Chapter 5

Oscillation of Impulsive Vector Partial Differential Equations with Distributed Deviating Arguments
5.1 Introduction

In 1970, Domšlak introduced the concept of H-oscillation to study the oscillation of the solutions of vector differential equations, where H is a unit vector in $\mathbb{R}^M$. In the papers [32, 33, 66] the study was based on vector ordinary differential equations, [96, 97, 102, 103, 109, 112] for vector partial differential equations and [81, 109] for impulsive vector partial differential equations. To have a clear idea, the study of monographs [16, 72, 116, 141, 148] and the references cited therein forms the essential background on the oscillation theory of differential equations. There has been very little work on the study of impulsive vector partial differential equations with functional arguments. This scarcity has been the motivation which led an attempt to initiate a research effort and make some progress, in the study of impulsive nonlinear vector partial differential equations with continuous distributed deviating arguments.

In this chapter, the impulsive nonlinear neutral delay vector partial differential equations with distributed deviating arguments is of the form...
\[
\frac{\partial}{\partial t} \left[ h(t) \frac{\partial}{\partial t} \left( U(x, t) + \int_a^b r(t, \zeta)U(x, \theta(t, \zeta))d\mu(\zeta) \right) \right] + g(t) \frac{\partial}{\partial t} \left( U(x, t) + \int_a^b r(t, \zeta)U(x, \theta(t, \zeta))d\mu(\zeta) \right) + \int_a^b p(x, t, \zeta)U(x, \sigma(t, \zeta))d\mu(\zeta) = a(t) \Delta U(x, t) \\
+ \sum_{j=1}^m b_j(t)\Delta U(x, \rho_j(t)) + F(x, t), \quad t \neq \tau_i \\
U(x, \tau_i^+) = c_i(x, \tau_i, U(x, \tau_i)), \\
\frac{\partial U(x, \tau_i^+)}{\partial t} = d_i(x, \tau_i, \frac{\partial U(x, \tau_i)}{\partial t}), \quad i = 1, 2, \ldots,
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with a piecewise smooth boundary \( \partial \Omega \) and \( \Delta \) is the Laplacian in the Euclidean space \( \mathbb{R}^N \).

The following conditions of Robin and Dirichlet boundary conditions also enhance in the equation (5.1.1)

\[
\frac{\partial}{\partial \eta} U(x, t) + \gamma(x, t)U(x, t) = 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}_+ 
\]

\[
U(x, t) = 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}_+ 
\]

where \( \eta \) is the unit exterior normal vector to \( \partial \Omega \), \( \gamma(x, t) \in C(\partial \Omega \times \mathbb{R}_+, \mathbb{R}_+) \) and \( \mathbb{R}_+ = [0, +\infty) \).

The problem (5.1.1) will be reduced to the following vector hyperbolic differential equations with deviating arguments

\[
\frac{\partial^2}{\partial t^2} U(x, t) = a(t) \Delta U(x, t) + \sum_{k=1}^m b_k(t)\Delta U(x, \tau_k(t)) - p(x, t)U(x, t) - \sum_{h=1}^l \int_a^b g_h(x, t, \xi)U(x, g_h(t, \xi))d\sigma(\xi) + F(x, t), \quad (x, t) \in \Omega \times [0, \infty) \equiv G
\]

and generalized the results obtained in [82] with impulse effects.
Next, present the following set of conditions which assume to hold, throughout the chapter.

\((A_1)\) \(g(t) \in C(\mathbb{R}_+, \mathbb{R}), h(t) \in C^1(\mathbb{R}_+, (0, +\infty))\) with \(h'(t) \geq 0\), \(\int_{t_0}^{+\infty} \frac{1}{\Pi(s)} \, ds = +\infty\), where \(\Pi(t) = \exp\left(\int_{t_0}^{t} \frac{h'(s) + g(s)}{h(s)} \, ds\right)\), and \(r(t, \zeta) \in C^2(\mathbb{R}_+ \times [a, b], \mathbb{R}_+)\).

\((A_2)\) \(p(x, t, \zeta) \in C(\bar{\Omega} \times \mathbb{R}_+ \times [a, b], \mathbb{R}_+), P(t, \zeta) = \min_{x \in \Omega} p(x, t, \zeta), \theta(t, \zeta), \sigma(t, \zeta) \in C(\mathbb{R}_+ \times [a, b], \mathbb{R}), 0 \leq t_0 = \tau_0 < \tau_1 < \cdots, \lim_{t \to \infty} \tau_i = \infty, \theta(t, \zeta) \leq t, \sigma(t, \zeta) \leq t \) for \(\zeta \in [a, b]\), \(\theta(t, \zeta)\) and \(\sigma(t, \zeta)\) are increasing with respect to \(t\) and \(\zeta\) respectively and

\[
\lim_{t \to +\infty, \zeta \in [a, b]} \theta(t, \zeta) = \lim_{t \to +\infty, \zeta \in [a, b]} \sigma(t, \zeta) = +\infty.
\]

There exists a function \(\varphi(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)\) that satisfies \(\varphi(t) \leq \sigma(t, a)\), with \(\varphi'(t) > 0\) and \(\lim_{t \to +\infty} \varphi(t) = +\infty\).

\((A_3)\) \(a(t), b_j(t) \in PC(\mathbb{R}_+, \mathbb{R}_+), j = 1, 2, \ldots, m,\)

(i) \(PC\) denotes the class of functions which are piecewise continuous in \(t\);

(ii) these functions are discontinuities of the first kind only at \(t = \tau_i, i = 1, 2, \cdots\) and

(iii) left continuous at \(t = \tau_i, i = 1, 2, \cdots\).

\((A_4)\) \(\rho_j(t) \in C(\mathbb{R}_+, \mathbb{R}), \lim_{t \to +\infty} \rho_j(t) = +\infty\) for \(j = 1, 2, \ldots, m, \mu(\zeta) : [a, b] \to \mathbb{R}\) is increasing and the integral which is obtained is called is stieltjes integral in [5.1.1], \(F \in C(\bar{G}, \mathbb{R}^M), f_H(x, t) \in C(\bar{G}, \mathbb{R})\) and \(\int_{\Omega} f_H(x, t) \, dx \leq 0\).

\((A_5)\) All the components of \(U(x, t)\) and their derivative \(\frac{\partial}{\partial t} U(x, t)\)

(i) are piecewise continuous in \(t\);
(ii) these functions are discontinuities of the first kind only at \( t = \tau_i \), 
\[ i = 1, 2, \cdots \] and

(iii) left continuous at \( t = \tau_i \),
\[ U(x, \tau_i) = U(x, \tau_i^-), \quad \frac{\partial}{\partial t} U(x, \tau_i) = \frac{\partial}{\partial t} U(x, \tau_i^-), \quad i = 1, 2, \cdots. \]

\((A_0)\) \( c_i, d_i \in PC(\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}) \) for \( i = 1, 2, \cdots \), and there exist constants \( \alpha_i, \alpha^*_i, \beta_i, \beta^*_i \) such that for \( i = 1, 2, \cdots \),
\[ \alpha^*_i \leq \frac{c_i(x, \tau_i, U(x, \tau_i))}{U(x, \tau_i)} \leq \alpha_i, \quad \beta^*_i \leq \frac{\frac{\partial}{\partial t} U(x, \tau_i)}{\frac{\partial}{\partial t} U(x, \tau_i)} \leq \beta_i. \]

We present some definitions and review some noteworthy results, from the literature which we will use throughout the chapter.

