Chapter 5

Fractional Variable Order Diffusion Model based Medical Image Denoising

5.1 Introduction

The field of fractional differential equations has drawn immense consideration towards theoretical [45, 51, 80] and applied researches [63, 67, 94]. Fractional derivatives provide useful tools for a description of memory and hereditary properties. Fractional differential equations are found to be an effective tool used in certain mathematical models such as hydrology [9], finance [28], physics [62] and signal and image processing etc. Various theories of fractional integrals and derivatives were developed by many authors for instance Riemann-Liouville [51], Grünwald [40], Caputo [53] and Riesz [55].

Recently, many researchers are focusing on image denoising based on total variation, wavelet transform, bilateral filter, histogram of gradient, fractional $\sin C_\alpha$ and anisotropic diffusion filter etc. [3, 6, 33, 39, 43, 44, 108]. However, the drawback of using total variation, second order and fourth order based image denoising suffered from staircase effect, too much of smoothness and preserving in discontinues. The constant order fractional derivatives are not capable of characterizing some complex diffusion processes. This work aims to further develop of variable fractional order model in the context of image denoising in order to overcome the above issues.
Now a days variable order fractional calculus is particularly recognized as a useful and promising approach in the modeling of diffusion process. The pioneering work of variable order operator can be traced. Chen et al. [18] by introducing Caputo fractional derivatives for noisy signal by wavelet denoising. In this research, we study the fractional variable order diffusion model for medical image denoising using Caputo finite difference scheme and derive the convergence and stability of the proposed problem. The experiments demonstrate the advantage of the fractional variable order model that achieves the improved PSNR value and it shows the quality enhancement of the medical images.

In this chapter, we study the fractional variable order diffusion model for medical image denoising using an efficient numerical technique and derive the stability and convergence of the model are discussed. The experiments demonstrate the advantage of the fractional variable order model that achieves the improved PSNR value and it allows the quality enhancement of the medical images.

5.2 Fractional Variable Order Operators

The research on fractional calculus is pursued over long time in different disciplines such as biomedical, computational biology, economics and control engineering. The usual calculus extends the definition of fractional calculus, where the orders need not to be positive integers. On the otherhand, the variable order calculus is a natural extension of the constant order calculus. In this sense, the order may function in any variable such as time and space variables or a system of other variables. The basic definitions for variable order fractional derivative are provided in the following sequel.
5.3 Fractional Variable Order Diffusion Equation based Image Denoising

**Definition 5.2.1.** [112] The Caputo space variable order derivative is defined as follows:

\[ D_x^{\alpha(x,y)} u = \frac{1}{\Gamma(n-\alpha(x,y))} \int_0^x \frac{1}{(x-\xi)^{\alpha(x,y)-n+1}} \frac{\partial^n u(\xi,y,t)}{\partial \xi^n} d\xi, \text{ where } n < \alpha(x,y) < n+1. \]

**Definition 5.2.2.** [112] The Caputo time variable order derivative is defined as follows:

\[ D_t^{\alpha(t)} u = \frac{1}{\Gamma(n-\alpha(t))} \int_0^t \frac{1}{(t-\xi)^{\alpha(t)-n+1}} \frac{\partial^n u(\xi,t)}{\partial \xi^n} d\xi, \text{ where } n < \alpha(t) < n+1. \]

### 5.3 Fractional Variable Order Diffusion Equation based Image Denoising

Let \( \Omega \) be a closed domain in \( \mathbb{R}^2 \). The formation of noisy image could be mathematically modeled as

\[ u_0(x,y) = u(x,y) + f(x,y) \quad (5.3.1) \]

where \( u_0(x,y) \) and \( u(x,y) \) are the observed and clear images respectively, \( f(x,y) \) is the Gaussian noise. The proposed model estimates the desired clean image \( u(x,y) \) by solving the following fractional variable order diffusion equation of the form

\[ \frac{\partial^{\alpha(x,y,t)} u}{\partial t^{\alpha(x,y,t)}} = \frac{\partial^{\beta(x,y,t)} u}{\partial x^{\beta(x,y,t)}} + \frac{\partial^{\gamma(x,y,t)} u}{\partial y^{\gamma(x,y,t)}}, \quad (5.3.2) \]

