CHAPTER 6
RESILIENT NONLINEAR FAULT-TOLERANT
CONTROLLER FOR T-S FUZZY FRACTIONAL
ORDER SYSTEM SUBJECT TO ACTUATOR
SATURATION

6.1 INTRODUCTION

Fractional order dynamical systems have a long history and recently many researchers are drawn to this field due to their extensive applications in various kinds of real-world problems (Moornani & Haeri 2009; Tepljakov et al. 2011; Zhong et al. 2016). Though an increase in interest is marked in the research area, the fractional order systems have not yet been fully exploited in the field of stability and stabilization. In (Trigeassou et al. 2011), a new set of conditions is obtained for the stability of fractional differential equations by using the Lyapunov approach. In particular, the approach employed in (Trigeassou et al. 2011) is based on continuous frequency distribution which has later been implemented in (Boroujeni & Momeni 2012; Lan & Zhou 2013) where an observer based control design has been investigated for a class of nonlinear fractional order systems.

On the other hand, extensive study has been done in recent years on nonlinear systems based on fuzzy models, since they can represent a broad class of practical systems. Moreover, among the different types of fuzzy models, T-S fuzzy model is considered as best approximation for nonlinear dynamical
systems. Some of the recent results on stability and stabilization of fractional order T-S fuzzy models are available in [Lin et al. 2016; Wang et al. 2016a; Wang et al. 2016b] where sufficient stability conditions are presented in terms of LMIs. Though several studies have been reported for fractional T-S fuzzy models, still there are many issues in modelling, analysis and control synthesis that have not yet been fully investigated.

This chapter proposes a resilient fault-tolerant control for T-S fuzzy model which is described by fractional order differential equations. In particular, the system under consideration is subject to actuator saturation and actuator faults. The system is also sensitive to uncertainties that affect the control gains. Utilizing the concept of continuous frequency distribution, a set of sufficient conditions for robust asymptotic stability of the closed-loop system will be derived using indirect Lyapunov approach.

6.1.1 Problem formulation and preliminaries

In this section, initially some definitions and lemmas are presented which will be used to develop the main results.

Definition 6.1. [Boroujeni & Momeni 2012] The $q$th order fractional integral of the function $f(t)$ with respect to $t$ and the initial value $t_0$ is given by

$$\int_{t_0}^{t} D^{-q}_t f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-q}} d\tau,$$

(6.1)

where $q > 0$ and $\Gamma(\cdot)$ is the Gamma function. ◻

Definition 6.2. [Boroujeni & Momeni 2012] The Riemann-Liouville fractional derivative of order $q$ of a function $f(t)$ with respect to $t$ and the initial value $t_0$
is given by
\[
I_0 D_q^t f(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{q-m+1}} d\tau,
\]
(6.2)
where \( q > 0, m - 1 \leq q \leq m \) and \( \Gamma(\cdot) \) is the Gamma function. □

**Definition 6.3.** (Trigeassou et al. 2011) Let \( g(t) \) be the impulse response of a linear system. The diffusive representation (or frequency weighting function) of \( g(t) \) is called \( \mu(\omega) \) with
\[
g(t) = \int_0^\infty \mu(\omega)e^{-\omega t}d\omega.
\]
(6.3)
□

**Remark 6.1.** The fractional order integral can be written as
\[
I_0 D_t^{-q} f(t) = g(t) * f(t),
\]
where * denotes the convolution operator, \( g(t) = \frac{t^{q-1}}{\Gamma(q)} \) and the diffusive representation of \( g(t) \) is introduced as
\[
\mu(\omega) = \frac{\sin(q\omega)}{\pi\omega} - q.
\]
(6.4)

**Lemma 6.1.** (Lan & Zhou 2013) The nonlinear fractional order differential equation
\[
I_0 D_t^q f(t) = f(x(t))
\]
due to continuous frequency distributed model of the fractional integrator can be written as
\[
\begin{cases}
\frac{\partial z(\omega,t)}{\partial t} = -\omega z(\omega,t) + f(x(t)), \\
x(t) = \int_0^\infty \mu(\omega)z(\omega,t)d\omega,
\end{cases}
\]
(6.5)
where \( \mu(\omega) \) is same as in (6.4).

