LAGRANGE SPACE AND GENERALIZED LAGRANGE SPACE ARISING FROM METRIC $e^{\sigma(x)}g_{ij}(x, y) + \frac{1}{c^2}y_i y_j$

7.1. Introduction

Various authors like R. Miron, M. Anastasiei, H. Shimada, T. Kawaguchi, U. P. Singh have studied Lagrange space and generalized Lagrange space in their papers [3], [2], [4], [5]. A generalized Lagrange space with metric tensor $\gamma_{ij}(x) + \frac{1}{c^2}y_i y_j$, where $\gamma_{ij}(x)$ is metric tensor of Riemannian space and $c$ is velocity of light has been studied by Beil in his paper [1]. In this chapter $\gamma_{ij}(x)$ has been replaced by $e^{\sigma(x)}g_{ij}(x, y)$, where $g_{ij}(x, y)$ is metric tensor of Finsler space $(M^n, F)$.

Let $M^n$ is n-dimensional smooth manifold and $F$ is Finsler function, the metric tensor $g_{ij}(x, y)$ is given by

$$g_{ij}(x, y) = \frac{\partial^2 F^2}{\partial y^i \partial y^j}. \quad (7.1.1)$$

Since $F$ is Finsler function of homogeneity one, so $g_{ij}(x, y)$ is homogeneous function of degree zero. The angular metric tensor of Finsler space $(M^n, F)$, $h_{ij}(x, y)$ is given by

$$h_{ij}(x, y) = g_{ij}(x, y) - l_i l_j, \quad (7.1.2)$$

where $l_i$ is unit vector given by

$$l_i = \frac{y_i}{F}. \quad (7.1.3)$$
7.2. Generalized Lagrange Space $L^n$ and Associated Lagrange Space $L^*n$

Consider a generalized Lagrange space $L^n = (M^n, G_{ij}(x, y))$ with metric tensor

$$G_{ij} = e^\sigma g_{ij}(x, y) + \frac{1}{c^2} y_i y_j. \quad (7.2.1)$$

Reciprocal metric tensor $G^{ij}$ of $G_{ij}$ is

$$G^{ij} = e^{-\sigma} \left( g^{ij} - \frac{1}{a_1 c^2} y^i y^j \right), \quad (7.2.2)$$

where

$$a_1 = e^\sigma + \frac{F^2}{C^2}, \quad F^2 = g_{ij} y^i y^j. \quad (7.2.3)$$

The d-tensor field $C^{ijk}$ of $L^n$ is defined as

$$C^{ijk} = \frac{1}{2} \left( \frac{\partial G_{jh}}{\partial y^k} + \frac{\partial G_{hk}}{\partial y^j} - \frac{\partial G_{jk}}{\partial y^h} \right). \quad (7.2.4)$$

Since $\frac{\partial y_i}{\partial y^j} = g_{ij}$ from (7.2.1) and (7.2.4), we have

$$C^{ijk} = e^\sigma C_{jkh} + \frac{1}{c^2} g_{jky_h}, \quad (7.2.5)$$

$$C^{ij} = G^{ih} C_{ijk} = C_{jkh} + \frac{1}{a_1 c^2} g_{jky^i}. \quad (7.2.6)$$

The metric tensor $G_{ij}$ is used to define the Lagrangian $L^*$ is given by

$$L^* = G_{ij} y^i y^j. \quad (7.2.7)$$

The Lagrangian gives a metric tensor $G^*_{ij}$, is given by

$$G^*_{ij} = \frac{1}{2} \frac{\partial^2 L^*}{\partial y^i \partial y^j}. \quad (7.2.8)$$

From (7.2.7) and (7.2.1), we have

$$L^* = e^\sigma F^2 + \frac{F^4}{c^2} = a_1 F^2, \quad (7.2.9)$$

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from (7.2.8) and (7.2.9), we have

\[
G_{ij} = a_2 g_{ij}(x, y) + \frac{4}{c^2} y_i y_j, \quad (7.2.10)
\]

\[
G^{*ij} = \frac{1}{a_2} \left( g^{ij} - \frac{1}{a_2 c^2} y^i y^j \right), \quad (7.2.11)
\]

\[
C^{*}_{jhk} = a_2 C_{jhk} + \frac{2}{c^2} (g_{hk} y_j + g_{jh} y_k + g_{jk} y_h), \quad (7.2.12)
\]

From (7.2.12) and (7.2.11), we have

\[
C^{*}_{jik} = C^{*}_{ijk} + \frac{2}{a_2 c^2} \left( \delta^i_j y_k + \delta^i_k y_j + \frac{a_2}{a_6} g_{jk} y^i - \frac{8}{a_6 c^2} y^i y_k y_j \right), \quad (7.2.13)
\]

where \( a_2 = e^\sigma + \frac{F^2}{c^2} \) and \( a_6 = e^\sigma + \frac{6F^2}{c^2} \).