where \( x_L < x < x_R, y_L < y < y_R, \alpha(x,y,t) \in (0,1] \) and \( \beta(x,y,t), \gamma(x,y,t) \in (1,2] \). Define \( t_n = n\Delta t \) as the integration time \( 0 \leq t_n \leq T \), \( \Delta x = h > 0 \) as the grid size in \( x- \) direction, \( \Delta y = k > 0 \) as the grid size in \( y- \) direction and \( (x,y,t) \in \Omega \) with initial condition

\[ u(x,y,0) = f(x,y,0) \quad (5.3.3) \]

and boundary conditions

\[
\begin{aligned}
&u_{i,N}^n = u_{i,N-1}^n, \\
&u_{N,j}^n = u_{N-1,j}^n,
\end{aligned}
\quad (5.3.4)
\]
where $M \times N$ is the size of the image $u(x, y)$. The solution of the proposed problem can be obtained by discretization of space and time variables. The discretization of the Caputo-type variable order space fractional derivative can be done as follows [112].

\[
\frac{\partial^{\alpha(x,y,t)} u}{\partial x^{\alpha(x,y,t)}} = \frac{h^{-\beta_{ij}}}{\Gamma(3 - \beta_{ij})} \sum_{s=0}^{i-1} (u_{i-s+1,j} - 2u_{i-s,j} + u_{i-s-1,j})(s + 1)^{2-\beta_{ij}} - s^{2-\beta_{ij}} (5.3.5)
\]

where $B_{ij}^n = (s + 1)^{2-\beta_{ij}} - s^{2-\beta_{ij}}$ and $Q_{ij} = \frac{h^{-\beta_{ij}}}{\Gamma(3 - \beta_{ij})}$.

\[
\frac{\partial^{\alpha(x,y,t)} u}{\partial y^{\alpha(x,y,t)}} = \frac{k^{-\gamma_{ij}}}{\Gamma(3 - \gamma_{ij})} \sum_{s=0}^{j-1} (u_{i,j-s+1} - 2u_{i,j-s} + u_{i,j-s-1})(s + 1)^{2-\gamma_{ij}} - s^{2-\gamma_{ij}} (5.3.6)
\]

where $C_{ij}^n = (s + 1)^{2-\gamma_{ij}} - s^{2-\gamma_{ij}}$ and $R_{ij} = \frac{k^{-\gamma_{ij}}}{\Gamma(3 - \gamma_{ij})}$.

Similarly the discretization of the Caputo type variable order time fractional derivative can be stated as follows [62].

\[
\frac{\partial^{\alpha(x,y,t)} u}{\partial t^{\alpha(x,y,t)}} = \frac{\tau^{-\alpha_{ij}}}{\Gamma(2 - \alpha_{ij})} \sum_{s=0}^{n} (u_{i,j}^{n+1-s} - u_{i,j}^{n-s}) \cdot (s + 1)^{1-\alpha_{ij}} - s^{1-\alpha_{ij}},
\]

where $A_{ij}^n = (s + 1)^{1-\alpha_{ij}} - s^{1-\alpha_{ij}}$ and $P_{ij} = \frac{\Gamma(2-\alpha_{ij})}{\Gamma(2-\alpha_{ij})}$. Discretize (5.3.2) in the following form:

\[
\frac{\tau^{-\alpha_{ij}+1}}{\Gamma(2 - \alpha_{ij})} \sum_{s=0}^{n} (u_{i,j}^{n+1-s} - u_{i,j}^{n-s}) A_{ij}^s = Q_{ij} \sum_{s=0}^{i-1} (u_{i-s+1,j} - 2u_{i-s,j} + u_{i-s-1,j}) B_{ij}^s + R_{ij} \sum_{s=0}^{j-1} (u_{i,j-s+1} - 2u_{i,j-s} + u_{i,j-s-1}) C_{ij}^s.
\]

