \phi \) and \( C_{m-3}^{j-1} \neq \phi \), then \( C_{m}^{j} = C_{m}^{m/3} = \{(1,4,...m-2), (2,5,...m-1), (3,6,...m)\} \).

(iii) If \( C_{m-1}^{j-1} = \phi \), \( C_{m-2}^{j-1} = C_{m-3}^{j-1} \neq \phi \), then
\[
C_{m}^{j} = \{(1,4,...m-4,m-1), (2,5,...m-3,m), (3,6,...m-2,m)\} \cup R
\]

Where
\[
R = \begin{cases} \{m-2\}, & \text{if } 1 \in \mathbb{P} \\ \{m-1\}, & \text{if } 1 \notin \mathbb{P}, 2 \in \mathbb{P} / \mathbb{P} \in C_{m-3}^{j-1} \\ \{m\}, & \text{otherwise} \end{cases}
\]

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(iv) If $C_{m-1}^{j-1} = \phi$, $C_{m-2}^{j-1} = C_{m-1}^{j-1} \neq \phi$, then $C_{m}^{j} = C_{m}^{m-1} = \{[m] \setminus \{p\} : P \in [m]\}$

(v) If $C_{m-1}^{j-1} = C_{m-2}^{j-1} = C_{m-3}^{j-1} \neq \phi$, then

$C_{m}^{j} = \{[m] \cup P : P \in C_{m-1}^{j-1}\} \cup R_{1} \cup R_{2}$, where

\[
R_{1} = \begin{cases} 
\{m\}, & \text{if } m - 2 \text{ or } m - 3 \in P_{1} \text{ for } P_{1} \in C_{m-2}^{j-1} / C_{m-1}^{j-1} \\
\{m-1\}, & \text{if } m - 2 \text{ or } P_{1}, m - 3 \notin P_{1} \text{ or } P_{1} \in C_{m-1}^{j-1} \cap C_{m-2}^{j-1} 
\end{cases}
\]

\[
R_{2} = \begin{cases} 
\{m-2\}, & \text{if } l \in P_{2}, \text{ for } P_{2} \in C_{m-3}^{j-1} \\
\{m-1\}, & \text{if } m - 3 \in P_{2} \text{ or } m - 4 \in P_{2}, \text{ for } P_{2} \in C_{m-3}^{j-1} / C_{m-2}^{j-1} 
\end{cases}
\]

Proof :

(i) $C_{m-2}^{j-1} = C_{m-3}^{j-1} = \phi$ and $C_{m-1}^{j-1} \neq \phi$, using Lemma 4.2.3, we get $j = m$.

Therefore $C_{m}^{j} = C_{m}^{m} = \{[m]\}$.

(ii) $C_{m-1}^{j-1} = C_{m-2}^{j-1} = \phi$ and $C_{m-3}^{j-1} \neq \phi$, using Lemma 4.2.3, we get $m = 3n$, $j = n$, for some $n \in \mathbb{N}$. Therefore, $C_{m}^{j} = C_{m}^{m/3} = \{(1, 4, \ldots 3n-2), (2, 5, \ldots 3n-1), (3, 6, 9, \ldots m)\}$.

(iii) $C_{m-1}^{j-1} = \phi$, $C_{m-2}^{j-1} = C_{m-3}^{j-1} \neq \phi$, using Lemma 4.2.3, we get $m = 3n + 2$, $j = n + 1$, for some $n \in \mathbb{N}$. We denote the families \{(1, 4, 3n-2, 3n+1), (2, 5, 3n-1, 3n+2), (3, 6, \ldots 3n, 3n+2)\} and \(P \cup \{3n, \text{ if } 1 \in P, 3n+1, \text{ if } 1 \notin P, 2 \in P / P \in C_{3n-1}^{n}\}\) by $Q_{1}$ and $Q_{2}$ respectively. We shall prove that $C_{3n-1}^{n+1} = Q_{1} \cup Q_{2}$. Since $C_{3n}^{n} = \{(1, 4, \ldots 3n-2), (2, 5, \ldots 3n-1), (3, 6, \ldots 3n)\}$, then $Q_{1} \subseteq C_{3n+2}^{n+1}$.

Also it is obvious that $Q_{2} \subseteq C_{3n+2}^{n+1}$. Therefore $Q_{1} \cup Q_{2} \subseteq C_{3n+2}^{n+1}$. Now let $Q \in C_{3n+2}^{n+1}$, then by Lemma 4.2.1, atleast one of the vertices labelled $3n+2$, $3n+1$ or $3n$ is in $Q$. Suppose that $3n+2 \in Q$, then by Lemma 4.2.1, atleast one of the vertices marked 1, 2 or 3 and $3n+1$, $3n$ or $3n-1$ are in $Q$. If $3n+1$ and atleast one of \{1, 2, 3\} and also $3n$ and atleast one of \{1, 2\} are in $Q$
then \( Q - \{3n+2\} \in C_{3n+1} \), a contradiction. If \( \{3, 3n\} \) or \( \{2, 3n-1\} \) is a subset of \( Q \), then \( Q = P \cup \{3n+2\} \) for some \( P \in C^n \). Hence \( Q \in Q_2 \). If \( \{1, 3n-1\} \) is a subset of \( Q \), then \( Q = P \cup \{3n\} \in C_{3n} \). Hence \( Q \in Q_1 \) if \( 3n+1 \) or \( 3n \) is in \( Q \), we also have the result by applying the same procedure.

(iv) Using Lemma 4.2.3, \( j = m - 1 \), Therefore, \( C_{m}^{j} = C_{m}^{m+1} = \{[m] - \{p\} / P \in [m]\} \).