Consider the fractional order T-S fuzzy model with saturating
actuator described by the following form

\[
\text{Rule } R_i : \text{ IF } \xi_1(t) \text{ is } N_{i1}, \xi_2(t) \text{ is } N_{i2}, \ldots, \xi_p(t) \text{ is } N_{ip},
\]

THEN \[D^q x(t) = A_i x(t) + B_i sat(u^f(t)), \quad (i = 1, 2, \cdots, r), \quad (6.6)\]

where \( r \) is the number of fuzzy rules, \( N_{ia}(a = 1, 2, \cdots, p) \) are the fuzzy sets, \( \xi_1(t), \xi_2(t), \cdots, \xi_p(t) \) are the premise variables, \( D^q \) denotes the Riemann-Liouville derivative, \( q \) \( (0 < q < 1) \) is the fractional order, \( x(t) \in \mathbb{R}^n \) is the state vector, \( u^f(t) \in \mathbb{R}^m \) is the control input, \( A_i, B_i \) are known constant real matrices, \( sat : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is the saturation function which is defined by

\[
sat(u^f) = [sat(u^f_1) \ sat(u^f_2) \cdots sat(u^f_m)]^T, \quad \text{where } sat(u^f_j) = \text{sign}(u^f_j)\text{min}\{1,|u^f_j|\}.
\]

Further the fractional order T-S fuzzy model can be written as

\[
D^q x(t) = \sum_{i=1}^{r} h_i(\xi(t))[A_i x(t) + B_i sat(u^f(t))], \quad (6.7)
\]

where \( h_i(\xi(t)) = \frac{\eta_i(\xi(t))}{\sum_{a=1}^{p} \eta_{ia}(\xi(t))} \), \( \eta_i(\xi(t)) = \prod_{a=1}^{p} N_{ia}(\xi_a(t)) \), \( N_{ia}(\xi_a(t)) \) is the grade of membership of \( \xi_a(t) \) in \( N_{ia} \) which satisfy \( \sum_{i=1}^{r} \eta_i(\xi(t)) > 0 \) with \( \eta_i(\xi(t)) \geq 0 \quad (i = 1, 2, \cdots, r) \). It is noted that \( \sum_{i=1}^{r} h_i(\xi(t)) = 1 \) and \( h_i(\xi(t)) \geq 0 \quad (i = 1, 2, \cdots, r) \) where \( h_i(\xi(t)) \) represents IF-THEN fuzzy rule normalized weight.

In this work, a fault-tolerant controller with presence of nonlinearity is implemented which is represented by the form

\[
u^f(t) = Gu(t) + \chi(u(t)), \quad (6.8)
\]

where \( G \) is the actuator fault matrix defined as in Section [2.2.1] without switching signal ‘i’.

Further \( u(t) = \bar{K}_i x(t) \). Here \( \bar{K}_i \) is assumed to take the structure \( \bar{K}_i = K_i + \Delta K_i(t) \), where \( K_i \)’s are the control gain matrices to be determined and \( \Delta K_i(t) \)’s are the possible gain variations. The gain variations are assumed to take the additive
form $\Delta K_i(t) = L_{ki}F_{ki}(t)M_{ki}$, where $L_{ki}, M_{ki}$ are known constant matrices and $F_{ki}(t)$ are unknown matrix functions satisfying $F_{ki}^T(t)F_{ki}(t) \leq I$.

The function $\chi(\cdot)$ represents the nonlinear characteristic of actuators which inevitably occur in a large number of realistic engineering applications. The vector function $\chi(u(t)) = [\chi_1(u(t)), \chi_2(u(t)), \ldots, \chi_m(u(t))]^T$ satisfies $|\chi_k(u(t))| \leq \sqrt{\varsigma_k}|u(t)|$, $k = 1, 2, \ldots, m$, $\varsigma_k > 0$ and from which it follows that

$$
\chi^T(u(t))\chi(u(t)) \leq u^T(t)\Xi u(t), \quad (6.9)
$$

where $\Xi = \text{diag}\{\varsigma_1, \varsigma_2, \ldots, \varsigma_m\}$.