Equation (5.3.7) can be written as

\[
u_{i,j}^{n+1} = u_{i,j}^n - \sum_{s=1}^{n} (u_{i,j}^{n+1-s} - u_{i,j}^{n-s}) A_{ij}^s
\]

\[
+ P_{ij} Q_{ij} \left[ \sum_{s=0}^{i-1} (u_{i-s+1,j} - 2u_{i-s,j} + u_{i-s-1,j}) B_{ij}^s \right]
\]

\[
+ P_{ij} R_{ij} \left[ \sum_{s=0}^{j-1} (u_{i,j-s+1} - 2u_{i,j-s} + u_{i,j-s-1}) C_{ij}^s \right].
\]

(5.3.8)
5.3. Fractional Variable Order Diffusion Equation based Image Denoising

Now (5.3.8) can be written at time $t = t_n$ in the matrix form

$$U_{ij}^{n+1} = T_{ij}^n u_{ij}^n + F_{ij}^n,$$

(5.3.9)

where $F_{ij}^n = [A_{ij}^n f(u_{m-1,n-1}^n, x_{m-1,n-1}, y_{m-1,n-1}, t_{mn}), \ldots, A_{ij}^n f(u_{1,1}^n, x_{1,1}, y_{1,1}, t_{mn})]^T$, $U_{ij}^n = (u_{m-1,n-1}^n, u_{m-2,n-2}^n, \ldots, u_{1,1}^n)^T$ and $T_{ij}^n = a_{mn}$ is a matrix with the following coefficients:

$$a_{mn} = \begin{cases} 
0, & \text{where } m > n + 2, \\
\theta Q_{ij} B_{ij}^0 + \theta R_{ij} C_{ij}^0, & \text{where } m = n + 2, \\
1 + \theta Q_{ij} (B_{ij}^n - 2B_{ij}^0) + \theta R_{ij} (C_{ij}^n - C_{ij}^0), & \text{where } m = n + 1, \\
\theta Q_{ij} (B_{ij}^n - 2B_{ij}^{n-1} + \theta B_{ij}^{n-2}), \\
\theta R_{ij} (C_{ij}^n - 2C_{ij}^{n-1} + \theta C_{ij}^{n-2}), & \text{where } m \leq n, \\
(P_{ij})_n (Q_{ij})_n(B_{ij})_0, & \text{where } m = 1, \\
\end{cases}$$

where

$$\theta = \begin{cases} 
0, & \text{where } m = 2, \\
1 & \text{otherwise.} 
\end{cases}$$

The numerical solutions $u_{ij}^{n+1}$ and $u_{ij}^n$ are defined from the given initial and symmetric boundary conditions (5.3.3) and (5.3.4). The solution $u_{ij}^{n+1}$ of (5.3.8) is represented as a denoised image with better PSNR values. The numerical scheme is expressed by the following algorithm:

1. Initialize the iteration process by setting $u_0(x, y)$
2. Initialize $\alpha \in (0, 1]$ and $\beta, \gamma \in (1, 2]$, $u_0 = f$ and time step $\tau = 0.1$
3. Compute $u_{ij}^{n+1}$ from (5.3.9) where $i=1,2,\ldots,M$, $j=1,2,\ldots,N$, for $n=1,2,\ldots,250$.
4. Check if $\frac{\|u_{ij}^{n+1} - u^n\|}{\|u^n\|} \leq \text{total}$; then stop.
5. Set $u_{ij}^{n+1} = u(x, y)$
6. Display output $u(x, y)$
5.4 Stability and Convergence Analysis

Consider $W^{n+1}$ and $U^{n+1}$ to be two different numerical solutions of (5.3.9) with initial values given by $W^{0}_{ij}$ and $U^{0}_{ij}$ respectively.