(v) \( C_{m-1}^{j-1} = C_{m-2}^{j-1} = C_{m-3}^{j-1} \neq \emptyset \). Suppose that \( P \in C_{m-1}^{j-1} \), then \( P \cup \{m\} \in C_m^j \). So \( Q_1 = \{\{m\} \cup P / P \in C_{m-1}^{j-1}\} \subseteq C_m^j \). Now suppose that \( C_{m-2}^{j-1} \neq \emptyset \). Let \( P_1 \in C_{m-2}^{j-1} \).

We denote \( P_1 \cup \left\{ \begin{array}{l}
\text{m, if } m - 2 \text{ or } m - 3 \in P_1, P_1 \in C_{m-2}^{j-1} | C_{m-1}^{j-1} \\
m - 1, \text{if } m - 2 \notin P_1, m - 3 \notin P_1 \text{ or } P_1 \in C_{m-1}^{j-1} \cup C_{m-2}^{j-1}
\end{array} \right. \).

Simply by \( Q_2 \). By Lemma 4.2.1, atleast one of the vertices marked \( m - 3, m - 2 \) or \( 1 \) is in \( P_1 \). If \( m - 2 \) or \( m - 3 \) is in \( P_1 \), then \( P_1 \cup \{m\} \in C_m^j \), otherwise \( P_1 \cup \{m-1\} \in C_m^j \). Hence \( Q_2 \subseteq C_m^j \). Here we shall consider \( C_{m-3}^{j-1} \neq \emptyset \). Let \( P_2 \in C_{m-3}^{j-1} \).

We denote \( P_2 \cup \left\{ \begin{array}{l}
m - 2, \text{if } 1 \in P_2, P_2 \in C_{m-3}^{j-1} \text{ or } P_2 \in C_{m-3}^{j-1} \cap C_{m-2}^{j-1} \\
m - 1, \text{if } m - 3 \in P_2 \text{ or } m - 4 \in P_2, P_2 \in C_{m-3}^{j-1} | C_{m-2}^{j-1}
\end{array} \right. \).

Simply by \( Q_3 \). If \( m - 3 \) or \( m - 4 \) is in \( P \), then \( P \cup \{m-1\} \in C_m^j \), otherwise \( P_2 \cup \{m-2\} \in C_m^j \). Hence \( Q_3 \subseteq Q \). Therefore we have \( Q_1 \cup Q_2 \cup Q_3 \subseteq C_m^j \). Now suppose that \( Q \in C_m^j \), so by Lemma 4.2.1, \( y \) contain atleast one of the vertices marked \( m, m - 1 \) or \( m - 2 \). If \( m \in Q \), using Lemma 4.2.1 atleast one of the vertices marked \( m - 1, m - 2 \) or \( m - 3 \) is in \( Q \), then \( Q = P \cup \{m\} \), for some \( P \in C_{m-2}^{j-1} \). Hence \( Q \in Q_2 \). Otherwise \( Q = P \cup \{m-1\} \) for some \( P \in C_{m-2}^{j-1} \). Hence \( Q \in Q_2 \). If \( m - 1 \) or \( m - 2 \) is in \( Q \), using the same procedure we get the result.
Theorem 4.2.5

If \( C^j_m \) is the family of strong dominating set of \( C_m \) with the number of the elements in the set \( j \), then \( \text{Sd} (C_m, j) = \text{Sd} (C_{m-1}, j - 1) + \text{Sd} (C_{m-2}, j - 1) + \text{Sd} (C_{m-3}, j - 1) \).

Proof

We can prove this theorem by using Theorem 4.2.4 and we rewrite Theorem 4.2.4. in the following form.

(i) If \( C^{j-1}_{m-2} = C^{j-1}_{m-3} = \phi \) and \( C^{j-1}_m \neq \phi \), then \( C^j_m = \left\{ (m) \cup P \mid P \in C^{j-1}_{m-1} \right\} \)

(ii) If \( C^{j-1}_{m-1} = C^{j-1}_{m-2} = \phi \) and \( C^{j-1}_m \neq \phi \), then

\[
C^j_m = \left\{ (n-2) \cup P_1, (n-1) \cup P_2, (n) \cup P_3 \mid 1 \in P_1, 2 \in P_2, 3 \in P_3, P_1, P_2, P_3 \in C^{j-1}_{m-3} \right\}
\]

(iii) If \( C^{j-1}_{m-1} = \phi \), \( C^{j-1}_{m-2} \neq \phi \) and \( C^{j-1}_m \neq \phi \), then

\[
C^j_m = \left\{ (m) \cup P_1, (m-1) \cup P_2 \mid P_1, P_2 \in C^{j-1}_{m-2}, 1 \in P_2 \right\} \cup \left\{ m, \text{ otherwise} \right\}
\]

where \( P \in C^{j-1}_{m-3} \).

(iv) If \( C^{j-1}_{m-3} = \phi \) and \( C^{j-1}_{m-2} = C^{j-1}_m \neq \phi \), then

\[
C^j_m = \left\{ (m) \cup P_1, \{m-1\} \cup P_2 \mid P_1, P_2 \in C^{j-1}_{m-1}, P_2 \in C^{j-1}_{m-2} \right\}
\]

(v) If \( C^{j-1}_{m-1} = C^{j-1}_{m-2} = C^{j-1}_m \neq \phi \), then

\[
C^j_m = \left\{ (m) \cup P \mid P \in C^{j-1}_{m-1} \right\} \cup \left\{ P_1 \cup \left\{ m, \text{ if } m-2 \in P_1, P_1 \in C^{j-1}_{m-2}, C^{j-1}_{m-1} \right\} \right\}
\]

where \( P_1 \in C^{j-1}_{m-2} \) and \( P_2 \in C^{j-1}_{m-3} \), \( C^{j-1}_{m-2} \cap C^{j-1}_{m-1} \).
By above construction, in every cases, we have \( S_d \( (C_m, j) = S_d \( (C_{m-1}, j - 1) + S_d \( (C_{m-2}, j - 1) + S_d \( (C_{m-3}, j - 1). \)

4.3. Strong domination polynomial of Cycle graphs

In this section, we introduce and establish the strong domination polynomial of cycle graphs.