At the same time, to stabilize the fractional order T-S fuzzy model (6.7) via resilient fault-tolerant control technique, we consider the following $i$th control rule

Rule $R_i$: IF $\xi_1(t)$ is $N_{i1}$, $\xi_2(t)$ is $N_{i2}$, $\ldots$, $\xi_p(t)$ is $N_{ip}$,
THEN $u^f(t) = Gu(t) + \chi(u(t)), \quad (i = 1, 2, \ldots, r). \quad (6.10)$

Then the overall controller can be inferred as

$$
u^f(t) = \sum_{j=1}^{r} h_j(\xi(t))(Gu(t) + \chi(u(t))). \quad (6.11)$$

Next an expression for the controller is derived when it is saturated. For this purpose, the following few notions are defined which will be used in further derivation.

Denote the $l$th row of matrix $F \in \mathbb{R}^{m \times n}$ as $f_l$ and define $L(F) = \{x(t) \in \mathbb{R}^n : |f_l(x(t))| \leq 1, l = 1, 2, \cdot\cdot\cdot m\}$. Let $P \in \mathbb{R}^{n \times n}$ be a symmetric matrix and define the ellipsoid $\mathcal{E}(P, 1) = \{x(t) \in \mathbb{R}^n : x^T(t)Px(t) \leq 1\}$.
From Lemma 1.9, if there exists an auxiliary matrix $F \in \mathbb{R}^{m \times n}$ which satisfies the condition $|f_i(t)| \leq 1, \ l = 1, 2, \cdots, m$, then for $\zeta_s \geq 0, \sum_{r=1}^{2m} \zeta_s = 1$, we have

$$sat(u^f(t)) = \sum_{s=1}^{2m} \zeta_s (D_s u^f(t) + D_s^- F x(t)). \quad (6.12)$$

Now considering (6.11) and (6.12), the overall resilient fault-tolerant controller with actuator saturation can be described by

$$sat(u^f(t)) = \sum_{j=1}^{r} \sum_{s=1}^{2m} h_j(\xi(t)) \zeta_s \left( D_s G \tilde{K}_j x(t) + D_s \chi(u(t)) + D_s^- F_j x(t) \right). \quad (6.13)$$

By substituting (6.13) into (6.7), we have the nonlinear fractional order T-S fuzzy model in the following form

$$\mathcal{D}^q x(t) = \sum_{s=1}^{2m} \zeta_s \left[ (\mathcal{A}_s + \mathcal{B}_s^A) x(t) + \mathcal{B}_s^L \chi(u(t)) \right], \quad (6.14)$$

where

$$\mathcal{A}_s = \sum_{i=1}^{r} \sum_{j=1}^{s} h_i(\xi(t)) h_j(\xi(t)) (A_i + B_i D_s G K_j + D_s^- F_j),$$

$$\mathcal{B}_s^A = \sum_{i=1}^{r} \sum_{j=1}^{s} h_i(\xi(t)) h_j(\xi(t)) B_i D_s G L K_j \tilde{M}_{kj}(t)$$

and

$$\mathcal{B}_s^L = \sum_{i=1}^{r} \sum_{j=1}^{s} h_i(\xi(t)) h_j(\xi(t)) B_i D_s.$$

### 6.1.2 Stability analysis

In the following theorem, sufficient conditions for the solvability of the resilient fault-tolerant control problem is derived when the actuator fault matrix is known.