In general, \( a_\gamma = e^\sigma + \frac{\gamma F^2}{c^2} \).

**Theorem 7.2.1.** If the metric tensor of generalized Lagrange space given by \( G_{ij} \) in (7.2.1) then the metric tensor of associated Lagrange space \( G^{*}_{ij} \) is given by (7.2.10) and reciprocal metric tensor of generalized Lagrange space and associated Lagrange space are given by (7.2.2) and (7.2.11) respectively.

### 7.3. Angular Metric Tensor of \( L^n \) and \( L^m \)

For a Finsler space \( F^n \) the angular metric tensor \( h_{ij} \) is

\[
h_{ij} = F \frac{\partial^2 F}{\partial y^i \partial y^j} = g_{ij} - l_i l_j, \quad (7.3.1)
\]

where \( l_i = \frac{y_i}{L} \).

The generalized Lagrange space is not obtained from a Lagrangian therefore its angular metric tensor \( H_{ij} \)

\[
H_{ij} = G_{ij} - L_i L_j. \quad (7.3.2)
\]
Now,

\[ L_i = G_{ij} L_j = \left\{ e^\sigma g_{ij}(x, y) + \frac{1}{c^2} y_i y_j \right\} \frac{y^j}{L^*}. \]  

(7.3.3)

From (7.2.9)

\[ L_i = G_{ij} L_j = \left( e^\sigma g_{ij}(x, y) + \frac{1}{c^2} y_i y_j \right) \frac{y^j}{\sqrt{a_1} F} = \left( e^\sigma l_i + \frac{F^2}{c^2} y_i \right) \frac{1}{\sqrt{a_1}} \]

\[ = \left( e^\sigma + \frac{F^2}{c^2} \right) \frac{l_i}{\sqrt{a_1}} = a_1 \frac{l_i}{\sqrt{a_1}} = \sqrt{a_1} l_i. \]  

(7.3.4)

From (7.3.4) and (7.3.2) and (7.2.1)

\[ H_{ij} = e^\sigma g_{ij}(x, y) + \frac{1}{c^2} y_i y_j - a_1 l_i l_j. \]  

(7.3.5)

Putting the value of \( a_1 \) in (7.3.5), we have

\[ H_{ij} = e^\sigma h_{ij}. \]  

(7.3.6)

The angular metric tensor of Lagrange space \( L^* \) is given by

\[ H_{ij}^* = L^* \frac{\partial^2 L^*}{\partial y^i \partial y^j}. \]

The successive differentiation of (7.2.9) w.r.t. \( y^i \) and \( y^j \) gives

\[ L^* \frac{\partial L^*}{\partial y^j} = a_1 y_j + \frac{F^2}{c^2} y_j; \]  

(7.3.7)

\[ L^* \frac{\partial^2 L^*}{\partial y^i \partial y^j} + \frac{\partial L^*}{\partial y^i} \frac{\partial L^*}{\partial y^j} = \left( \frac{2}{c^2} y_i \right) y_j + a_1 g_{ij} + \frac{F^2}{c^2} g_{ij} + \frac{2}{c^2} y_i y_j; \]  

(7.3.8)

or

\[ L^* \frac{\partial^2 L^*}{\partial y^i \partial y^j} + \frac{\partial L^*}{\partial y^i} \frac{\partial L^*}{\partial y^j} = \frac{4}{c^2} y_i y_j + a_2 g_{ij}, \]

or

\[ L^* \frac{\partial^2 L^*}{\partial y^i \partial y^j} = \frac{4F^2}{c^2} l_i l_j - L_i^* L_j^* + a_2 g_{ij}, \]  

(7.3.9)

Now, from (7.3.7)

\[ L_i^* L_j^* = a_2 y_j \quad \Rightarrow \quad L_j^* = \frac{a_2 y_j}{L^*}. \]  

(7.3.10)
From (7.3.9) and (7.3.10), we get

\[ H_{ij}^* = (a_4 - e^a) l_i l_j - \frac{a^2}{a_1} l_i l_j + a_2 g_{ij}, \]

\[ H_{ij} = a_2 h_{ij} + \left( a_6 - \frac{a^2}{a_1} \right) l_i l_j. \] (7.3.11)

**Theorem 7.3.1.** If the metric tensor of generalized Lagrange space given by \( G_{ij} \) in (7.2.1), the angular metric tensor of generalized Lagrange space and associated Lagrange space are given by (7.3.6) and (7.3.11) respectively.