Let \( C^j_m \) be the family of strong dominating sets of a cycle \( C_m \) with the number of elements in the set \( j \), and let \( S_d \( (C_m, j) = |C^j_m| \) and since \( \gamma_{Sd} \( (C_m) = \left\lfloor \frac{m}{3} \right\rfloor \). Then the strong domination polynomial \( SD \( (C_m, x) \) of \( C_m \) is defined as \( SD \( (C_m, x) = \sum_{j=\gamma_{Sd}(C_m)}^{m} S_d \( (C_m, j) \cdot x^j. \)

**Theorem 4.3.1**

For every \( m \geq 4 \) and \( \left\lfloor \frac{m}{3} \right\rfloor \leq j \leq m, m \in \mathbb{N} \), \( S_d \( (C_m, j) \) is the coefficient of \( x^m \cdot y^j \) in the expansion of the function \( g(x, y) = \frac{x^4 \cdot y^2 \cdot (6 + 4y + y^2 + 5x + 4xy + xy^2 + 3x^2 + 3x^2y + x^2y^2)}{1 - xy - x^2y - x^3y} \)

**Proof**

Set \( g(x, y) = \sum_{m=4}^{\infty} \sum_{j=2}^{\infty} x^m \cdot y^j \). By receive formula for \( S_d \( (C_m, j) \), using theorem 4.3.2 we can write \( g(x, y) \) in the following form

\[
g(x, y) = \sum_{m=4}^{\infty} \sum_{j=2}^{\infty} \left( \binom{m-1}{j-1} + \binom{m-2}{j-1} + \binom{m-3}{j-1} \right) x^m \cdot y^j
\]

\[
= xy \sum_{m=4}^{\infty} \sum_{j=2}^{\infty} \binom{m-1}{j-1} x^{m-1} \cdot y^{j-1} + x^2 y \sum_{m=4}^{\infty} \sum_{j=2}^{\infty} \binom{m-2}{j-1} x^{m-2} \cdot y^{j-1} +
\]

\[
x^3 y \sum_{m=4}^{\infty} \sum_{j=2}^{\infty} \binom{m-3}{j-1} x^{m-3} \cdot y^{j-1}
\]
\[ g(x, y) = xy \left[ \frac{3}{1} x^3 y + \frac{3}{2} x^3 y^2 + \frac{3}{3} x^3 y^3 \right] + xy g(x, y) + x^2 y \left[ \frac{2}{1} x^2 y + \frac{2}{2} x^2 y^2 + \frac{1}{1} x^3 y + \frac{3}{2} x^3 y^2 + \frac{3}{3} x^3 y^3 \right] + x^2 y g(x, y) + \]
\[ x^1 y \left[ \frac{1}{1} x y + \frac{2}{2} x^2 y^2 + \frac{1}{1} x^3 y + \frac{3}{2} x^3 y^2 + \frac{3}{3} x^3 y^3 \right] + x^3 y g(x, y) + \]
\[ g(x, y) - xy g(x, y) - x^2 y g(x, y) - x^3 y g(x, y) = xy (3x^3 y + 3x^3 y^2 + x^3 y^3) + x^2 y (2x^2 y + x^3 y + 3x^3 y^2 + x^3 y^3) \]
\[ g(x, y) [1 - xy - x^2 y - x^3 y] = x^4 y^2 (6 + 4y + y^2 + 5x + 4xy + x^2 y + 3x^2 y + 3y^2 + x^2 y^2). \]

Therefore, \[ g(x, y) = \frac{x4y^2 (6+4y+y^2+5x+4xy+x^2 y+3x^2 y+x^2 y^2)}{1-xy-x^2 y-x^3 y} \]

Using Theorem 4.2.4, we obtain the coefficient of SD (m, x) for 1 \( \leq m \leq 20 \) in Table 4.1.

Let SD (C_m, j) = |C_m^j|. There are interesting relationships between the numbers SD (C_m, j), \( (1 \leq j \leq 20) \) in the table.
Table 4.1 SD table

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Theorem 4.3.2

The following properties satisfied for SD $(C_m, x)$:

1. For every $m \geq 4$, $j \geq \left\lceil \frac{m}{3} \right\rceil$, $S_d(C_m, j) = \binom{m-1}{j-1} + \binom{m-2}{j-1} + \binom{m-3}{j-1}$

2. $S_d(C_m, m) = 1$, for every $m \geq 3$

3. $S_d(C_{3m}, m) = 3$, for every $m \in \mathbb{N}$

4. $S_d(C_{3m+2}, m+1) = 3m + 2$, for every $m \in \mathbb{N}$

5. $S_d(C_{3m+1}, m+1) = \frac{(3m+1)(m+2)}{2}$, for every $m \in \mathbb{N}$

6. $S_d(C_m, m-1) = m$, for every $m \geq 3$

7. $S_d(C_m, m-2) = \frac{m(m-1)}{2}$, for every $m \geq 3$

8. For every $m \geq 3$, $1 = S_d(C_m, m) < S_d(C_{m+1}, m) < S_d(C_{m+2}, m) < \ldots < (S_d(C_{2m-1}, m) < S_d(C_{2m}, m))$

9. $S_d(C_{2m+1}, m) > \ldots > S_d(C_{3m-1}, m) > S_d(C_m, m) = 3$

10. If $S_m = \sum_{j=1}^{m} S_d(C_m, j)$, then for every $m \geq 4$, $S_m = S_{m-1} + S_{m-2} + S_{m-3}$ combining with initial values $S_1 = 1$, $S_2 = 3$ and $S_3 = 7$.