**Definition 5.1.1.** By a solution of \((5.1.1), (5.1.2)\) we mean a function \( U(x, t) \in C^2(\Omega \times [t_{-1}, +\infty), \mathbb{R}^M) \cap C^1(\hat{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R}^M) \) which satisfies \((5.1.1)\), where
\[ t_{-1} := \min \left\{ 0, \min_{1 \leq i \leq m} \left[ \inf_{t \geq 0} \rho_i(t) \right], \min_{\zeta \in [a,b]} \left\{ \inf_{t \geq 0} \theta(t, \zeta) \right\} \right\} \quad \text{and} \]
\[ \hat{t}_{-1} := \min \left\{ 0, \min_{\zeta \in [a,b]} \left\{ \inf_{t \geq 0} \sigma(t, \zeta) \right\} \right\}. \]

Now based on this definition of a solution, we can precisely define what we mean by \( H \)-oscillation.

**Definition 5.1.2.** \([148]\) Let \( H \) be a fixed unit vector in \( \mathbb{R}^M \). A solution \( U(x, t) \) of \((5.1.1)\) is said to be \( H \)-oscillatory in \( G \) if the inner product \( \langle U(x, t), H \rangle \) has a zero in \( \Omega \times [t, +\infty) \) for \( t > 0 \). Otherwise it is a \( H \)-nonoscillatory.

For convenience, we introduce the following notations:
\[
\begin{align*}
  u_H(x, t) &= \langle U(x, t), H \rangle, \quad F(t) = r_0 \int_a^b P(t, \zeta) d\mu(\zeta), \quad f_H(x, t) = \langle F(x, t), H \rangle, \\
  V_H(t) &= \frac{1}{|\Omega|} \int_\Omega u_H(x, t) dx, \quad \tilde{V}_H(t) = K_\Phi \int_\Omega u_H(x, t) \Phi(x) dx.
\end{align*}
\]
where $r_0 = 1 - \int_a^b r(\sigma(t, \zeta), \zeta) d\mu(\zeta)$, $|\Omega| = \int_\Omega dx$, $K_\Phi = (\int_\Omega \Phi(x) dx)^{-1}$.

In Section 5.2, we discuss the $H$-oscillation of problem (5.1.1), (5.1.2). In Section 5.3, we discuss the $H$-oscillation of problem (5.1.1), (5.1.3). In Section 5.4 we present an example to point up the main results.

### 5.2 $H$-Oscillation Problem of (5.1.1) with Robin Boundary Condition

In this section, establish sufficient conditions for the $H$-oscillation of all solutions of problem (5.1.1).

#### Lemma 5.2.1

Let $H$ be a fixed unit vector in $\mathbb{R}^M$ and $U(x, t)$ be a solution of (5.1.1).

(i) If $u_H(x, t)$ is eventually positive, then $u_H(x, t)$ satisfies the scalar impulsive partial differential inequality

\[
\begin{aligned}
\frac{\partial}{\partial t} \left( h(t) \frac{\partial}{\partial t} \left( u_H(x, t) + \int_a^b r(t, \zeta) u_H(x, \theta(t, \zeta)) d\mu(\zeta) \right) \right) \\
+ g(t) \frac{\partial}{\partial t} \left( u_H(x, t) + \int_a^b r(t, \zeta) u_H(x, \theta(t, \zeta)) d\mu(\zeta) \right) \\
+ \int_a^b P(t, \zeta) u_H(x, \sigma(t, \zeta)) d\mu(\zeta) - a(t) \Delta u_H(x, t) \\
- \sum_{j=1}^{n} b_j(t) \Delta u_H(x, \rho_j(t)) \leq f_H(x, t), \quad t \neq \tau_i
\end{aligned}
\]  

\[\alpha_i^* \leq \frac{u_H(x, \tau_i^+)}{u_H(x, \tau_i)} \leq \alpha_i, \quad \beta_i^* \leq \frac{u_H'(x, \tau_i^+)}{u_H'(x, \tau_i)} \leq \beta_i, \quad i = 1, 2, \ldots.
\]

(ii) If $u_H(x, t)$ is eventually negative, then $u_H(x, t)$ satisfies the scalar impulsive partial differential inequality
\[
\frac{\partial}{\partial t} \left( h(t) \frac{\partial}{\partial t} \left( u_H(x, t) + \int_a^b r(t, \zeta) u_H(x, \theta(t, \zeta)) d\mu(\zeta) \right) \right) \\
+ g(t) \frac{\partial}{\partial t} \left( u_H(x, t) + \int_a^b r(t, \zeta) u_H(x, \theta(t, \zeta)) d\mu(\zeta) \right) \\
+ \int_a^b P(t, \zeta) u_H(x, \sigma(t, \zeta)) d\mu(\zeta) - a(t) \Delta u_H(x, t) \\
- \sum_{j=1}^m b_j(t) \Delta u_H(x, \rho_j(t)) = f_H(x, t), \quad t \neq \tau_i
\]

\[
\alpha_i^* \geq \frac{u_H(x, \tau_i^+)}{u_H(x, \tau_i)} \geq \alpha_i, \quad \beta_i^* \geq \frac{u_H' (x, \tau_i^+)}{u_H'(x, \tau_i)} \geq \beta_i, \quad i = 1, 2, \ldots
\]

**Proof.** (i) Let \( u_H(x, t) \) be eventually positive.

**Case 1:** \( t \neq \tau_i, \ i = 1, 2, \ldots \). Taking the inner product of (5.1.1) and \( H \), we get

\[
\frac{\partial}{\partial t} \left( h(t) \frac{\partial}{\partial t} \left( \langle U(x, t), H \rangle + \int_a^b r(t, \zeta) \langle U(x, \theta(t, \zeta)), H \rangle d\mu(\zeta) \right) \right) \\
+ g(t) \frac{\partial}{\partial t} \left( \langle U(x, t), H \rangle + \int_a^b r(t, \zeta) \langle U(x, \theta(t, \zeta)), H \rangle d\mu(\zeta) \right) \\
+ \int_a^b p(x, t, \zeta) \langle U(x, \sigma(t, \zeta)), H \rangle d\mu(\zeta) = a(t) \Delta \langle U(x, t), H \rangle \\
+ \sum_{j=1}^m b_j(t) \Delta \langle U(x, \rho_j(t)), H \rangle + \langle F(x, t), H \rangle,
\]

that is

\[
\frac{\partial}{\partial t} \left( h(t) \frac{\partial}{\partial t} \left( u_H(x, t) + \int_a^b r(t, \zeta) u_H(x, \theta(t, \zeta)) d\mu(\zeta) \right) \right) \\
+ g(t) \frac{\partial}{\partial t} \left( u_H(x, t) + \int_a^b r(t, \zeta) u_H(x, \theta(t, \zeta)) d\mu(\zeta) \right) \\
+ \int_a^b p(x, t, \zeta) u_H(x, \sigma(t, \zeta)) d\mu(\zeta) = a(t) \Delta u_H(x, t) \\
+ \sum_{j=1}^m b_j(t) \Delta u_H(x, \rho_j(t)) + f_H(x, t).
\]