**Theorem 6.1.** Given the fault matrix $G$ and $K_i$, the fractional order T-S fuzzy system (6.7) is robustly asymptotically stable with resilient fault-tolerant controller (6.13) within the ellipsoid $\mathcal{E}(P, 1)$, if there exists symmetric positive definite matrix $P$ and real positive scalars $\varepsilon_1, \varepsilon_2$ such that for $s = 1, 2, \cdots, 2^m,$
and \( i, j = 1, 2, \cdots, r \), the following conditions hold with \( 1 \leq i \leq j \leq r \):

\[
\begin{align*}
\left[ \Phi_{ij}^s \right] + \left[ \Phi_{ji}^s \right] &< 0, \quad i = j, \\
\left[ \Phi_{ij}^s \right] + \left[ \Phi_{ji}^s \right] &< 0, \quad i < j,
\end{align*}
\]

(6.15)

\( \mathcal{E}(P, 1) \subset L(F_i) \),

(6.16)

where \( \left[ \Phi_{ij}^s \right] \) takes the form

\[
\begin{bmatrix}
P A_i + P B_i D_s G K_j + P D_s^c F_j & \varepsilon_1 P B_i D_s & K_j^T & \varepsilon_2 P B_i D_s G L_{k_j} & M_{k_j}^T \\
* & -\varepsilon_1 I & 0 & 0 & 0 \\
* & * & -\varepsilon_1 \Xi^{-1} & \varepsilon_2 L_{k_j} & 0 \\
* & * & * & -\varepsilon_2 I & 0 \\
* & * & * & * & -\varepsilon_2 I
\end{bmatrix}
\]

Proof. By using Lemma 6.1, fractional order T-S fuzzy system (6.14) can be written as

\[
\begin{align*}
\frac{\partial z(\wp, t)}{\partial t} &= -\wp z(\wp, t) + \sum_{i=1}^m \zeta_i \left[ (A_s + B_s^\Delta) x(t) + B_s^\chi \chi(u(t)) \right], \\
x(t) &= \int_0^\infty \mu(\wp) z(\wp, t) d\wp.
\end{align*}
\]

(6.17)

To continue with the proof of the theorem, we consider two Lyapunov functions: first, the monochromatic Lyapunov function \( \psi(\wp, t) \) corresponding to the elementary function \( \wp \) which is given by \( \psi(\wp, t) = z^T(\wp, t) P z(\wp, t) \): second, the Lyapunov function \( V(t) \) obtained by integrating all the monochromatic function \( \psi(\wp, t) \) with the weighting function \( \mu(\wp) \). Then we have

\[
V(t) = \int_0^\infty \mu(\wp) \psi(\wp, t) d\wp = \int_0^\infty \mu(\wp) z^T(\wp, t) P z(\wp, t).
\]

(6.18)

Taking the time derivative of (6.18) along the trajectories of system (6.17), we
have

\[ \dot{V}(t) = \int_0^\infty \mu(t) \left\{ -z^T(t) + x^T(t) \left( \sum_{s=1}^{2^m} \zeta_s (A_s + B_s^A) \right)^T + \sum_{s=1}^{2^m} \zeta_s (B_s^B)^T \right\} P \int_0^\infty \mu(t) \right\} - z^T(t) + x^T(t) \left( \sum_{s=1}^{2^m} \zeta_s (A_s + B_s^A) \right) x(t) \]

By using Lyapunov theory, (6.17) can be robustly asymptotically stable if \( \dot{V}(t) < 0 \). Further, form (6.19), it is clear that the inequality \( \dot{V}(t) < 0 \) holds for \( s = 1, 2, \ldots, 2^m \) if

\[ 2x^T(t)P \left( \sum_{s=1}^{2^m} \zeta_s (A_s + B_s^A) \right) x(t) + 2x^T(t)P \left( \sum_{s=1}^{2^m} \zeta_s B_s^B \right) \chi(u(t)) < 0 \]

\[ \Rightarrow \sum_{s=1}^{2^m} \zeta_s \left( 2x^T(t)P(A_s + B_s^A)x(t) + 2x^T(t)P B_s^B \chi(u(t)) \right) < 0 \]

\[ \Rightarrow 2x^T(t)P(A_s + B_s^A)x(t) + 2x^T(t)P B_s^B \chi(u(t)) < 0. \]

(6.20)