Proof

1. By using the Theorem 4.2.4, we get the result $S_d(C_m, j) = \binom{m-1}{j-1} + \binom{m-2}{j-1} + \binom{m-3}{j-1}$.

2. Since, for any $m$ vertices graph $G$, $S_d(G, m) = 1$, then we get $S_d(C_m, m) = 1$.

3. Since $C_m^m = \{(1, 4, 7, \ldots 3m-2), (2, 5, \ldots 3m-1), (3, 6, 9, \ldots 3m)\}$. So, $S_d(C_{3m}, m) = 3$. 
4. We prove this by induction method. The result is true for \( m = 1 \), because \( C_5^3 = \{(1, 3), (1, 4), (2, 4), (2, 5), (3, 5)\} \). Now we assume the result is true for less than \( m \), \( m \in \mathbb{N} \). Now prove it for \( m \). By induction hypothesis, we have
\[
\text{Sd} (C_{3m+2}, m + 1) = \text{Sd} (C_{m+1}, m) + \text{Sd} (C_{3m}, m) + \text{Sd} (C_{3m-1}, m) = 3m + 2.
\]

5. We prove this by induction method. The result is true for \( m = 1 \), since \( C_4^3 = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\} \), So \( \text{Sd} (C_4, 2) = 6 \). Now we assume the result is true for less than \( m \), \( m \in \mathbb{N} \). now prove it for \( m \). By induction hypothesis, we get
\[
\text{Sd} (C_{3m+1}, m + 1) = \text{Sd} (C_{3m}, m) + \text{Sd} (C_{3m-1}, m) + \text{Sd} (C_{3m-2}, m)
\]
\[
= 3 + \text{Sd} (C_{3(m-1)+2}, m) + \text{Sd} (C_{3(m-1)+1}, m)
\]
\[
= 3 + 3(m - 1) + 2 + \frac{(m-1)[3(m-1)+7]+2}{2}
\]
\[
= \frac{3m^2 + 7m + 2}{2}
\]
\[
= \frac{(3m+1) (m+2)}{2}
\]

6. We know that for any graph having \( m \) vertices, \( \text{Sd} (G, m - 1) = m \), hence we have \( \text{Sd} (C_m, m - 1) = m \).

7. We prove this by induction method. The result is true for \( m = 1 \), since \( \text{Sd} (C_3, 1) = 3 \).
Now we assume the result is true for less than \( m \), \( m \in \mathbb{N} \). Now prove it for \( m \). By induction hypothesis, we get
\[
\text{Sd} (C_m, m - 2) = \text{Sd} (C_{m-1}, m - 3) + \text{Sd} (C_{m-2}, m - 3) + \text{Sd} (C_{m-3}, m - 3)
\]
\[
= \frac{(m-2)(m-1)}{2} + m - 2 + 1
\]
\[
= \frac{m(m-1)}{2}
\]

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We prove that $S_d(C_j, m) < S_d(C_{j+1}, m)$, for every $m \leq j \leq 2m - 1$ and $S_d(C_j, m) > S_d(C_{j+1}, m)$ for, $2m \leq j \leq 3m - 1$. We prove this by induction on $m$. The result is true for $m = 3$. Now assume the result is true for less than $m$. Now prove it for $m = k + 1$, that is $S_d(C_j, k + 1) < S_d(C_{j+1}, k + 1)$, for $k + 1 < j < 2k + 1$ using induction hypothesis, we have

$$S_d(C_j, k + 1) = S_d(C_{j-1}, k) + S_d(C_{j-2}, k) + S_d(C_{j-3}, k)$$

Similarly, we can prove the other inequality.

9. $S_m = \sum_{j=\lfloor \frac{m}{4} \rfloor}^{m} S_d(C_m, j)$

\[
= \sum_{j=\lfloor \frac{m}{4} \rfloor}^{m} \left[ \begin{array}{c} m-1 \\ j-1 \end{array} \right] + \begin{array}{c} m-2 \\ j-1 \end{array} + \begin{array}{c} m-3 \\ j-1 \end{array}
\]

\[
= \sum_{j=\lfloor \frac{m}{4} \rfloor}^{m-1} (m-1)j + \sum_{j=\lfloor \frac{m}{4} \rfloor}^{m-2} (m-2)j + \sum_{j=\lfloor \frac{m}{4} \rfloor}^{m-3} (m-3)j
\]

\[
= S_{m-1} + S_{m-2} + S_{m-3}.
\]

10. $S_d(C_{m+2}, m) = \begin{array}{c} m+1 \\ m-1 \end{array} + \begin{array}{c} m \\ m-1 \end{array} + \begin{array}{c} m-1 \\ m-1 \end{array}$

\[
= \frac{(m+1)!}{(m-1)!2!} + \frac{m!}{(m-1)!1!} + 1
\]

\[
= \frac{m^2 + 3m + 2}{2}
\]

\[
= \frac{(m+2)(m+1)}{2}.
\]
Theorem 4.3.3

For every \( m > 4 \), \( \text{SD}(m, x) = x \left[ \text{SD}(C_{m-1}, x) + \text{SD}(C_{m-2}, x) + \text{SD}(C_{m-3}, x) \right] \) with the initial values \( \text{SD}(C_1, x) = x \), \( \text{SD}(C_2, x) = x^2 + 2x \), \( \text{SD}(C_3, x) = x^3 + 3x^2 + 3x \).

Proof:

We prove this theorem by using Theorem 4.2.4 and using the definition of the strong domination polynomial.