**Theorem 5.4.1.** The explicit approximation method defined by (5.3.9) to variable order space time diffusion equation (5.3.2) is unconditionally stable, that is

$$|W^{n+1} - U^{n+1}| \leq C|W^{0} - U^{0}| \quad \text{for any } n$$

**Proof.** Define $W^{n+1} - U^{n+1} = \epsilon^{n+1}$. From (5.3.9),

$$\epsilon^{n+1}_{ij} = T^{n}_{ij} \epsilon^{n}_{ij} + F^{n}_{\epsilon}, \quad (5.4.1)$$

where

$$F^{n}_{\epsilon} = \begin{bmatrix} A^{n}_{m-1,n-1} f(U^{n}_{1,m-1}, x^{n}_{1,m-1}, y^{n}_{1,n-1}, t_{m,n}) \\ -A^{n}_{m-1,n-1} f(W^{n}_{1,m-1}, x^{n}_{1,m-1}, y^{n}_{1,n-1}, t_{m,n}) \end{bmatrix} \cdot \cdots \cdot \begin{bmatrix} A^{n}_{1,1} f(U^{n}_{1,1}, x^{n}_{1,1}, y^{n}_{1,1}, t_{1,1}) \\ -A^{n}_{1,1} f(W^{n}_{1,1}, x^{n}_{1,1}, y^{n}_{1,1}, t_{1,1}) \end{bmatrix}^{T}$$

$$\leq \begin{bmatrix} A^{n}_{m-1,n-1} L^{n}_{m-1,n-1} \epsilon^{n}_{m-1,n-1} \cdot \cdots \cdot A^{n}_{1,1} L^{n}_{1,1} \epsilon^{n}_{1,1} \end{bmatrix} \Delta F^{n}_{\epsilon}, \quad (5.4.2)$$

$\Delta F^{n} = \text{diag}(A^{n}_{m-1,n-1} L^{n}_{m-1,n-1}, \cdots, A^{n}_{1,1} L^{n}_{1,1})$. We note that $|L^{n}_{ij}| \leq L$ for any $i, j$.

Let $\overline{A} = \max \{A^{n}_{m-1,n-1}, \cdots, A^{n}_{1,1}\}$. From (5.4.1), $\|T^{n} + \Delta F^{n}\|_{m,n} \leq (2 + \overline{A}L)$. Then

$$\|\epsilon^{n+1}_{ij}\|_{\infty} \leq \|T^{n} + \Delta F^{n}\|_{\infty} \|\epsilon^{n}_{ij}\|_{\infty}. \quad (5.4.3)$$

The stability via mathematical induction is analyzed in [85]. From (5.3.3), $\|\epsilon^{0}_{ij}\|_{\infty} \leq C \|\epsilon^{0}_{ij}\|_{\infty}$, where $C$ is a constant.

From (5.4.2), $\|\epsilon^{n+1}_{ij}\|_{\infty} \leq C(2 + \overline{A}L) \|\epsilon^{0}_{ij}\|_{\infty} \leq C_{1} \|\epsilon^{0}_{ij}\|_{\infty}$. Hence the theorem holds. \qed
5.4.1 Convergence Analysis

Equations (5.3.5) and (5.3.6) are shown to be smooth functions and stable. Therefore according to the Lax Equivalence theorem [75], they converge at $O(\tau + h + k)$. So the proposed model is consistent with a local truncation error at this rate.

5.5 Numerical Results and Discussion

The performance of the proposed method is evaluated by computing the numerical criteria PSNR value which is commonly applied to determine the quality of the processed image and also as a measure of excellence of reconstruction in image firmness and image. The PSNR is defined via the Mean Square Error (MSE) for two images, namely, $u_0$ (corrupted image) and $u$ (denoised image) respectively with

$$MSE = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} (u_0(i,j) - u(i,j))^2,$$