Now, using Lemma 1.8 and (6.9), \( 2x^T(t)P B_s^B \chi(u(t)) < 0 \) can be equivalently written as

\[ \varepsilon_1 x^T(t)P B_s^B (B_s^B)^T P x(t) + \varepsilon_1^{-1} \chi^T(u(t)) \chi(u(t)) < 0 \]

\[ \Rightarrow \varepsilon_1 x^T(t)P B_s^B (B_s^B)^T P x(t) + \varepsilon_1^{-1} x^T(t) \mathcal{K}^T \mathcal{K} x(t) < 0, \]

(6.21)

where \( \mathcal{K} = \sum_{j=1}^r (K_j + L_k \Xi_{k_j}(t) M_{k_j}) \) and \( \varepsilon_1 > 0 \).

Combining (6.20) and (6.21), the robust asymptotic stability of (6.17) is
guaranteed when

\[ 2x^T(t)P(A_s + B_s^a)x(t) + \varepsilon_1 x^T(t)P(B_s^a)(B_s^a)^TPx(t) \]
\[ + \varepsilon_1^{-1} x^T(t)K^T \Xi K x(t) < 0, \quad s = 1, 2, \cdots, 2^m. \] (6.22)

Further, by applying Schur complement and Lemma 1.4 to (6.22), we obtain

\[ \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\xi(t)) h_j(\xi(t)) \left( x^T(t)[\Phi_{ij}^s]x(t) \right) < 0, \quad s = 1, 2, \cdots, 2^m, \] (6.23)

which can be equivalently written as

\[ \sum_{i=1}^{r} h_i^2(\xi(t)) \left( x^T(t)[\Phi_{ii}^s]x(t) \right) + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} h_i(\xi(t)) h_j(\xi(t)) \]
\[ \left( x^T(t)[\Phi_{ij}^s + \Phi_{ji}^s]x(t) \right) < 0, \quad s = 1, 2, \cdots, 2^m, \] (6.24)

where \([\Phi_{ij}^s]\) is defined in as the statement of the theorem: then the fractional order T-S fuzzy system (6.14) is robustly asymptotically stable. This completes the proof.

\[ \square \]

6.1.3 Control design

In the following theorem, we deal with the resilient fault-tolerant controller design for system (6.7). To be precise, we obtain sufficient conditions for the existence of controller in terms of LMIs such that the closed-loop system (6.14) is robustly asymptotically stable. This completes the proof.

**Theorem 6.2.** For given fault matrix \(G\), if there exist symmetric matrix \(X > 0\), any matrices \(Y_i, Z_i\) and real positive scalars \(\varepsilon_1, \varepsilon_2\) such that for \(s = 1, 2, \cdots, 2^m\)
and \( i, j = 1, 2, \ldots, r \), the following LMIs hold:

\[
\begin{align*}
\tilde{\Phi}_{ii} + \begin{bmatrix} \tilde{\Phi}_{ij} \\ \tilde{\Phi}_{ji} \end{bmatrix} & < 0, \quad i = j, \\
1 \begin{bmatrix} \begin{bmatrix} \tilde{\Phi}_{ij} \\ \tilde{\Phi}_{ji} \end{bmatrix} \end{bmatrix} & < 0, \quad i < j,
\end{align*}
\]

(6.25)

\[
\begin{bmatrix} 1 & w_{il} \\ \ast & X \end{bmatrix} > 0, \quad l = 1, 2, \ldots, m,
\]

(6.26)

where \( \tilde{\Phi}_{ij} \) takes the form

\[
\begin{bmatrix}
A_iX + B_iD_sGY_j + D_sW_j & \varepsilon_1B_iD_sY_j^T & \varepsilon_2B_iD_sGL_k & X^TM_k^T \\
\ast & -\varepsilon_1I & 0 & 0 & 0 \\
\ast & \ast & -\varepsilon_1\Xi^{-1} & \varepsilon_2L_k & 0 \\
\ast & \ast & \ast & -\varepsilon_2I & 0 \\
\ast & \ast & \ast & \ast & -\varepsilon_2I
\end{bmatrix}
\]

and \( w_{il} \) is the \( l \)th row of the matrix \( W_i \), then the fractional order T-S fuzzy system (6.14) is robustly asymptotically stable. Further the controller gain matrices can be computed by \( K_i = Y_iX^{-1} \).