Example 4.3.4

Let \( C_7 \) be cycle with order 7, then by Theorem 4.3.2, we have

\[
\text{SD}(C_7, x) = \sum_{j=\lfloor \frac{m}{4} \rfloor}^{m} \left( \binom{m-1}{j-1} + \binom{m-2}{j-1} + \binom{m-3}{j-1} \right) x^j = 14x^3 + 28x^4 + 21x^5 + 7x^6 + x^7.
\]

4.4. Weak Domination sets of Cycle graphs

Let \( C_m, m \geq 3 \) be the cycle graph with \( m \) vertices, \( V(C_m) = [m] \) and \( E(C_m) = \{(v, u) : \forall u, v \in V(C_m)\} \). Let \( C_m^j \) be the family of weak dominating sets of \( C_m \) with the number of
elements $j$. We shall explore the weak dominating sets of cycle graphs. To prove our main results we need the Lemma 4.2.1, 4.2.2 and 4.2.3.

**Theorem 4.4.1**

If $C^j_m$ is the family of weak dominating set of $C_m$ with the number of the elements in the set $j$, then $\text{Wd} (C_m, j) = \text{Wd} (C_{m-1}, j-1) + \text{Wd} (C_{m-2}, j-1 + \text{Wd} (C_{m-3}, j-1)$.

**Proof**

We can prove this theorem by using Theorem 4.2.4 and we rewrite Theorem 4.2.4 in the following form.

(i) If $C^{j-1}_{m-2} = C^{j-1}_{m-3} = \emptyset$ and $C^{j-1}_{m-3} \neq \emptyset$, then $C^j_m = \{\{m\} \cup P / P \in C^{j-1}_{m-3} \}$

(ii) If $C^{j-1}_{m-1} = C^{j-1}_{m-2} = \emptyset$ and $C^{j-1}_{m-3} \neq \emptyset$, then

$$C^j_m = \{\{m-2\} \cup P_2, \{m-1\} \cup P_2, \{m\} \cup P_3 / 1 \in P_2, 2 \in P_2, 3 \in P_3, P_2, P_3 \in C^{j-1}_{m-3} \}$$

(iii) If $C^{j-1}_{m-1} = \emptyset$, $C^{j-1}_{m-2} \neq \emptyset$ and $C^{j-1}_{m-3} \neq \emptyset$, then

$$C^j_m = \{\{m\} \cup P_1, \{m-1\} \cup P_2 | P_1, P_2 \in C^{j-1}_{m-2}, 1 \in P_2 \} \cup \left\{P \cup \begin{cases} m-2, & \text{if } 1 \in P \\ m-1, & \text{if } 1 \not\in P, 2 \in P \\ m, & \text{otherwise} \end{cases} \right\}, \text{where } P \in C^{j-1}_{m-3}$$

(iv) If $C^{j-1}_{m-3} = \emptyset$ and $C^{j-1}_{m-2} = C^{j-1}_{m-1} \neq \emptyset$, then

$$C^j_m = \{\{m\} \cup P_1, \{m-1\} \cup P_2 | P_1 \in C^{j-1}_{m-1}, P_2 \in C^{j-1}_{m-2} \}$$

(v) If $C^{j-1}_{m-1} = C^{j-1}_{m-2} = C^{j-1}_{m-3} \neq \emptyset$, then

$$C^j_m = \{\{m\} \cup P | P \in C^{j-1}_{m-1} \} \cup \left\{P_1 \cup \begin{cases} m, & \text{if } m-2 \text{ or } m-3 \in P_1, P_1 \in C^{j-1}_{m-2} | C^{j-1}_{m-1} \\ m-1, & \text{if } m-2 \not\in P_1, m-3 \not\in P_1 \text{ or } P_1 \in C^{j-1}_{m-1} \cap C^{j-1}_{m-2} \end{cases} \right\}$$
U \left\{ P_2 \cup \begin{cases} \{m-2\}, & \text{if } 1 \in P_2, \text{for } P_2 \in C_{m-3}^{j-1} \text{ or } P_2 \in C_{m-3}^{j-1} \cap C_{m-2}^{j-1} \\ \{m-1\}, & \text{if } m-3 \in P_2 \text{ or } m-4 \in P_2, \text{ } P_2 \in C_{m-3}^{j-1} \cap C_{m-2}^{j-1} \end{cases} \right\}

where $P_1 \in C_{m-2}^{j-1} | C_{m-1}^{j-1}$ and $P_2 \in C_{m-3}^{j-1} | C_{m-2}^{j-1} \cap C_{m-1}^{j-1}$.

By above construction, in every cases, we have $Wd(C_m, j) = Wd(C_{m-1}, j - 1) + Wd(C_{m-2}, j - 1) + Wd(C_{m-3}, j - 1)$.

### 4.5. Weak domination polynomial of Cycle graphs

In this section, we introduce and establish the weak domination polynomial of cycle graphs.

Let $C_m^j$ be the family of weak dominating sets of a cycle $C_m$ with the elements in the set $j$, and let $Wd(C_m, j) = |C_m^j|$ and since $\gamma_{Wd}(C_m) = \left[ \frac{m}{3} \right]$. Then the weak domination polynomial $WD(C_m, x)$ of $C_m$ is defined as $WD(C_m, x) = \sum_{j=\gamma_{Wd}(C_m)}^{m} Wd(C_m, j) x^j$.