$$PSNR = 10 \log_{10} \frac{\max(u_0, u) \max(u_0, u)}{MSE}.$$

The PSNR and MSE values of the denoised image are computed which act as a quantitative standard for comparison of the proposed model by fractional order and integer order diffusion models (shown in Table 5.1). Figure 5.5 shows the relation between $\beta, \gamma$ and PSNR values. The computational time (CPU) is shown in Table 5.1. To this end, we performed some image data bases where a noisy image is evolved by using the fractional order; integer order diffusion models and the proposed model with $\alpha$ ($0 < \alpha \leq 1$) fractional order derivative could achieve the detection of the inflexion point without any shift. The parameter $\sigma$ indicates Gaussian noise; each pixel in the image will be changed from its original value by a small amount. The stopping criterion for the iteration is $\frac{||u^{n+1} - u^n||}{||u^n||} \geq \epsilon$. 
We considered $\epsilon = 10^{-5}$ in the numerical experiment. The proposed model is analysed with four types of medical images, namely, OCT(Optical Coherence Tomography) image, MRI(Magnetic Resonance Image), CT(Computed Tomography) scan and X-ray image. From Figures 5.1(e), 5.2(e), 5.3(e) and 5.4(e), the proposed model provides affirmative evidence for a better performance in preserving the details and edge information than the fractional and integer order models.

Figure 5.1(OCT image) shows beneath the surface of the retina. The OCT is the most valuable advancement in retinal diagnostic image and we can better understand the very fine changes which can indicate abnormality. It constructs a cross sectional view of ocular structures accurate to less than 10 micron. So denoising of these types of images is still challenging. Hence the retinal diagnostic image Figure 5.1(a) is used for the first experiment. Figure 5.1(a) is corrupted image by an additive Gaussian noise at four different levels such as 10, 15, 20 and 25 which are shown in Figure 5.1(b). The denoised image 5.1(e) by proposed model is clearer and it has larger PSNR value than those in Figure 5.1(c) and Figure 5.1(d). Also Figure 5.1(d) gives the information of retina that is invaluable in picking up the most subtle changes from Figure 5.1(a). Therefore the proposed model can be used for enhanced eye examination in addition to all the normal tests.

Figure 5.2(MRI image) shows brain tumor image. Recently biomedical images are playing a vital role in diagnosing anatomy and the physiological process of the body in both health and disease. This MRI image has proven successful for the diagnosis of all parts of the body including cancer, heart and vascular disease, stroke, breast disease and musculoskeletal disorders etc. Detection of tumor from MRI data is tedious for physicians and challenging for computers in all medical disciplines.
5.5. Numerical Results and Discussion

MRI scans detect a greater number of lesions and define the location more readily and they are also better at detecting spread to the meninges, the lining around the brain (and spinal cord). Figure 5.2(a) is the brain image affected by tumor. Figure 5.2(b) is corrupted by additive Gaussian noise at four different levels such as 10, 15, 20 and 25. Figures 5.2(c) and 5.2(d) have lost the information as well as appearance of tumor is not clear. Figure 5.2(e) provides original information without any loss and can be used for diagnosis and surgical purpose.

Figure 5.3(CT image) shows lung cancer image. This CT image is a diagnostic image which is used to test and create detailed images of internal organs, bones, soft tissues and blood vessels. A chest CT scan can help determine the cause of lung symptoms such as shortness of breathe or chest pain or check lung problems such as a tumor, excess fluid around the lung. Shown in Figure 5.3(a) is lung image affected by cancer and CT lung screening is capable of detecting lung nodules as small as 2 or 3 millimeters. Figure 5.3(b) is corrupted by additive Gaussian noise at four different levels such as 10, 15, 20 and 25. In the Figures 5.3(c) and Figure 5.3(d), small nodules and affected area not visible. Figure 5.3(e) is showing malignant tumors when they are still small and can be removed before disease spreads to other areas of the body.

Figure 5.4(X-ray image) displays spider image. X-ray image has become an important method for visualizing cellular and histological structures in a wide range of biological and medical studies. Figure 5.4(a) is high resolution X-ray image of spider. Figure 5.4(b) is perturbed by additive Gaussian noise at 10, 15, 20 and 25. Figures 5.4(c) and 5.4(d) are not making possible to see smaller details. But Figure 5.4(e) is making it possible to see similar details of Figure 5.4(a).
Figure 5.1: OCT-Retina image with $\sigma = 20$, $\alpha = 0.5$ and $\beta = \gamma = 1.5$
Figure 5.2: MRI-Brain image with $\sigma = 15$, $\alpha=0.6$ and $\beta = \gamma = 1.6$
Figure 5.3: CT-Lung image with $\sigma = 10$, $\alpha = 0.7$ and $\beta = \gamma = 1.7$
Figure 5.4: X-ray-Spider image with $\sigma = 25$, $\alpha = 0.8$ and $\beta = \gamma = 1.8$
### Table 5.1: The PSNR and MSE values obtained by applying different test images