**Proof.** In view of Theorem 6.1, pre and post multiply \( [\Phi_{ij}] \) with \( \text{diag}\{P^{-1}, I, I, I\} \) and let \( P^{-1} = X, Y_i = K_iX \) and \( W_i = F_iX \). Subsequently, we obtain \( \tilde{\Phi}_{ij} \). Besides, following the same method as in (Fu & Ma 2016), it is easy to show that (6.16) is equivalent to (6.26). This completes the proof. \( \square \)

In the forthcoming theorem, we extend the results obtained in Theorem 6.2 to a case when the actuator fault matrix is unknown.

**Theorem 6.3.** Consider the fractional order T-S fuzzy system (6.14) and the unknown fault matrix \( G \). If there exist symmetric matrix \( X > 0 \), any matrices \( Y_i, Z_i \) and real positive scalars \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) such that for \( s = 1, 2, \ldots, 2^m \) and \( i, j = \)
1, 2, \ldots, r, the inequality (6.26) and the following LMIs hold:
\[
\begin{align*}
\begin{bmatrix}
\Phi_{ii}^s & i = j, \\
\Phi_{ij} & i < j,
\end{bmatrix} & < 0,
\end{align*}
\]
(6.27)
where \( \Phi_{ij}^s \) takes the form
\[
\begin{bmatrix}
\mathcal{L} & \epsilon_1 B_i D_s Y_j^T & \epsilon_2 B_i D_s G_0 L_{kj} & X^T M_{kj}^T & \epsilon_3 B_i D_s G_1 & Y_j^T \\
* & -\epsilon_1 I & 0 & 0 & 0 & 0 \\
* & * & -\epsilon_1 \mathcal{L}^{-1} & \epsilon_2 L_{kj} & 0 & 0 \\
* & * & * & -\epsilon_2 I & 0 & 0 \\
* & * & * & * & -\epsilon_3 I & 0 \\
* & * & * & * & * & -\epsilon_3 I \\
\end{bmatrix}
\]
\( \mathcal{L} = A_i X + B_i D_s G_0 Y_j + D_s W_j \), then system (6.14) is robustly asymptotically stable, while controller gain matrices are designed as \( K_i = Y_i X^{-1} \).

\textbf{Proof.} The proof of this theorem immediately follows from that of Theorem 6.2 by considering the expression \( G = G_0 + G_1 \Sigma \), as in (2.4) and applying Lemma 1.4. \(\square\)

**6.1.4 Numerical examples**

In this section, two examples including a hydro-turbine governing system are considered to demonstrate the effectiveness of the derived results for fuzzy fractional order differential systems.

**Example 6.1.** In this example, we consider the hydro-turbine governing system
as follows (Wang et al. 2016b)

\[
\begin{align*}
\dot{x}_1 &= \omega_0 x_2, \\
\dot{x}_2 &= \frac{1}{T_{ab}} \left( x_3 - D x_2 - \frac{E'q}{x_\Sigma} \sin x_1 - \frac{V_s^2}{2} \frac{x_{\Sigma} - x_\Sigma^q}{x_\Sigma x_\Sigma^q} \sin 2x_1 \right), \\
\dot{x}_3 &= \frac{1}{e_q T_w} \left( -x_3 + e_y x_4 + \frac{e e_T \omega}{T_w} x_4 \right), \\
D'x_4 &= -\frac{1}{T_y} x_4,
\end{align*}
\]

where \( \omega_0 = 314, T_{ab} = 9.0s, E'q = 1.35, x_{d\Sigma} = 1.15, x_{q\Sigma} = 1.474, T_w = 0.8s, T_y = 0.1s, V_s = 1.0, e_q = 0.5, e_y = 1.0, e = 0.7 \) and \( q = 0.9 \). By borrowing the approach adopted in (Wang et al. 2016b), the hydro-turbine governing system (6.28) can be alternatively written in the formulation of (6.7), with the following parameter values