**Theorem 4.5.1**

For every $m \geq 4$ and $\left[ \frac{m}{3} \right] \leq j \leq m$, $m \in N$, $Wd(C_m, j)$ is the coefficient of $x^m y^j$ in the expansion of the function

$$g(x, y) = \frac{x^4 y^2 (6 + 4y + y^2 + 5x + 4xy + xy^2 + 3x^2 + 3x^2y + x^2y^2)}{1 - xy - x^2y - x^3y}$$

**Proof**

Set $g(x, y) = \sum_{m=4}^{\infty} \sum_{j=2}^{m} x^m y^j$. By receive formula for $Wd(C_m, j)$, using theorem 4.4.1 we can write $g(x, y)$ in the following form
\[ g(x, y) = \sum_{m=4}^{\infty} \sum_{j=2}^{m} \left( \frac{m-1}{j-1} \right) \left( \frac{m-2}{j-1} \right) \left( \frac{m-3}{j-1} \right) x^m y^j \]

\[ = xy \sum_{m=4}^{\infty} \sum_{j=2}^{m} \left( \frac{m-1}{j-1} \right) x^{m-1} y^{j-1} + x^2 y \sum_{m=4}^{\infty} \sum_{j=2}^{m} \left( \frac{m-2}{j-1} \right) x^{m-2} y^{j-1} + x^3 y \sum_{m=4}^{\infty} \sum_{j=2}^{m} \left( \frac{m-3}{j-1} \right) x^{m-3} y^{j-1} \]

\[ = xy \left[ \frac{3}{1} x^3 y + \frac{3}{2} x^3 y^2 + \frac{3}{3} x^3 y^3 \right] + xy g(x, y) + x^2 y \left[ \frac{2}{1} x^2 y + \frac{2}{2} x^2 y^2 + \frac{3}{1} x^2 y^3 + \frac{3}{2} x^2 y^4 + \frac{3}{3} x^2 y^5 \right] + x^2 y g(x, y) + x^3 y \left[ \frac{1}{1} x y + \frac{2}{1} x^2 y^2 + \frac{3}{1} x^3 y^3 + \frac{3}{2} x^3 y^4 + \frac{3}{3} x^3 y^5 \right] + x^3 y g(x, y) \]

\[ g(x, y) = xy g(x, y) - x^2 y g(x, y) - x^3 y g(x, y) = xy (3x^3 y + 3x^3 y^2 + x^3 y^3) + x^2 y \]

\[ (2x^2 y + x^2 y^2 + 3x^2 y + 3x^2 y^2 + x^3 y^3) + x^3 y (xy + 2x^2 y + x^2 y^2 + 3x^3 y + 3x^3 y^2 + x^3 y^3) \]

\[ g(x, y) [1 - xy - x^2 y - x^3 y] = x^4 y^2 (6 + 4y + y^2 + 5x + 4xy + xy^2 + 3x^2 + 3xy^2 + x^2 y + x^3 y) \]

Therefore,

\[ g(x, y) = \frac{x^4 y^2 (6 + 4y + y^2 + 5x + 4xy + xy^2 + 3x^2 + 3xy^2 + x^2 y + x^3 y)}{1 - xy - x^2 y - x^3 y} \]

Using Theorem 4.2.5 we obtain the coefficient of WD (m, x) for 1 \( \leq m \leq 20 \) in Table 4.2. Let WD (C_m, j) = \(| C_m^j | \). There are interesting relationships between the numbers WD (C_m, j), (1 \( \leq j \leq 20 \)) in the table.
|   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | 1   | 2   | 1   | 3   | 3   | 1   | 5   | 1   | 5   | 10  | 1   | 7   | 1   | 9   | 1   | 11  | 1   | 20  | 1   |
| 2 | 0   | 6   | 4   | 1   | 3   | 3   | 1   | 5   | 1   | 5   | 10  | 1   | 7   | 1   | 9   | 1   | 11  | 1   | 20  | 1   |
| 3 | 0   | 5   | 10  | 9   | 11  | 13  | 5   | 1   | 9   | 11  | 13  | 5   | 1   | 9   | 1   | 11  | 1   | 20  | 1   |
| 4 | 0   | 3   | 14  | 15  | 6   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 20  | 1   |
| 5 | 0   | 2   | 8   | 28  | 7   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 20  | 1   |
| 6 | 0   | 1   | 28  | 21  | 7   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 20  | 1   |
| 7 | 0   | 0   | 14  | 28  | 7   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 20  | 1   |
| 8 | 0   | 0   | 8   | 28  | 28  | 8   | 1   | 4   | 1   | 4   | 1   | 4   | 1   | 4   | 1   | 4   | 1   | 4   | 1   | 20  |
| 9 | 0   | 0   | 3   | 81  | 75  | 36  | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 20  | 1   |
| 10| 0   | 0   | 0   | 75  | 28  | 8   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 9   | 1   | 20  |
| 11| 0   | 0   | 0   | 11  | 99  | 231 | 154 | 16   | 1   | 16  | 1   | 16  | 1   | 16  | 1   | 16  | 1   | 16  | 1   | 20  |
| 12| 0   | 0   | 0   | 0   | 3   | 72  | 282 | 456  | 396  | 396  | 13  | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 20  |
| 13| 0   | 0   | 0   | 0   | 0   | 3   | 72  | 282 | 456  | 396  | 396  | 13  | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 20  |
| 14| 0   | 0   | 0   | 0   | 0   | 0   | 14  | 210 | 786  | 1372 | 1372 | 861 | 91  | 14  | 1   | 1   | 1   | 1   | 1   | 20  |
| 15| 0   | 0   | 0   | 0   | 0   | 0   | 0   | 14  | 210 | 786  | 1372 | 1372 | 861 | 91  | 14  | 1   | 1   | 1   | 1   | 20  |
| 16| 0   | 0   | 0   | 0   | 0   | 0   | 0   | 17  | 867 | 2954 | 3487 | 4976 | 3921 | 1962 | 787 | 136 | 1   | 1   | 1   | 20  |
| 17| 0   | 0   | 0   | 0   | 0   | 0   | 0   | 17  | 867 | 2954 | 3487 | 4976 | 3921 | 1962 | 787 | 136 | 1   | 1   | 1   | 20  |
| 18| 0   | 0   | 0   | 0   | 0   | 0   | 0   | 3   | 1001 | 3677 | 5688 | 4314 | 2046 | 1021 | 986 | 748 | 13  | 18  | 1   | 20  |
| 19| 0   | 0   | 0   | 0   | 0   | 0   | 0   | 3   | 1001 | 3677 | 5688 | 4314 | 2046 | 1021 | 986 | 748 | 13  | 18  | 1   | 20  |
| 20| 0   | 0   | 0   | 0   | 0   | 0   | 0   | 3   | 1001 | 3677 | 5688 | 4314 | 2046 | 1021 | 986 | 748 | 13  | 18  | 1   | 20  |
Theorem 4.5.2