### 7.4. C-reducibility of \( L^n \) and \( L'^{n} \):

**Definition 7.3.1.** A generalized Lagrange space \( L^n \) is called C-reducible space if

\[ \overline{C}_{ijk} = (M_j H_{hk} + M_h H_{jk} + M_k H_{jh}), \] (7.4.1)

where \( M_j \) are component of a covariant vector field.

Suppose generalized Lagrange space \( L^n \) is C-reducible, then (7.4.1) holds. Using (7.2.5) and (7.3.6) and relation \( y_h = Fl_h \), (7.4.1) can be written as

\[ e^a C_{jhk} + \frac{F}{c^2} g_{jk} l_h = (M_j h_{hk} + M_h h_{jk} + M_k h_{jh}) e^a. \] (7.4.2)

Contracting (7.4.2) by \( l^h l^j l^k \) and using (7.4.1), we get

\[ \frac{F}{c^2} = 0 \quad \Rightarrow \quad F = 0, \]

which is contradiction.

**Theorem 7.4.1.** The generalized Lagrange space \( L^n = (M^n, G_{ij}) \) can not be C-reducible.

Now consider the space \( L'^{n} \), its C-reducibility is given by

\[ C'^{*}_{jhk} = (M_j^* H'^{*}_{hk} + M_h^* H'^{*}_{jk} + M_k^* H'^{*}_{jh}). \] (7.4.3)
$M^*_h$ are component of covariant vector field using (7.2.12), (7.3.11), (7.4.3) and
$y_h = Fl_h$ in (7.4.3), we get

$$a_2 C_{jhk} + \frac{2F}{c^2} (g_{hk} l_j + g_{jh} l_k + g_{jk} l_h) = a_2 (M^*_j h_{hk} + M^*_h h_{jk} + M^*_k h_{jh})$$

$$+ \left(a_6 - \frac{a_2}{a_1}\right) (M^*_j l_h l_k + M^*_h l_j l_k + M^*_k l_l l_j), \quad (7.4.4)$$

Contracting (7.4.4) by $l^j$ and putting $\rho^* = M^*_i l^i$, we have

$$\frac{2F^2}{c^2} (g_{hk} + 2l_h l_k) = a_2 \rho^* h_{hk} + \left(a_6 - \frac{a_2^2}{a_1}\right) (\rho^* l_h l_k + M^*_h l_k + M^*_k l_h). \quad (7.4.5)$$

Contracting (7.4.5) by $l^h$, we have

$$\frac{6F^2}{c^2} l_k = \left(a_6 - \frac{a_2^2}{a_1}\right) (\rho^* l_k + \rho^* l_k + M^*_k). \quad (7.4.6)$$

Again contracting (7.4.6) by $l^k$, which gives

$$\rho^* = \frac{2F^2}{c^2} \left(\frac{a_1}{a_1 a_6 - a_2^2}\right). \quad (7.4.7)$$

From (7.4.6) and (7.4.7), we have

$$\frac{2F^2}{c^2} l_k = \left(a_6 - \frac{a_2^2}{a_1}\right) M^*_k. \quad (7.4.8)$$

From (7.4.8) and (7.4.5), we have

$$\frac{2F^2}{c^2} g_{hk} = a_2 \rho^* h_{hk} + \frac{2F^2}{c^2} l_h l_k. \quad (7.4.9)$$

Using $g_{hk} = h_{hk} + l_h l_k$ in (7.4.9), we get

$$\left(\frac{2F^2}{c^2} - a_2 \rho^*\right) h_{hk} = 0.$$

It gives $\rho^* = \frac{2F^2}{c^2 a_2}$, which contradict (7.4.7). Hence

**Theorem 7.4.2.** The Lagrange space $L^{*n} = (M^n, L^*)$ can not be C-reducible.
References


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5. U. P. Singh : Motion and Affine motion in generalized Lagrange space and Lagrange space arising from metric tensor $\gamma_{ij}(x) + \frac{1}{c^2}y_iy_j$, J. Nat. Acad. Maths., 13 (1999), 41-52.