Using condition \((A_2)\), we have

\[
\int_a^b p(x, t, \zeta) u_H(x, \sigma(t, \zeta)) d\mu(\zeta) \geq \int_a^b P(t, \zeta) u_H(x, \sigma(t, \zeta)) d\mu(\zeta).
\]
From (5.2.3) and (5.2.4), it follows that
\[
\begin{align*}
&\frac{\partial}{\partial t}\left(h(t)\frac{\partial}{\partial t}\left(u_H(x,t) + \int_a^b r(t,\zeta)u_H(x,\theta(t,\zeta))d\mu(\zeta)\right)\right) \\
&+ g(t)\frac{\partial}{\partial t}\left(u_H(x,t) + \int_a^b r(t,\zeta)u_H(x,\theta(t,\zeta))d\mu(\zeta)\right) \\
&+ \int_a^b P(t,\zeta)u_H(x,\sigma(t,\zeta))d\mu(\zeta) - a(t)\Delta u_H(x,t) \\
&- \sum_{j=1}^m b_j(t)\Delta u_H(x,\rho_j(t)) \leq f_H(x,t), \quad t \neq \tau_i.
\end{align*}
\]

(5.2.5)

**Case 2:** \( t = \tau_i, \; i = 1, 2, \cdots \). Taking the inner product of (5.1.1) and \( H \), and using \((A_6)\), we obtain

\[
\begin{align*}
\alpha^*_i \leq \frac{U(x,\tau_i^+)}{U(x,\tau_i)} \leq \alpha_i, \quad &\beta^*_i \leq \frac{U'(x,\tau_i^+)}{U'(x,\tau_i)} \leq \beta_i \\
\alpha^*_i \leq \langle U(x,\tau_i), H \rangle \leq \alpha_i, \quad &\beta^*_i \leq \langle U'(x,\tau_i), H \rangle \leq \beta_i
\end{align*}
\]

that is

\[
\alpha^*_i \leq \frac{u_H(x,\tau_i^+)}{u_H(x,\tau_i)} \leq \alpha_i, \quad \beta^*_i \leq \frac{u_H'(x,\tau_i^+)}{u_H'(x,\tau_i)} \leq \beta_i, \quad i = 1, 2, \cdots .
\]

(5.2.6)

Therefore, combining (5.2.5) and (5.2.6) we immediately obtain (5.2.1), which shows that \( u_H(x,t) \) satisfies the scalar impulsive partial differential inequality (5.2.1).

(ii) The following proof is similar to that of (i) and we omit it. The lemma is complete. \( \square \)

The inner product of boundary conditions (5.1.2) \((5.1.3)\) and \( H \) yields the following boundary conditions:

\[
\mathcal{D}_\eta u_H(x,t) + \gamma(x,t)u_H(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+.
\]

(5.1.2)′

\[
u_H(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+.
\]

(5.1.3)′

**Lemma 5.2.2.** Let \( H \) be a fixed unit vector in \( \mathbb{R}^M \). If the scalar impulsive partial differential inequality (5.2.1) has no eventually positive solutions and the scalar impulsive partial differential inequality (5.2.2) has no eventually negative
solutions, satisfying the boundary condition \((5.1.2)' [(5.1.3)']\), then each solution \(U(x,t)\) of \((5.1.1), (5.1.2)\) is \(H\)-oscillatory in \(\Omega \times \mathbb{R}_+\).

**Proof.** Suppose to the contrary that there is a \(H\)-nonoscillatory solution \(U(x,t)\) of \((5.1.1), (5.1.2)\) in \(G\), then \(u_H(x,t)\) is eventually positive or eventually negative. If \(u_H(x,t)\) is eventually positive, by Lemma 5.2.1 clearly that \(u_H(x,t)\) satisfies the scalar impulsive partial differential inequality \((5.2.1)\). In addition to this, it is clearly see that \(u_H(x,t)\) satisfies the boundary conditions \((5.1.2)' [(5.1.3)']\). This is a contradiction to the hypothesis.

Similarly, if \(u_H(x,t)\) is eventually negative using Lemma 5.2.1, clearly that \(u_H(x,t)\) satisfies the scalar impulsive partial differential inequality \((5.2.2)\). It is obvious that \(u_H(x,t)\) satisfies, the boundary conditions \((5.1.2)' [(5.1.3)']\). This is a contradiction. Thus the lemma is proved.

**Theorem 5.2.1.** Let \(H\) be a fixed unit vector in \(\mathbb{R}^M\). If the impulsive differential inequality

\[
(h(t)Z_H'(t))' + g(t)Z_H'(t) + F(t)Z_H(\varphi(t)) \leq 0, \quad t \neq \tau_i
\]

\[
\alpha_i^* \leq \frac{Z_H(\tau_i^+)}{Z_H(\tau_i)} \leq \alpha_i, \quad \beta_i^* \geq \frac{Z_H'(\tau_i^+)}{Z_H'(\tau_i)} \geq \beta_i, \quad i = 1, 2, \ldots
\]

(5.2.7)

has no eventually positive solutions and the impulsive differential inequality

\[
(h(t)Z_H'(t))' + g(t)Z_H'(t) + F(t)Z_H(\varphi(t)) \geq 0, \quad t \neq \tau_i
\]

\[
\alpha_i^* \geq \frac{Z_H(\tau_i^+)}{Z_H(\tau_i)} \geq \alpha_i, \quad \beta_i^* \leq \frac{Z_H'(\tau_i^+)}{Z_H'(\tau_i)} \leq \beta_i, \quad i = 1, 2, \ldots
\]

(5.2.8)

has no eventually negative solutions, then each solution \(U(x,t)\) of the problem \((5.1.1), (5.1.2)\) is \(H\)-oscillatory in \(G\).

**Proof.** Suppose to the contrary that there exists a solution \(U(x,t)\) of \((5.1.1), (5.1.2)\) which is not \(H\)-oscillatory in \(G\). Without loss of generality, we may assume that \(u_H(x,t) > 0\) in \(\Omega \times [t_0, +\infty)\), for some \(t_0 > 0\). By the assumption that there
exists a $\tau_1 > t_0$ such that $\theta(t, \zeta) \geq t_0$, $\sigma(t, \zeta) \geq t_0$ for $(t, \zeta) \in [\tau_1, +\infty) \times [a, b]$ and $\rho_j(t) \geq t_0$, $j = 1, 2, \ldots, m$ for $t \geq \tau_1$, we have that

$$u_H(x, \theta(t, \zeta)) > 0, \quad u_H(x, \sigma(t, \zeta)) > 0 \quad \text{and} \quad u_H(x, \rho_j(t)) > 0,$$

for $x \in \Omega$, $t \in [\tau_1, +\infty)$, $\zeta \in [a, b]$, $j = 1, 2, \ldots, m$.