The following properties satisfied for WD (Cₘ, x) :

1. For every \( m \geq 4, \ j \geq \left\lceil \frac{m}{3} \right\rceil \), \( Wd(C_m, j) = \frac{(m-1)}{j-1} + \frac{(m-2)}{j-1} + \frac{(m-3)}{j-1} \)

2. \( Wd(C_m, m) = 1 \), for every \( m \geq 3 \)

3. \( Wd(C_{3m}, m) = 3 \), for every \( m \in \mathbb{N} \)

4. \( Wd(C_{3m+2}, m + 1) = 3m + 2 \), for every \( m \in \mathbb{N} \)

5. \( Wd(C_{3m+1}, m + 1) = \frac{(3m+1)(m+2)}{2} \), for every \( m \in \mathbb{N} \)

6. \( Wd(C_m, m - 1) = m \), for every \( m \geq 3 \)

7. \( Wd(C_m, m - 2) = \frac{m(m-1)}{2} \), for every \( m \geq 3 \)

8. For every \( m \geq 3 \), \( 1 = Wd(C_m, m) < Wd(C_{m+1}, m) < Wd(C_{m+2}, m) < \cdots < Wd(C_{2m-1}, m) < Wd(C_{2m}, m) > Wd(C_{2m+1}, m) > \cdots > Wd(C_{3m-1}, m) > Wd(C_m, m) = 3 \)

9. If \( S_m = \sum_{j=\left\lceil \frac{m}{3} \right\rceil}^{m} Wd(C_m, j) \), then for every \( m \geq 4 \), \( S_m = S_{m-1} + S_{m-2} + S_{m-3} \) combining with initial values \( S_1 = 1 \), \( S_2 = 3 \) and \( S_3 = 7 \).

10. \( \sum_{j=m}^{3m} Wd(C_j, m) = 3 \sum_{j=m-1}^{3m-3} Wd(C_j, m-1) \), for every \( m \geq 4 \).

11. \( Wd(C_{m+2}, m) = \frac{(m + 2)(m + 1)}{2} \) for every \( m \in \mathbb{N} \).

Proof

1. By using the Theorem 4.2.4 we get the result \( Wd(C_m, j) = \frac{(m-1)}{j-1} + \frac{(m-2)}{j-1} + \frac{(m-3)}{j-1} \)

2. Since, for any \( m \) vertices graph \( G \), \( Wd(G, m) = 1 \), then we get \( Wd(C_m, m) = 1 \).
3. Since \( C_{3m}^m = \{(1, 4, 7, \ldots, 3m-2), (2, 5, \ldots, 3m-1), (3, 6, 9, \ldots, 3m)\} \). So, \( \text{Wd} (C_{3m}, m) = 3 \).

4. We prove this by induction method. The result is true for \( m = 1 \), because \( C_5^2 = \{(1, 3), (1, 4), (2, 4), (2, 5), (3, 5)\} \). Now we assume the result is true for less than \( m, m \in \mathbb{N} \).

Now prove it for \( m \). By induction hypothesis, we have \( \text{Wd} (C_{3m+2}, m+1) = \text{Wd} (C_{3m+1}, m) + \text{Wd} (C_{3m}, m) + \text{Wd} (C_{3m-1}, m) = 3m + 2 \).

5. We prove this by induction method. The result is true for \( m = 1 \), since \( C_4^2 = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\} \). So \( \text{Wd} (C_4, 2) = 6 \). Now we assume the result is true for less than \( m, m \in \mathbb{N} \). now prove it for \( m \). By induction hypothesis, we get

\[
\text{Wd} (C_{3m+1}, m + 1) = \text{Wd} (C_{3m}, m) + \text{Wd} (C_{3m-1}, m) + \text{Wd} (C_{3m-2}, m)
= 3 + \text{Wd} (C_{3(m-1)+2}, m) + \text{Wd} (C_{3(m-1)+1}, m)
= 3 + 3 (m - 1) + 2 + \frac{(m-1) [3(m-1)+7]+2}{2}
= \frac{3m^2 + 7m + 2}{2}
= \frac{(3m+1) (m+2)}{2}
\]

6. We know that for any graph having \( m \) vertices, \( \text{Wd} (G, m - 1) = m \), hence we have \( \text{Wd} (C_m, m - 1) = m \).

7. We prove this by induction method. The result is true for \( m = 1 \), since \( \text{Wd} (C_3, 1) = 3 \).

Now we assume the result is true for less than \( m, m \in \mathbb{N} \). Now prove it for \( m \). By induction hypothesis, we get

\[
\text{Wd} (C_m, m - 2) = \text{Wd} (C_{m-1}, m - 3) + \text{Wd} (C_{m-2}, m - 3) + \text{Wd} (C_{m-3}, m - 3)
= \frac{(m-2)(m-1)}{2} + m - 2 + 1
\]
We prove that \( W_d(C_j, m) < W_d(C_{j+1}, m) \), for every \( m \leq j \leq 2m - 1 \) and \( W_d(C_j, m) > W_d(C_{j+1}, m) \) for, \( 2m \leq j \leq 3m - 1 \). We prove this by induction on \( m \). The result is true for \( m = 3 \). Now assume the result is true for less than \( m \). Now prove it for \( m = k + 1 \), that is \( W_d(C_j, k + 1) < W_d(C_{j+1}, k + 1) \), for \( k + 1 \leq j \leq 2k + 1 \) using induction hypothesis, we have

\[
W_d(C_j, k + 1) = W_d(C_{j-1}, k) + W_d(C_{j-2}, k) + W_d(C_{j-3}, k)
\]

Similarly, we can prove the other inequality.