\[
A_1 = \begin{bmatrix}
0 & 314 & 0 & 0 \\
-\frac{17231}{16951} & -\frac{2}{9} & \frac{1}{9} & 0 \\
0 & 0 & -\frac{5}{2} & \frac{33}{2} \\
0 & 0 & 0 & -10
\end{bmatrix}, \quad \quad A_2 = \begin{bmatrix}
0 & 314 & 0 & 0 \\
-\frac{1577}{16951} & -\frac{2}{9} & \frac{1}{9} & 0 \\
0 & 0 & -\frac{5}{2} & \frac{33}{2} \\
0 & 0 & 0 & -10
\end{bmatrix}, \quad \quad B_1 = B_2 = I_{4 \times 4}.
\]

Take the membership functions as \( h_1(\xi(t)) = \frac{1}{2} (1 + \frac{\xi(t)}{4}) \) and \( h_2(\xi(t)) = \frac{1}{2} (1 - \frac{\xi(t)}{4}) \), where \( \xi(t) = x_1(t) \). In order to stabilize system (6.28) via resilient fault-tolerant controller, the following parameters are taken: \( L_{k1} = \begin{bmatrix} 0.2 & 0.1 & 0 & 0.1 \end{bmatrix}^T, \quad L_{k2} = \begin{bmatrix} 0.2 & 0.1 & 0 & 0.1 \end{bmatrix}^T, \quad M_{k1} = \begin{bmatrix} 0.02 & 0 & 0 & 0.01 \end{bmatrix}, \quad M_{k2} = \begin{bmatrix} 0.01 & 0.1 & 0 & 0.02 \end{bmatrix} \) and \( F_{k1}(t) = F_{k2}(t) = \sin(t) \). Further, take \( q = 0.9, \phi(u(t)) = \sin(6.1 u(t)), \Xi = 0.01 \) and the range of the fault as \( 0.4I_{4 \times 4} \leq G \leq 0.8I_{4 \times 4} \). Then, by solving the LMIs in Theorem 6.3 with the above parameter values, we obtain feasible solution. Then the associated controller gains are given by
\[ K_1 = \begin{bmatrix} -2.6466 & -74.2160 & -0.2590 & -0.2773 \\ -13.7771 & -389.9279 & -1.3610 & -1.6470 \\ -0.2340 & -9.3995 & -0.0144 & -0.1124 \\ -0.5308 & -18.1793 & -0.0972 & 0.2823 \\ -2.2738 & -71.3558 & -0.2217 & -0.3000 \\ -13.9554 & -424.4920 & -1.5436 & -1.8218 \\ -0.2311 & -10.3745 & -0.0160 & -0.1107 \\ -0.5098 & -16.6609 & -0.0934 & 0.2938 \end{bmatrix} \]

and

\[ K_2 = \begin{bmatrix} -2.2738 & -71.3558 & -0.2217 & -0.3000 \\ -13.9554 & -424.4920 & -1.5436 & -1.8218 \\ -0.2311 & -10.3745 & -0.0160 & -0.1107 \\ -0.5098 & -16.6609 & -0.0934 & 0.2938 \end{bmatrix} \]

The state trajectories of the system (6.28) with the control input \( u(t) = 0 \) is shown in Figure 6.1 which shows that the system is not stable.

![Figure 6.1: State responses of the open-loop fuzzy fractional order system](image)
The responses of states of the closed-loop system are shown in Figure 6.2. The corresponding saturated control response is shown in Figure 6.3.