For $t \geq t_0$ and $t \neq \tau_i$ for $i = 1, 2, \ldots$, inequality (5.2.1) is multiplied both sides by $\frac{1}{|\Omega|}$ and integrate with respect to $x$ over the domain $\Omega$ to attain

$$\begin{align*}
\frac{d}{dt} \left( h(t) \frac{d}{dt} \left( \frac{1}{|\Omega|} \int_{\Omega} u_H(x, t) dx + \frac{1}{|\Omega|} \int_{\Omega} \int_a^b r(t, \zeta) u_H(x, \theta(t, \zeta)) d\mu(\zeta) dx \right) \right) \\
+ g(t) \frac{d}{dt} \left( \frac{1}{|\Omega|} \int_{\Omega} u_H(x, t) dx + \frac{1}{|\Omega|} \int_{\Omega} \int_a^b r(t, \zeta) u_H(x, \theta(t, \zeta)) d\mu(\zeta) dx \right) \\
+ \frac{1}{|\Omega|} \int_{\Omega} \int_a^b P(t, \zeta) u_H(x, \sigma(t, \zeta)) d\mu(\zeta) dx - a(t) \frac{1}{|\Omega|} \int_{\Omega} \Delta u_H(x, t) dx \\
- \sum_{j=1}^m b_j(t) \frac{1}{|\Omega|} \int_{\Omega} \Delta u_H(x, \rho_j(t)) dx \leq \frac{1}{|\Omega|} \int_{\Omega} f_H(x, t) dx, \quad t \neq \tau_i.
\end{align*}$$

By applying Green’s formula and (5.1.2)’ is the boundary condition, we get

$$\int_{\Omega} \Delta u_H(x, t) dx = \int_{\partial\Omega} \frac{\partial}{\partial \eta} u_H(x, t) dS = - \int_{\partial\Omega} \gamma(x, t) u_H(x, t) dS \leq 0, \quad (5.2.10)$$

and for $j = 1, 2, \ldots, m$

$$\int_{\Omega} \Delta u_H(x, \rho_j(t)) dx = \int_{\partial\Omega} \frac{\partial}{\partial \eta} u_H(x, \rho_j(t)) dS$$

$$= - \int_{\partial\Omega} \gamma(x, \rho_j(t)) u_H(x, \rho_j(t)) dS \leq 0, \quad t \geq t_0 \quad (5.2.11)$$

where $dS$ is the surface element on $\partial\Omega$. Moreover, by $(A_4)$, $\int_{\Omega} f_H(x, t) dx \leq 0$.

Combining (5.2.9)-(5.2.11), we obtain

$$\begin{align*}
\frac{d}{dt} \left[ h(t) \frac{d}{dt} \left( V_H(t) + \int_a^b r(t, \zeta)V_H(\theta(t, \zeta)) d\mu(\zeta) \right) \right] \\
+ g(t) \frac{d}{dt} \left( V_H(t) + \int_a^b r(t, \zeta)V_H(\theta(t, \zeta)) d\mu(\zeta) \right) \\
+ \int_a^b P(t, \zeta)V_H(\sigma(t, \zeta)) d\mu(\zeta) \leq 0, \quad t \geq t_0.
\end{align*}$$
Setting $Z_H(t) = V_H(t) + \int_a^b r(t, \zeta)V_H(\theta(t, \zeta))d\mu(\zeta)$, we have
\[(h(t)Z_H'(t))' + g(t)Z_H'(t) + \int_a^b P(t, \zeta)V_H(\sigma(t, \zeta))d\mu(\zeta) \leq 0. \quad (5.2.12)\]

Obviously $Z_H(t) > 0$ for $t \geq \tau_1$. Next verify that $Z_H'(t) > 0$ for $t \geq \tau_2$. In fact assume there exists $T \geq \tau_2$ such that $Z_H'(T) \leq 0$. We have
\[h(t)Z_H''(t) + (h'(t) + g(t))Z_H'(t) \leq 0, \quad t \geq \tau_2 \quad (5.2.13)\]

and from $(A_1)$, it follows that $\Pi'(t) = \Pi(t) \left(\frac{h'(t) + g(t)}{h(t)}\right)$, $\Pi(t) > 0$ and $\Pi'(t) \geq 0$ for $t \geq \tau_2$. Multiplying both sides of this inequality by $\frac{\Pi(t)}{h(t)}$, we obtain
\[\Pi(t)Z_H''(t) + \Pi'(t)Z_H'(t) = (\Pi(t)Z_H'(t))' \leq 0. \quad (5.2.14)\]

From $(5.2.14)$ we have $\Pi(t)Z_H'(t) \leq \Pi(T)Z_H'(T) \leq 0, \quad t \geq T$. Thus
\[Z_H(t) \leq Z_H(T) + \Pi(T)Z_H'(T) \int_T^t ds \frac{d}{\Pi(s)} \quad \text{for} \quad t \geq T. \]

Again, from $(A_1)$, we have $\lim_{t \to \infty} Z_H(t) = -\infty$ which contradicts the fact that $Z_H(t) > 0$ for $t > 0$. Hence $Z_H'(t) > 0$ and since $\theta(t, \zeta) \leq t$ for $t \geq \tau_1$, we have
\[V_H(t) = Z_H(t) - \int_a^b r(t, \zeta)V_H(\theta(t, \zeta))d\mu(\zeta) \geq Z_H(t) - \int_a^b r(t, \zeta)Z_H(\theta(t, \zeta))d\mu(\zeta) \geq Z_H(t) \left(1 - \int_a^b r(t, \zeta)d\mu(\zeta)\right)\]

and
\[V_H(\sigma(t, \zeta)) \geq r_0 Z_H(\sigma(t, \zeta)).\]

Therefore from $(5.2.12)$, we have
\[(h(t)Z_H'(t))' + g(t)Z_H'(t) + r_0 \int_a^b P(t, \zeta)Z_H(\sigma(t, \zeta))d\mu(\zeta) \leq 0, \quad t \geq t_0.\]
From (A2) and $Z'_H(t) > 0$, we have

$$Z_H(\sigma(t, \zeta)) \geq Z_H(\sigma(t, a)) > 0, \quad \zeta \in [a, b] \quad \text{and} \quad \varphi(t) \leq \sigma(t, a) \leq t.$$ 

Thus, $Z_H(\varphi(t)) \leq Z_H(\sigma(t, a))$ and therefore

$$(h(t)Z'_H(t))' + g(t)Z'_H(t) + F(t)Z_H(\varphi(t)) \leq 0, \quad t \geq \tau_1. \tag{5.2.15}$$

For $t \geq t_0, t = \tau_i, i = 1, 2, \cdots$, inequality (5.2.1) is multiplied both sides by $\frac{1}{|\Omega|}$ and integrating with respect to $x$ over the domain $\Omega$,

$$\alpha^*_i \leq \frac{V_H(\tau^+_i)}{V_H(\tau_i)} \leq \alpha_i, \quad \beta^*_i \leq \frac{V'_H(\tau^+_i)}{V'_H(\tau_i)} \leq \beta_i.$$ 

Since $Z_H(t) = V_H(t) + \int_a^b r(t, \zeta)V_H(\theta(t, \zeta))d\mu(\zeta)$, we have that

$$\alpha^*_i \leq \frac{Z_H(\tau^+_i)}{Z_H(\tau_i)} \leq \alpha_i, \quad \beta^*_i \leq \frac{Z'_H(\tau^+_i)}{Z'_H(\tau_i)} \leq \beta_i. \tag{5.2.16}$$

Therefore (5.2.15) and (5.2.16) show that $Z_H(t) > 0$ is a positive solution of the impulsive differential inequality (5.2.7). This is a contradiction.