9. \( S_m = \sum_{j=\lfloor m/2 \rfloor}^{m} W_d(C_m, j) \)

\[
= \sum_{j=\lfloor m/2 \rfloor}^{m} \left[ \binom{m-1}{j-1} + \binom{m-2}{j-1} + \binom{m-3}{j-1} \right]
\]

\[
= \sum_{j=\lfloor m/2 \rfloor}^{m-1} \left[ \binom{m-1}{j} + \binom{m-2}{j} + \binom{m-3}{j} \right] + \binom{m-3}{j-1}
\]

\[
= S_{m-1} + S_{m-2} + S_{m-3}
\]

10. We prove this by induction method. The result is true for \( m = 3 \). Then

\[
\sum_{j=3}^{9} W_d(C_j, 3) = 54 = 3 \sum_{j=2}^{6} W_d(C_j, 2)
\]

Now we assume the result is true for less than \( m \), \( m \in \mathbb{N} \). Now prove it for \( m = k \).

\[
\sum_{j=k}^{M} W_d(C_j, k) = \sum_{j=k}^{M} \left[ \binom{j-1}{k-1} + \binom{j-2}{k-1} + \binom{j-3}{k-1} \right]
\]
Theorem 4.5.3

For every m > 4, \( WD(m, x) = x[WD(C_{m-1}, x) + WD(C_{m-2}, x) + WD(C_{m-3}, x)] \) with the initial values \( WD(C_1, x) = x, WD(C_2, x) = x^2 + 2x, WD(C_3, x) = x^3 + 3x^2 + 3x. \)

Proof:

We prove this theorem by using Theorem 4.5.1 and using the definition of the weak domination polynomial.

Example 4.5.4

Let \( C_7 \) be cycle with order 7, then by Theorem 4.5.2, we have

\[
WD(C_7, x) = \sum_{j=\lfloor \frac{m}{2} \rfloor}^{m} \left( \begin{array}{c} m-1 \\ j-1 \end{array} \right) + \left( \begin{array}{c} m-2 \\ j-1 \end{array} \right) + \left( \begin{array}{c} m-3 \\ j-1 \end{array} \right) \right) x^j = 14x^3 + 28x^4 + 21x^5 + 7x^6 + x^7.
\]
4.6. Private dominating set of Cycle

Let $C_m$, $m \geq 3$ be the cycle with m vertices. $V(C_m) = [m]$ and $E(C_m) = \{1, 2\}, (2, 3), (3, 4) \ldots (m-1, m), (m, 1)\}$. Let the private dominating set of $C_m$ with the number of elements in the set $j$ is denoted by $Pd(C_m^j)$. Also, $Pd(C_m, j) = |Pd(C_m^j)|$.

For Finding the private dominating set of cycles we use the interesting relationship among $3 \leq m \leq 8$ and $1 \leq j \leq 8$.

| Table 4.3 $Pd(C_m, j)$ Private dominating set with number of elements in the set $j$ |
|---|---|---|---|---|---|---|---|
| $j$ | $m$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
| 3  | 3  | 3  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 4  | 0  | 0  | 4  | 0  | 0  | 0  | 0  | 0  | 0  |
| 5  | 0  | 0  | 5  | 0  | 0  | 0  | 0  | 0  | 0  |
| 6  | 0  | 0  | 3  | 0  | 0  | 0  | 0  | 0  | 0  |
| 7  | 0  | 0  | 0  | 9  | 0  | 0  | 0  | 0  | 0  |
| 8  | 0  | 0  | 0  | 0  | 8  | 0  | 0  | 0  | 0  |

Here we have private domination number of cycle is $\Gamma_{vt}(C_m) = \left\lceil \frac{m}{3} \right\rceil, m \geq 3$.

Using this private domination number we can determine Table $PD(S_m, j)$. We can’t generalize the private dominating set using recursive formula.
4.7. Private domination Polynomial :

**Definition 4.7.1**

Let $Pd(C^\downarrow_m)$ be the family of private dominating set of a cycle with the number of elements in the set $j$ and $\Gamma_{pv1}(C_m) = \left\lceil \frac{m}{3} \right\rceil, m \geq 3$. Then private domination polynomial is

$$PD(C_m, x) = \sum_{j=\Gamma_{pv1}(C_m)} Pd(C_m, j) x^j.$$ 

Here we can’t use the recursive formula for finding private domination polynomial, but we use the table value for finding private domination polynomial of cycle graphs.

Now, we can prove this private domination polynomial using examples.

**Example 4.7.2**

Let $C_4$ be the cycle with 4 vertices, then $PD(C_4, x) = 4x^2$.

![Figure 4.3 C_4 - Cycle graph with 4 vertices](image)

Here $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$ are not a private dominating set. Now $\{12\}$, $\{14\}$, $\{34\}$, $\{2, 3\}$ are a private dominating set. Since each vertex has at least one external private neighbor. But $\{1, 3\}$, $\{2, 4\}$ are not a private dominating set. Since each vertex has no private neighbor.
Also \{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}\} are not a private dominating set, since no external private neighbor. This is true for \{1, 2, 3, 4\} also.

Therefore, only the private dominating set is \{(1, 2), (1, 4), (3, 4), (2, 3)\}. Then the value of private dominating set is 4. Hence PD (C_4, x) = 4x^2.

**Example 4.7.3**

Let C_7 be the cycle with 7 vertices, then PD (C_7, x) = 9x^3.

**Figure 4.4 C_7 - Cycle graph with 7 vertices**