Figure 6.2: State responses of the closed-loop system with the proposed controller

Figure 6.3: Saturated control responses

From Figure 6.2, one can view that the system states are converging to origin proving asymptotic stability of the system (6.28). It is noted that the slight
fluctuations in the trajectories may be due to the presence of uncertainties and nonlinearities. □

**Example 6.2.** In this example we examine the performance of a three dimensional fractional order T-S fuzzy system. Consider the parameters as given below

\[
A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 30 \\ 0 & 4 & -4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 70 \\ 0 & 4 & -4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 0 & 25 \\ 0 & -1 & 50 \\ 0 & 4 & -4 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 0 & -25 \\ 0 & -1 & 50 \\ 0 & 4 & -4 \end{bmatrix}, \quad B_1 = B_2 = B_3 = B_4 = I_{3 \times 3}.
\]

The fuzzy sets membership function will be taken as

\[
h_1(\xi(t)) = \frac{1}{2}(1 + \frac{\xi_1(t)}{20}),
\]

\[
h_2(\xi(t)) = \frac{1}{2}(1 - \frac{\xi_1(t)}{20}),
\]

\[
h_3(\xi(t)) = \frac{1}{2}(1 + \frac{\xi_2(t)}{25}) \quad \text{and} \quad h_4(\xi(t)) = \frac{1}{2}(1 - \frac{\xi_2(t)}{25}),
\]

where \(\xi_1(t) = x_1(t)\) and \(\xi_2(t) = x_2(t)\).

Here \(\Delta K_i(t) = 0\) and the other parameters are chosen as \(q = 0.9\), \(\phi(u(t)) = 3.5\sin(u(t))\) and \(\Xi = 0.1\). The range of the fault is taken as \(0.7I_{3 \times 3} \leq G \leq 0.9I_{3 \times 3}\). By solving the obtained LMIs in Theorem 6.3, we will get the controller gains as

\[
K_1 = \begin{bmatrix} 0.0474 & 0.0000 & -0.0000 \\ 0.0000 & -16.8244 & -54.8679 \\ 0.0000 & -29.7294 & -109.6650 \end{bmatrix},
\]

\[
K_2 = \begin{bmatrix} 0.0474 & -0.0000 & -0.0000 \\ 0.0000 & -13.0985 & -48.5087 \\ 0.0000 & -26.8879 & -101.9281 \end{bmatrix},
\]

\[
K_3 = K_4 = \begin{bmatrix} 0.0477 & 0.4994 & -2.3880 \\ 0.0493 & -14.9325 & -52.6474 \\ -0.0369 & -28.5194 & -106.4968 \end{bmatrix}.
\]
The state responses of the open-loop and closed-loop system are given in Figure 6.4. From Figure 6.4(a), it can be easily seen that the state variables of the open-loop system do not converge to zero, but later when the controller is applied, it is evident from Figure 6.4(b), that the states of the system asymptotically converges to equilibrium point, which guarantees that the proposed controller can effectively tolerate actuator saturation and failures.

**Figure 6.4: State responses of open-loop and closed-loop system for Example 6.2**

The invariant ellipsoid and the state response of the closed-loop system is shown in Figure 6.5. We notice from Figure 6.5 that the trajectory starting from the invariant ellipsoid converges to the origin while remaining inside the ellipsoid itself.

**Figure 6.5: Invariant ellipsoid and state trajectory of Example 6.2**
Figure 6.6 shows the invariant ellipsoids for closed-loop system of this example for different values of $\Xi$. It is seen from Figure 6.6 that the invariant sets of the considered system differs significantly from each other for different choice of $\Xi$. Hence, we can conclude that the proposed resilient controller allows to make trade-off between the size of invariant sets by adjusting the tuning parameters.

Figure 6.6: Invariant ellipsoids for different values of $\Xi$ for Example 6.2

Figure 6.7 represents the invariant ellipsoids when the actuator fault matrix $G$ is known with $G = I$ (normal) and $G = 0.8$ (defective). It can be seen from Figure 6.7 that the ellipsoid obtained when the actuator is defective lies inside the ellipsoid obtained when the actuator is normal. Hence, the area of the invariant ellipsoid decreases when the actuator is defective which indicates that the component failures influence systems stability region.
Figure 6.7: Invariant ellipsoids when the actuator is normal (black & white) and defective (colour) for Example 6.2

In short, the simulation results of the numerical examples reveal that the derived sufficient conditions for the proposed controller can stabilize the fractional order T-S fuzzy control system even in the presence of nonlinearities, gain fluctuations, actuator saturation and actuator failures. □