Suppose, now, that $u_H(x, t) < 0$ is a negative solution of the impulsive partial differential inequality (5.2.2) satisfying the boundary condition (5.1.2), $(x, t) \in \Omega \times [t_0, +\infty), \ t_0 > 0$. Using the above procedure, easily, we can reach a contradiction. The proof is complete. \hfill \Box

**Theorem 5.2.2.** Assume that if there exists a function $\psi(t) \in C^1(\mathbb{R}_+, (0, +\infty))$ which is increasing with respect to $t$, such that

$$\int_{t_0}^{+\infty} \prod_{t_0 \leq \tau_i < s} \left( \frac{\beta_i}{\alpha_i} \right)^{-1} \left[ \psi(s)F(s) - \frac{E^2(s)}{4G(s)} \right] ds = +\infty, \tag{5.2.17}$$

where

$$E(t) = \frac{\psi'(t)}{\psi(t)} - \frac{g(t)}{h(t)} \quad \text{and} \quad G(t) = \frac{\varphi'(t)}{\psi(\varphi(t))h(\varphi(t))},$$

then each solution of the problem (5.1.1), (5.1.2) is H-oscillatory in $G$. 
Proof. We prove that the inequality \((5.2.7)\) has no eventually positive solution if the conditions of Theorem \(5.2.1\) hold. Suppose that \(Z_H(t)\) is an eventually positive solution of the inequality \((5.2.7)\) then there exists a number \(\tau_1 \geq t_0\) such that \(Z_H(\varphi(t)) > 0\) for \(t \geq \tau_1\). Thus we have

\[
(h(t)Z_H'(t))' + g(t)Z_H'(t) + F(t)Z_H(\varphi(t)) \leq 0. \tag{5.2.18}
\]

To define

\[B(t) := \psi(t) \frac{h(t)Z_H'(t)}{Z_H(\varphi(t))},\]

then \(B(t) \geq 0\) and

\[B'(t) \leq \left(\frac{\psi'(t)}{\psi(t)} - \frac{g(t)}{h(t)}\right) B(t) - \psi(t) F(t) - \frac{B^2(t)}{\psi(\varphi(t))} \frac{\varphi'(t)}{h(\varphi(t))}.\]

Thus \(B'(t) \leq E(t)B(t) - F(t)\psi(t) - B^2(t)G(t)\) and \(B(\tau_1^+) \leq \frac{\beta_i}{\alpha_i^*} B(\tau_i)\). Let

\[A(t) = \prod_{t_0 \leq \tau_i < t} \left(\frac{\beta_i}{\alpha_i^*}\right)^{-1} B(t).\]

It is verify that \(B(t)\) is continuous in each interval \((\tau_i, \tau_{i+1}]\), and since \(B(\tau_1^+) \leq \frac{\beta_i}{\alpha_i^*} B(\tau_i)\), it follows that

\[A(\tau_i^+) = \prod_{t_0 \leq \tau_j \leq \tau_i} \left(\frac{\beta_i}{\alpha_i^*}\right)^{-1} B(\tau_i^+) \leq \prod_{t_0 \leq \tau_j < \tau_i} \left(\frac{\beta_i}{\alpha_i^*}\right)^{-1} B(\tau_i) = A(\tau_i)\]

and for all \(t \geq t_0\),

\[A(\tau_i^-) = \prod_{t_0 \leq \tau_j \leq \tau_i} \left(\frac{\beta_i}{\alpha_i^*}\right)^{-1} B(\tau_i^-) \leq \prod_{t_0 \leq \tau_j < \tau_i} \left(\frac{\beta_i}{\alpha_i^*}\right)^{-1} B(\tau_i) = A(\tau_i)\]

which implies \(A(t)\) is continuous on \([t_0, +\infty)\).

\[
A'(t) + \prod_{t_0 \leq \tau_i < t} \left(\frac{\beta_i}{\alpha_i^*}\right)^{-1} A^2(t)G(t) + \prod_{t_0 \leq \tau_i < t} \left(\frac{\beta_i}{\alpha_i^*}\right)^{-1} F(t)\psi(t) - A(t) E(t)
= \prod_{t_0 \leq \tau_i < t} \left(\frac{\beta_i}{\alpha_i^*}\right)^{-1} [B'(t) + B^2(t)G(t) - B(t)E(t) + F(t)\psi(t)] \leq 0,
\]
that is
\[
A'(t) \leq - \prod_{t_0 \leq \tau_i < t} \left( \frac{\beta_i}{\alpha_i^2} \right) G(t) A^2(t) + A(t) E(t) - \prod_{t_0 \leq \tau_i < t} \left( \frac{\beta_i}{\alpha_i^2} \right)^{-1} F(t) \psi(t). \tag{5.2.19}
\]

Taking
\[
X(t) = \left( \prod_{t_0 \leq \tau_i < t} \left( \frac{\beta_i}{\alpha_i^2} \right) G(t) \right)^{\frac{1}{2}} A(t) \quad \text{and} \quad Y(t) = \frac{E(t)}{2} \left( \prod_{t_0 \leq \tau_i < t} \left( \frac{\beta_i}{\alpha_i^2} \right)^{-1} \frac{1}{G(t)} \right)^{\frac{1}{2}},
\]
applying Lemma \textbf{4.1.3} with
\[
E(t) A(t) - \prod_{t_0 \leq \tau_i < t} \left( \frac{\beta_i}{\alpha_i^2} \right) G(t) A^2(t) \leq \frac{E^2(t)}{4G(t)} \prod_{t_0 \leq \tau_i < t} \left( \frac{\beta_i}{\alpha_i^2} \right)^{-1}.
\]
Thus
\[
A'(t) \leq - \prod_{t_0 \leq \tau_i < t} \left( \frac{\beta_i}{\alpha_i^2} \right)^{-1} \left[ F(t) \psi(t) - \frac{E^2(t)}{4G(t)} \right].
\]

Integrating both sides from \(t_0\) to \(t\), we have
\[
A(t) \leq A(t_0) - \int_{t_0}^{t} \prod_{t_0 \leq \tau_i < s} \left( \frac{\beta_i}{\alpha_i^2} \right)^{-1} \left[ F(s) \psi(s) - \frac{E^2(s)}{4G(s)} \right] ds.
\]
Letting \(t \to \infty\) and using \textbf{(5.2.17)} we have \( \lim_{t \to \infty} A(t) = -\infty \), which leads to a contradiction with \( A(t) \geq 0 \) and completes the proof. \( \square \)

\textbf{Theorem 5.2.3.} Assume that there exist functions \( \psi \) and \( \delta \in C^1(\mathbb{R}_+, (0, +\infty)) \) where \( \psi \) is increasing, and functions \( k, K \in C^1(\mathbb{D}, \mathbb{R}) \), where \( \mathbb{D} = \{(t, s): t \geq s \geq t_0 > 0\} \) such that

\begin{align*}
(A_7) & \quad K(t, t) = 0 \text{ and } K(t, s) > 0 \text{ for all } t > s \geq t_0, \\
(A_8) & \quad \frac{\partial K(t, s)}{\partial t} \geq 0 \text{ and } \frac{\partial K(t, s)}{\partial s} \leq 0, \\
(A_9) & \quad -\frac{\partial K(t, s)}{\partial s} = k(t, s) \sqrt{K(t, s)}.
\end{align*}
If
\[
\limsup_{t \to +\infty} \frac{1}{K(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \leq \tau_i < s} \left( \frac{\beta_i}{\alpha_i^*} \right)^{-1} \left( F(s)\psi(s)K(t,s)\delta(s) - \frac{1}{4} \left[ k(t,s)\delta(s) - \delta'(s) \sqrt{K(t,s)} - E(s)\delta(s) \sqrt{K(t,s)} \right]^2 \right) ds = +\infty,
\]
(5.2.20)
then each solution of the problem (5.1.1), (5.1.2) is H-oscillatory in G.