<table>
<thead>
<tr>
<th>Models</th>
<th>$\sigma = 10$</th>
<th>$\sigma = 15$</th>
<th>$\sigma = 20$</th>
<th>$\sigma = 25$</th>
<th>Avg.PSNR</th>
<th>MSE</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Image a</strong>: OCT-Retina image ($\alpha = 0.5$ and $\beta = \gamma = 1.5$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Integer order diffusion</td>
<td>31.97</td>
<td>36.5</td>
<td>34.05</td>
<td>31.45</td>
<td>33.49</td>
<td>0.99</td>
<td>0.69</td>
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<td>39.95</td>
<td>35.27</td>
<td>34.81</td>
<td>37.04</td>
<td>0.67</td>
<td>0.67</td>
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<tr>
<td>Proposed model</td>
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<td>44.63</td>
<td>44.05</td>
<td>41.29</td>
<td>42.75</td>
<td>0.44</td>
<td>0.53</td>
</tr>
<tr>
<td><strong>Image b</strong>: MRI-Brain image ($\alpha = 0.6$ and $\beta = \gamma = 1.6$)</td>
<td></td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Integer order diffusion</td>
<td>24.19</td>
<td>24.16</td>
<td>24.19</td>
<td>24.15</td>
<td>24.17</td>
<td>0.25</td>
<td>0.68</td>
</tr>
<tr>
<td>Fractional order diffusion</td>
<td>36.14</td>
<td>36.18</td>
<td>36.19</td>
<td>36.15</td>
<td>36.17</td>
<td>0.15</td>
<td>0.66</td>
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<tr>
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<td>43.50</td>
<td>43.73</td>
<td>43.84</td>
<td>44.13</td>
<td>43.8</td>
<td>0.75</td>
<td>0.50</td>
</tr>
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<td><strong>Image c</strong>: CT-Lung image ($\alpha = 0.7$ and $\beta = \gamma = 1.7$)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Integer order diffusion</td>
<td>26.72</td>
<td>25.69</td>
<td>24.97</td>
<td>25.73</td>
<td>25.78</td>
<td>0.27</td>
<td>0.19</td>
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<tr>
<td>Fractional order diffusion</td>
<td>36.49</td>
<td>37.19</td>
<td>37.11</td>
<td>36.55</td>
<td>36.84</td>
<td>0.21</td>
<td>0.66</td>
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<tr>
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<td>44.50</td>
<td>44.29</td>
<td>44.36</td>
<td>0.19</td>
<td>0.64</td>
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<tr>
<td><strong>Image d</strong>: X-ray-Spider image ($\alpha = 0.8$ and $\beta = \gamma = 1.8$)</td>
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</tr>
<tr>
<td>Integer order diffusion</td>
<td>30.89</td>
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<td>32.75</td>
<td>32.26</td>
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<tr>
<td>Proposed model</td>
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<td>44.03</td>
<td>44.01</td>
<td>43.99</td>
<td>44.04</td>
<td>0.24</td>
<td>0.66</td>
</tr>
</tbody>
</table>
5.6. Conclusion

In this chapter, the stability and convergence for fractional variable order is discussed. The remarkable difference between integer, fractional and fractional variable order models of diffusion equation can be seen easily. The fractional variable order model describes the characteristics of denoising with more accuracy compares to integer order. Thus the fundamental goal of this work to construct an image denoising algorithm for variable order diffusion equation of fractional order in space and time by using finite difference approximation is achieved. The present work shows the validity and great potential of fractional variable order diffusion equation for denoising the noised images. The experimental results show that the quality of denoising images, highest PSNR and least MSE values are obtained by fractional variable order diffusion model.