**Proof.** Let $Z_H(t)$ is an eventually positive solution of (5.2.7). It is proved in the proceeding of Theorem 5.2.2, we obtain
\[
A'(t) \leq - \prod_{t_0 \leq \tau_i < t} \left( \frac{\beta_i}{\alpha_i^*} \right) G(t)A^2(t) + A(t)E(t) - \prod_{t_0 \leq \tau_i < t} \left( \frac{\beta_i}{\alpha_i^*} \right)^{-1} F(t)\psi(t)
\]
the above inequality is multiplied by $K(t,s)\delta(s)$ for $t \geq s \geq T$, and integrating from $T$ to $t$, we get
\[
\int_{T}^{t} A'(s)K(t,s)\delta(s) ds \leq - \int_{T}^{t} \prod_{t_0 \leq \tau_i < s} \left( \frac{\beta_i}{\alpha_i^*} \right) G(s)A^2(s)K(t,s)\delta(s) ds + \int_{T}^{t} A(s)E(s)K(t,s)\delta(s) ds - \int_{T}^{t} \prod_{t_0 \leq \tau_i < s} \left( \frac{\beta_i}{\alpha_i^*} \right)^{-1} F(s)\psi(s)K(t,s)\delta(s) ds.
\]
Thus we have
\[
\int_{T}^{t} \prod_{t_0 \leq \tau_i < s} \left( \frac{\beta_i}{\alpha_i^*} \right)^{-1} F(s)\psi(s)K(t,s)\delta(s) ds \leq A(T)K(t,T)\delta(T)
\]
\[
- \int_{T}^{t} \left[ - \frac{\partial K(t,s)}{\partial s} \delta(s) - K(t,s)\delta'(s) - E(s)K(t,s)\delta(s) \right] A(s) ds
\]
\[
- \int_{T}^{t} \prod_{t_0 \leq \tau_i < s} \left( \frac{\beta_i}{\alpha_i^*} \right) G(s)A^2(s)K(t,s)\delta(s) ds.
\]
Therefore

\[
\int_{t}^{t} \prod_{t_{0} \leq t_{1} < s} \left( \frac{\beta_{i}}{\alpha_{i}} \right)^{-1} F(s) \psi(s) K(t, s) \delta(s) ds
- \frac{1}{4} \int_{t}^{t} \prod_{t_{0} \leq t_{1} < s} \left( \frac{\beta_{i}}{\alpha_{i}} \right)^{-1} \left[ \frac{k(t, s) \delta(s) - \delta'(s) \sqrt{K(t, s)} - E(s) \delta(s) \sqrt{K(t, s)}}{G(s) \delta(s)} \right]^{2} \leq A(T) K(t, T) \delta(T). \tag{5.2.21}
\]

From (5.2.21) for \( t \geq T \geq t_{0} \), we have

\[
\frac{1}{K(t, t_{0})} \int_{t_{0}}^{t} \prod_{t_{0} \leq t_{1} < s} \left( \frac{\beta_{i}}{\alpha_{i}} \right)^{-1} \left[ F(s) \psi(s) K(t, s) \delta(s)
- \frac{1}{4} \left[ \frac{k(t, s) \delta(s) - \delta'(s) \sqrt{K(t, s)} - E(s) \delta(s) \sqrt{K(t, s)}}{G(s) \delta(s)} \right]^{2} \right] ds
\]

\[
= \frac{1}{K(t, t_{0})} \left[ \int_{t_{0}}^{T} + \int_{T}^{t} \right] \left\{ \prod_{t_{0} \leq t_{1} < s} \left( \frac{\beta_{i}}{\alpha_{i}} \right)^{-1} \left[ F(s) \psi(s) K(t, s) \delta(s)
- \frac{1}{4} \left[ \frac{k(t, s) \delta(s) - \delta'(s) \sqrt{K(t, s)} - E(s) \delta(s) \sqrt{K(t, s)}}{G(s) \delta(s)} \right]^{2} \right] \right\} ds
\]

\[
\leq \frac{1}{H(t, t_{0})} \int_{t_{0}}^{T} \prod_{t_{0} \leq t_{1} < s} \left( \frac{b_{k}}{a_{k}} \right)^{-1} F(s) \psi(s) B(t, s) \phi(s) ds + \phi(T) A(T)
\]

\[
\leq \int_{t_{0}}^{T} \prod_{t_{0} \leq t_{1} < s} \left( \frac{\beta_{i}}{\alpha_{i}} \right)^{-1} F(s) \psi(s) \delta(s) ds + \delta(T) A(T).
\]

Letting \( t \to +\infty \), we have

\[
\limsup_{t \to +\infty} \frac{1}{K(t, t_{0})} \int_{t_{0}}^{t} \prod_{t_{0} \leq t_{1} < s} \left( \frac{\beta_{i}}{\alpha_{i}} \right)^{-1} \left[ F(s) \psi(s) K(t, s) \delta(s)
- \frac{1}{4} \left[ \frac{k(t, s) \delta(s) - \delta'(s) \sqrt{K(t, s)} - E(s) \delta(s) \sqrt{K(t, s)}}{G(s) \delta(s)} \right]^{2} \right] ds < +\infty,
\]

which leads to a contradiction with \( 5.2.20 \) and completes the proof. \( \Box \)

Choosing \( \delta(s) = \psi(s) \equiv 1 \), in Theorem \( 5.2.3 \) we establish the following result.
Corollary 5.2.1. Assume that the conditions of Theorem 5.2.3 hold and

\[
\limsup_{t \to +\infty} \frac{1}{K(t, t_0)} \int_{t_0}^{t} \prod_{t_0 \leq \tau_i < s} \left( \frac{\beta_i}{\alpha_i^*} \right)^{-1} \left( F(s)K(t, s) \right) - \frac{1}{4} \left[ k(t, s) - E(s)\sqrt{K(t, s)} \right]^2 ds = +\infty,
\]

then each solution of the problem \((5.1.1),(5.1.2)\) is H-oscillatory in \(G\).

Theorem 5.2.3 and Corollary 5.2.1 various oscillatory criteria is obtained by different choices of the weighted function \(K(t, s)\).

For example, choosing \(K(t, s) = (t - s)^{\lambda-1}, t \geq s \geq t_0\), in which \(\lambda > 2\) is an integer, then \(k(t, s) = (\lambda - 1)(t - s)^{\lambda-3}/2, t \geq s \geq t_0\). Corollary 5.2.1 leads to the following result.

Corollary 5.2.2. If is an integer \(\lambda > 2\) exists such that

\[
\limsup_{t \to +\infty} \frac{1}{(t - t_0)^{\lambda-1}} \int_{t_0}^{t} \prod_{t_0 \leq \tau_i < s} \left( \frac{\beta_i}{\alpha_i^*} \right)^{-1} (t - s)^{\lambda-1} \left( F(s) - \frac{1}{4G(s)} \times \left[ E^2(s) - 2(\lambda - 1)E(s) \frac{(t - s)}{(t - s)^2} + \frac{(\lambda - 1)^2}{(t - s)^2} \right] \right) ds = +\infty,
\]

then each solution of the problem \((5.1.1),(5.1.2)\) is H-oscillatory in \(G\).

Now we consider \(K(t, s) = [Q(t) - Q(s)]^\lambda, t \geq s \geq t_0\), where \(Q(t) = \int_{t_0}^{t} \frac{1}{q(s)} ds\) and \(\lim_{t \to +\infty} Q(t) = +\infty\), then \(k(t, s) = \lambda (Q(t) - Q(s))^{\lambda-2}/2\). This leads to the result in the following Corollary.

Corollary 5.2.3. If is an integer \(\lambda > 2\) exist, such that

\[
\limsup_{t \to +\infty} \frac{1}{[Q(t) - Q(t_0)]^\lambda} \int_{t_0}^{t} \prod_{t_0 \leq \tau_i < s} \left( \frac{\beta_i}{\alpha_i^*} \right)^{-1} [Q(t) - Q(s)]^\lambda \left( F(s) - \frac{1}{4G(s)} \times \left[ E^2(s) - \frac{2\lambda E(s)}{Q(t) - Q(s)} + \frac{\lambda^2}{[Q(t) - Q(s)]^2} \right] \right) ds = +\infty,
\]

then each solution of the problem \((5.1.1),(5.1.2)\) is H-oscillatory in \(G\).
5.3 $H$ - Oscillation Problem of (5.1.1) with Dirichlet Boundary Condition

In this section, establish sufficient conditions for the $H$-oscillation of all solutions of the problem (5.1.1), (5.1.3).

**Theorem 5.3.1.** Let $H$ be a fixed unit vector in $\mathbb{R}^M$. If the impulsive differential inequality

\[
\left( h(t)\tilde{Z}_H'(t) \right)' + g(t)\tilde{Z}_H(t) + F(t)\tilde{Z}_H(\varphi(t)) \leq 0, \quad t \neq \tau_i \quad \left\{ \begin{array}{l}
\alpha_i^+ \leq \frac{\tilde{Z}_H'(\tau_i^+)}{\tilde{Z}_H(\tau_i)} \leq \alpha_i, \\
\beta_i^+ \leq \frac{\tilde{Z}_H'(\tau_i^+)}{\tilde{Z}_H'(\tau_i)} \leq \beta_i \quad i = 1, 2, \ldots,
\end{array} \right. \tag{5.3.1}
\]

has no eventually positive solution and the impulsive differential inequality

\[
\left( h(t)\tilde{Z}_H'(t) \right)' + g(t)\tilde{Z}_H(t) + F(t)\tilde{Z}_H(\varphi(t)) \geq 0, \quad t \neq \tau_i \quad \left\{ \begin{array}{l}
\alpha_i^+ \geq \frac{\tilde{Z}_H'(\tau_i^+)}{\tilde{Z}_H(\tau_i)} \geq \alpha_i, \\
\beta_i^+ \geq \frac{\tilde{Z}_H'(\tau_i^+)}{\tilde{Z}_H'(\tau_i)} \geq \beta_i \quad i = 1, 2, \ldots,
\end{array} \right. \tag{5.3.2}
\]

has no eventually negative solution, then each solution $U(x,t)$ of the problem (5.1.1), (5.1.3) is $H$-oscillatory in $G$.

**Proof.** Suppose to the contrary that there exists a solution $U(x,t)$ of (5.1.1), (5.1.3) which is not $H$-oscillatory in $G$. Without loss of generality, we may assume that $u_H(x,t) > 0$ in $\Omega \times [t_0, +\infty)$, for some $t_0 > 0$. By the assumption that there exists a $\tau_1 > t_0$ such that $\theta(t,\zeta) \geq t_0$, $\sigma(t,\zeta) \geq t_0$ for $(t,\zeta) \in [\tau_1, +\infty) \times [a,b]$ and $\rho_j(t) \geq t_0$, $j = 1, 2, \ldots, m$ for $t \geq \tau_1$, we have that

\[ u_H(x,\theta(t,\zeta)) > 0, \quad u_H(x,\sigma(t,\zeta)) > 0 \quad \text{and} \quad u_H(x,\rho_j(t)) > 0, \]

for $x \in \Omega$, $t \in [\tau_1, +\infty)$, $\zeta \in [a,b]$, $j = 1, 2, \ldots, m$.

For $t \geq t_0$ and $t \neq \tau_i$ for $i = 1, 2, \ldots$, inequality (5.2.1) is multiplied both sides
by \( K_\Phi \Phi(x) \) and integrate with respect to \( x \) over the domain \( \Omega \) to attain
\[
\frac{\partial}{\partial t} \left( h(t) \frac{\partial}{\partial t} \left( K_\Phi \int_\Omega \Phi(x) u_H(x,t)dx + K_\Phi \int_a^b \Phi(x) r(t,\zeta) u_H(x,\theta(t,\zeta)) d\mu(\zeta) dx \right) \right) \\
+ g(t) \frac{\partial}{\partial t} \left( K_\Phi \int_\Omega \Phi(x) u_H(x,t)dx + K_\Phi \int_a^b \Phi(x) r(t,\zeta) u_H(x,\theta(t,\zeta)) d\mu(\zeta) dx \right) \\
+ K_\Phi \int_a^b \int_a^b \Phi(x) P(t,\zeta) u_H(x,\sigma(t,\zeta)) d\mu(\zeta) dx - a(t) K_\Phi \int_\Omega \Phi(x) \Delta u_H(x,t) dx \\
- \sum_{j=1}^m b_j(t) K_\Phi \int_\Omega \Phi(x) \Delta u_H(x,\rho_j(t)) dx \leq K_\Phi \int_\Omega \Phi(x) f_H(x,t) dx, \quad t \neq \tau_i.
\] (5.3.3)

By using Green’s formula and \( (5.1.3)' \) is the boundary condition, we get
\[
\int_\Omega K_\Phi \Phi(x) \Delta u_H(x,t) dx = K_\Phi \int_{\partial \Omega} \left[ \Phi(x) \frac{\partial u_H(x,t)}{\partial \eta} - u_H(x,t) \frac{\partial \Phi(x)}{\partial \eta} \right] dS \\
+ K_\Phi \int_\Omega u_H(x,t) \Delta \Phi(x) dx \\
= 0 - \lambda_0 \tilde{V}_H(t) \leq 0.
\] (5.3.4)

For \( j = 1, 2, \cdots, m \)
\[
\int_\Omega K_\Phi \Phi(x) \Delta u_H(x,\rho_j(t)) dx = K_\Phi \int_{\partial \Omega} \left[ \Phi(x) \frac{\partial u_H(x,\rho_j(t))}{\partial \eta} - u_H(x,\rho_j(t)) \frac{\partial \Phi(x)}{\partial \eta} \right] dS \\
+ K_\Phi \int_\Omega u_H(x,\rho_j(t)) \Delta \Phi(x) dx \\
= 0 - \lambda_0 \tilde{V}_H(\rho_j(t)) \leq 0
\] (5.3.5)

By \( (A_4) \), \( \int_\Omega f_H(x,t) dx \leq 0 \). In view of \( (5.3.3)-(5.3.5) \), we get
\[
\frac{d}{dt} \left[ h(t) \frac{d}{dt} \left( \tilde{V}_H(t) + \int_a^b r(t,\zeta) \tilde{V}_H(\theta(t,\zeta)) d\mu(\zeta) \right) \right] \\
+ g(t) \frac{d}{dt} \left( \tilde{V}_H(t) + \int_a^b r(t,\zeta) \tilde{V}_H(\theta(t,\zeta)) d\mu(\zeta) \right) \\
+ \int_a^b P(t,\zeta) \tilde{V}_H(\sigma(t,\zeta)) d\mu(\zeta) \leq 0, \quad t \geq t_0.
\]

Setting \( \tilde{Z}_H(t) = \tilde{V}_H(t) + \int_a^b r(t,\zeta) \tilde{V}_H(\theta(t,\zeta)) d\mu(\zeta) \), we have
\[
\left( h(t) \tilde{Z}_H'(t) \right)' + g(t) \tilde{Z}_H'(t) + \int_a^b P(t,\zeta) \tilde{V}_H(\sigma(t,\zeta)) d\mu(\zeta) \leq 0.
\]

The remainder proof is similar to that of Theorem 5.2.1 and hence the details are omitted. The proof is complete. \( \square \)
Theorem 5.3.2. Let the conditions of Theorem 5.2.2 hold, then each solution of the problem \((5.1.1), (5.1.3)\) is \(H\)-oscillatory in \(G\).

Theorem 5.3.3. Let the conditions of Theorem 5.2.3 hold, then each solution of the problem \((5.1.1), (5.1.3)\) is \(H\)-oscillatory in \(G\).

Corollary 5.3.1. Let the conditions of Corollary 5.2.1 hold, then each solution of the problem \((5.1.1), (5.1.3)\) is \(H\)-oscillatory in \(G\).

Corollary 5.3.2. Let the conditions of Corollary 5.2.2 hold, then each solution of the problem \((5.1.1), (5.1.3)\) is \(H\)-oscillatory in \(G\).

Corollary 5.3.3. Let the conditions of Corollary 5.2.3 hold, then each solution of the problem \((5.1.1), (5.1.3)\) is \(H\)-oscillatory in \(G\).

5.4 Example

In this section we provide an example to illustrate our main results.

Example 5.4.1. Consider the following impulsive partial differential equations of the form

\[
\begin{aligned}
\frac{\partial}{\partial t} \left[ 4 \frac{\partial}{\partial t} \left( U(x,t) + \frac{1}{2} \int_{\pi/2}^{\pi} U(x,t - \zeta) d\zeta \right) \right] \\
+ \left( -\frac{4}{5} \right) \frac{\partial}{\partial t} \left( U(x,t) + \frac{1}{2} \int_{\pi/2}^{\pi} U(x,t - \zeta) d\zeta \right) \\
+ \frac{2}{5} \int_{\pi/2}^{\pi} U(x,t - \zeta) d\zeta = \frac{17}{5} \Delta U(x,t) \\
+ \frac{3}{5} \Delta U(x,t - \pi) + \frac{12}{5} \Delta U(x,t - \frac{\pi}{2}) + F(x,t), \quad t \neq \tau_i,
\end{aligned}
\]

\[ (5.4.1) \]

\[ U(x,\tau_i^+) = \frac{i + 1}{i} U(x,\tau_i), \]

\[ \frac{\partial}{\partial t} U(x,\tau_i^+) = \frac{\partial}{\partial t} U(x,\tau_i), \quad i = 1, 2, \cdots , \]
for \((x,t) \in (0, \pi) \times \mathbb{R}_+,\) with the boundary condition

\[
U(0,t) = U(\pi,t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \neq \tau_i, \quad i = 1, 2, \ldots .
\] (5.4.2)

Here \(\Omega = (0, \pi),\) \(N = 1,\) \(M = 2,\) \(m = 2,\) \(\alpha_i = \alpha_i^* = \frac{i+1}{i},\) \(\beta_i = \beta_i^* = 1,\) \(h(t) = 4,\)
\(r(t,\zeta) = \frac{1}{2},\) \(\theta(t,\zeta) = t - \zeta,\) \(g(t) = -\frac{4}{5},\) \(\sigma(t,\zeta) = t - \zeta,\) \(\mu(\zeta) = \zeta,\) \(P(t,\zeta) = \frac{2}{5},\)
\(a(t) = \frac{17}{5},\) \(b_1(t) = \frac{3}{5},\) \(b_2(t) = \frac{12}{5},\) \(\rho_1(t) = t - \pi,\) \(\rho_2(t) = t - \frac{\pi}{2},\) \([a,b] = [\pi/2, \pi],\)
\[
F(x,t) = \begin{pmatrix} \frac{6}{5} \sin x \sin t \\ \sin x e^{-t} \left( -\frac{7}{5} + \frac{17}{5} e^{\pi} - \frac{2}{5} e^{\frac{\pi}{2}} \right) \end{pmatrix}.
\]

Let \(H = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\) then we have \(f_H(x,t) = f_{e_1}(x,t) = \frac{6}{5} \sin x \sin t\) and

\[
\int_{\Omega} f_{e_1}(x,t) dx = \frac{6}{5} \int_{\Omega} \sin x \sin t dx = \frac{12}{5} \sin t \leq 0, \quad 0 \leq t \leq \pi.
\]

Take \(\lambda = 3,\) \(\varphi(t) = t,\) \(\varphi'(t) = 1,\) \(\psi(t) = 1.\) Since \(t_0 = 1,\) \(\tau_i = 2^i,\) \(E(s) = \frac{1}{5},\)
\(G(s) = \frac{1}{4},\) \(F(s) = \frac{4\pi - \pi^2}{5}.\) Then hypotheses \((A_1)-(A_6)\) hold,

\[
\lim_{t \to +\infty} \int_{t_0}^{t} \prod_{i_0 \leq \tau_i < s} \frac{\beta_i^*}{\alpha_i} ds = \int_{1}^{+\infty} \prod_{1 < \tau_i < s} \frac{i}{i+1} ds
\]
\[
= \int_{1}^{\tau_1} \prod_{1 < \tau_i < s} \frac{i}{i+1} ds + \int_{\tau_1}^{\tau_2} \prod_{1 < \tau_i < s} \frac{i}{i+1} ds + \int_{\tau_2}^{\tau_3} \prod_{1 < \tau_i < s} \frac{i}{i+1} ds + \ldots
\]
\[
= 1 + \frac{1}{2} \times 2 + \frac{1}{2} \times \frac{2}{3} \times 2^2 + \ldots
\]
\[
= \sum_{n=0}^{\infty} \frac{2^n}{n+1} = +\infty.
\]

Thus

\[
\limsup_{t \to +\infty} \frac{1}{(t - 1)^2} \left\{ \int_{1}^{t} \prod_{1 < \tau_i < s} \frac{i+1}{i} (t - s)^2 \times \right.\]
\[
\left[ \frac{4\pi - \pi^2}{5} - \frac{1}{400} + \frac{1}{5(t-s)} - \frac{1}{4(t-s)^2} \right] ds \right\} = +\infty.
\]
Now it is easily checked that the hypotheses of Corollary 5.3.2 are verified and hence each solution $U(x,t)$ of equation (5.4.1), (5.4.2) is $e_1$-oscillatory in $G$. In fact

$$U(x,t) = \begin{pmatrix} \sin x \cos t \\ \sin x \ e^{-t} \end{pmatrix},$$

is one such solution of the problem (5.4.1), (5.4.2). We note the above solution $U(x,t)$ is not $e_2$-oscillatory in $G$, where $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We conclude this chapter with the following remark.

**Remark 5.4.1.** In this chapter, we have established some new oscillation criteria for impulsive vector partial differential equations with continuous distributed deviating arguments. We have derived sufficient conditions for the $H$-oscillation of solutions, using impulsive differential inequalities and average technique with two different boundary conditions. The example we have used is illustrative of the application of Corollary 5.3.2. The present results complement and extend those established for problems without impulses.